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Michael Weise

A note on the second derivatives of rHCT basis functions

CSC/14-04

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TECHNISCHE UNIVERSITÄT CHEMNITZ

Chemnitz Scientific Computing
Preprints

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CSC/14-04

Abstract

We consider reduced Hsieh-Clough-Tocher basis functions with respect to a splitting into subtriangles at the barycenter of the original triangular element. This article gives a proof that the second derivatives of those functions, which in general may jump at the subtriangle boundaries, do not jump at the barycenter.

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$$
\begin{aligned}
&9x_{k+1,\mathrm{b}}y_{k,\mathrm{b}} + 9x_{k,\mathrm{b}}y_{k+1,\mathrm{b}} + 4x_{k+1,k}y_{k+1,k} \\
&\quad = (2x_{k+1} - x_k - x_{k-1})(2y_k - y_{k+1} - y_{k-1}) \\
&\qquad + (2x_k - x_{k+1} - x_{k-1})(2y_{k+1} - y_k - y_{k-1}) \\
&\qquad + 4(x_{k+1} - x_k)(y_{k+1} - y_k) \\
&\quad = 4x_{k+1}y_k - 2x_{k+1}y_{k+1} - 2x_{k+1}y_{k-1} - 2x_k y_k + x_k y_{k+1} + x_k y_{k-1} \\
&\qquad - 2x_{k-1}y_k + x_{k-1}y_{k+1} + x_{k-1}y_{k-1} \\
&\qquad + 4x_k y_{k+1} - 2x_{k+1}y_{k+1} - 2x_{k-1}y_{k+1} - 2x_k y_k + x_{k+1}y_k + x_{k-1}y_k \\
&\qquad - 2x_k y_{k-1} + x_{k+1}y_{k-1} + x_{k-1}y_{k-1} \\
&\qquad + 4x_{k+1}y_{k+1} - 4x_{k+1}y_k - 4x_k y_{k+1} + 4x_k y_k \\
&\quad = (x_{k-1}y_{k-1} - x_{k-1}y_{k+1} - x_k y_{k-1} + x_k y_{k+1}) \\
&\qquad + (x_{k-1}y_{k-1} - x_{k-1}y_k - x_{k+1}y_{k-1} + x_{k+1}y_k) \\
&\quad = x_{k-1,k}y_{k-1,k+1} + x_{k-1,k+1}y_{k-1,k},
\end{aligned}
$$

which shows $C_{2:} = A_{2:}$.

In summary we have shown $\underline{A} = \underline{B} = \underline{C}$ or

$$(D_2\Psi_k|_{T_1})(\hat{a}_\mathrm{b}) = (D_2\Psi_k|_{T_2})(\hat{a}_\mathrm{b}) = (D_2\Psi_k|_{T_3})(\hat{a}_\mathrm{b}) \quad \forall\, k = 1, 2, 3.$$

This proves the stated hypothesis

$$(D_2\Psi|_{T_1})(\hat{a}_\mathrm{b}) = (D_2\Psi|_{T_2})(\hat{a}_\mathrm{b}) = (D_2\Psi|_{T_3})(\hat{a}_\mathrm{b}).$$

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1 Introduction

Some relevant problems such as the biharmonic problem or the plate problem can be described by a partial differential equation of fourth order. The weak formulation of any such problem features functions from the Sobolev space H^2. Thus, the functions themselves as well as their first and second generalised derivatives have to be square-integrable over the considered domain. The natural approach to solving such problems numerically by the finite element method is to use conforming finite elements. This means that the FE basis functions belong to a finite-dimensional subspace of the appropriate space H^2. This is fulfilled for FE basis functions which are globally C^1-continuous.

One example of C^1-continuous elements is the reduced Hsieh–Clough–Tocher (rHCT) element, which goes back to [1]. It is a triangular element with piecewise cubic shape functions defined on three subtriangles. The shape functions are constructed in such a way that the resulting global basis functions are C^1-continuous. The element uses the values of the function and both first derivatives at all three vertices as degrees of freedom, which sums up to 9 in total. Global C^1-continuity is achieved by inner C^1-continuity and the condition that the restriction of any shape function to any element edge has to be linear. The splitting into three subtriangles may be based on an arbitrary interior point. For simplicity we only consider the barycenter of the whole element as splitting point, which is also the most popular choice in the literature.

The goal of this article is to show the following remarkable property. While the second derivatives of rHCT shape functions based on a barycenter splitting may jump across internal edges, they do not jump at the barycenter of the element. We suppose that this property also holds at an arbitrary splitting point, but we have not checked if this is actually true. Only the proof for the barycenter is given in the course of the article.

Our practical motivation for this article comes from remarks 7.10 and 7.11 in [3]. The above property was used there to get rid of nodal jump terms in the construction of an a posteriori error estimator for rHCT elements for plate and laminate problems, but no proof was given.

2 Shape functions

There exist several approaches to the definition of rHCT shape functions. They all lead to the same functions eventually; only the formulations differ. We consider the method given in [2], which was also the basis for the implementation used in [3]. The construction of shape functions from [2] is recapitulated in this section.

Consider a split of the original triangle T with the vertices

$$a_j = [x_j, y_j]^\mathsf{T}, \quad j = 1, 2, 3$$

based on the barycenter

$$a_\mathrm{b} = [x_\mathrm{b}, y_\mathrm{b}]^\mathsf{T} := \tfrac{1}{3}(a_1 + a_2 + a_3).$$

Shape functions that belong to node a_j of the triangle T are written as a row vector

$$\Psi_j(a) = [\psi_j^{(0)}(a), \psi_j^{(1)}(a), \psi_j^{(2)}(a)]$$

and the full vector of all shape functions takes the form

$$\Psi(a) = [\Psi_1(a), \Psi_2(a), \Psi_3(a)]$$

at an arbitrary point $a = [x, y]^\mathsf{T}$. Shape functions with superscript (0) are related to the function value at the respective node and those with superscripts (1) and (2) are related to the function derivative with respect to x and y at the respective node.

In order to shorten the following expressions we introduce some abbreviations which will be used throughout the article. We use $x_{i,j}$ and $y_{i,j}$ to denote $x_i - x_j$ and $y_i - y_j$, respectively. This implies $x_{i,j} = -x_{j,i}$ and $y_{i,j} = -y_{j,i}$. Furthermore, all indices $k, k-1, k+1$ run from 1 to 3 and $k \pm 1$ is always understood implicitly as

$$k \pm 1 \;\mapsto\; \big((k \pm 1 - 1) \mod 3\big) + 1$$

to stay in the admissible index set $\{1, 2, 3\}$. Formulas that use k as an index are valid for $k = 1, 2, 3$.

The outer edges of the element are denoted by E_k and the inner edges by f_k. Their orientation is as given in Figure 1, which leads to the formulas

$$E_k = \begin{bmatrix} x_{k-1,k+1} \\ y_{k-1,k+1} \end{bmatrix} \text{ and } f_k = \begin{bmatrix} x_{k,\mathrm{b}} \\ y_{k,\mathrm{b}} \end{bmatrix}.$$

We define normals of the outer edges with the same length by

$$N_k = \begin{bmatrix} -y_{k-1,k+1} \\ x_{k-1,k+1} \end{bmatrix}.$$

The subtriangle containing E_k is denoted T_k. The Jacobians of the mappings from the reference triangle to the three subtriangles, confer also section 3, are

$$\underline{J}_k = \begin{bmatrix} x_{k+1,\mathrm{b}} & x_{k-1,\mathrm{b}} \\ y_{k+1,\mathrm{b}} & y_{k-1,\mathrm{b}} \end{bmatrix}.$$

As above, a comparison of coefficients gives

$$\begin{aligned}
&-9x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}} + x_{k,k-1}y_{k,k-1} - x_{k,k-1}y_{k-1,k+1} - x_{k-1,k+1}y_{k,k-1} \\
&\quad = -(x_{k-1,k} + x_{k-1,k+1})(y_{k-1,k} + y_{k-1,k+1}) + x_{k,k-1}y_{k,k-1} \\
&\qquad - x_{k,k-1}y_{k-1,k+1} - x_{k-1,k+1}y_{k,k-1} \\
&\quad = -x_{k-1,k}y_{k-1,k} - x_{k-1,k}y_{k-1,k+1} - x_{k-1,k+1}y_{k-1,k} - x_{k-1,k+1}y_{k-1,k+1} \\
&\qquad + x_{k,k-1}y_{k,k-1} - x_{k,k-1}y_{k-1,k+1} - x_{k-1,k+1}y_{k,k-1} \\
&\quad = -x_{k-1,k+1}y_{k-1,k+1}, \\
&9x_{k-1,\mathrm{b}}y_{k,\mathrm{b}} + 9x_{k,\mathrm{b}}y_{k-1,\mathrm{b}} + 4x_{k,k-1}y_{k,k-1} \\
&\quad = (2x_{k-1} - x_k - x_{k+1})(2y_k - y_{k+1} - y_{k-1}) \\
&\qquad + (2x_k - x_{k+1} - x_{k-1})(2y_{k-1} - y_k - y_{k+1}) + 4(x_k - x_{k-1})(y_k - y_{k-1}) \\
&\quad = 4x_{k-1}y_k - 2x_{k-1}y_{k+1} - 2x_{k-1}y_{k-1} - 2x_k y_k + x_k y_{k+1} + x_k y_{k-1} \\
&\qquad - 2x_{k+1}y_k + x_{k+1}y_{k+1} + x_{k+1}y_{k-1} \\
&\qquad + 4x_k y_{k-1} - 2x_{k+1}y_{k-1} - 2x_{k-1}y_{k-1} - 2x_k y_k + x_{k+1}y_k + x_{k-1}y_k \\
&\qquad - 2x_k y_{k+1} + x_{k+1}y_{k+1} + x_{k-1}y_{k+1} \\
&\qquad + 4x_k y_k - 4x_k y_{k-1} - 4x_{k-1}y_k + 4x_{k-1}y_{k-1} \\
&\quad = (x_{k-1}y_k - x_{k-1}y_{k+1} - x_{k+1}y_k + x_{k+1}y_{k+1}) \\
&\qquad + (x_k y_{k-1} - x_k y_{k+1} - x_{k+1}y_{k-1} + x_{k+1}y_{k+1}) \\
&\quad = x_{k-1,k+1}y_{k,k+1} + x_{k,k+1}y_{k-1,k+1}, \\
&2x_{k,k-1}y_{k,k-1} + x_{k,k-1}y_{k+1,k} + x_{k+1,k}y_{k,k-1} \\
&\quad = x_{k,k-1}(y_{k,k-1} + y_{k+1,k}) + (x_{k,k-1} + x_{k+1,k})y_{k,k-1} \\
&\quad = x_{k,k-1}y_{k+1,k-1} + x_{k+1,k-1}y_{k,k-1} \\
&\quad = x_{k-1,k}y_{k-1,k+1} + x_{k-1,k+1}y_{k-1,k},
\end{aligned}$$

which shows $B_{2:} = A_{2:}$. Similarly, one gets

$$\begin{aligned}
&-9x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}} + x_{k+1,k}y_{k+1,k} - x_{k+1,k}y_{k-1,k+1} - x_{k-1,k+1}y_{k+1,k} \\
&\quad = -(x_{k+1,k} + x_{k+1,k-1})(y_{k+1,k} + y_{k+1,k-1}) + x_{k+1,k}y_{k+1,k} \\
&\qquad - x_{k+1,k}y_{k-1,k+1} - x_{k-1,k+1}y_{k+1,k} \\
&\quad = -x_{k+1,k}y_{k+1,k} - x_{k+1,k}y_{k+1,k-1} - x_{k+1,k-1}y_{k+1,k} - x_{k-1,k+1}y_{k-1,k+1} \\
&\qquad + x_{k+1,k}y_{k+1,k} - x_{k+1,k}y_{k-1,k+1} - x_{k-1,k+1}y_{k+1,k} \\
&\quad = -x_{k-1,k+1}y_{k-1,k+1}, \\
&2x_{k+1,k}y_{k+1,k} + x_{k+1,k}y_{k,k-1} + x_{k,k-1}y_{k+1,k} \\
&\quad = x_{k+1,k}(y_{k+1,k} + y_{k,k-1}) + (x_{k+1,k} + x_{k,k-1})y_{k+1,k} \\
&\quad = x_{k+1,k}y_{k+1,k-1} + x_{k+1,k-1}y_{k+1,k} \\
&\quad = x_{k-1,k+1}y_{k,k+1} + x_{k,k+1}y_{k-1,k+1},
\end{aligned}$$

$A_{3:} = B_{3:} = C_{3:}$ follows analogously with y replaced by x; a double '$-$' cancels out.

Finally, we consider $A_{2:}$, $B_{2:}$, and $C_{2:}$ and get

$$\begin{aligned}
A_{2:} &= \Big(\underline{F}_k\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_k\,\underline{M}_k\Big)_{2:}\\
&= \tfrac{1}{54\mu^2}\Big(6x_{k-1,k+1}y_{k-1,k+1}(c^k + 2b^k_{k+1} + 2b^k_{k-1})^\mathsf{T}\\
&\qquad + 6x_{k-1,k+1}(-y_{k-1,k+1}c^k + y_{k,k-1}b^k_{k+1} + y_{k+1,k}b^k_{k-1})^\mathsf{T}\\
&\qquad - 6y_{k-1,k+1}(x_{k-1,k+1}c^k - x_{k,k-1}b^k_{k+1} - x_{k+1,k}b^k_{k-1})^\mathsf{T}\Big)\\
&= \tfrac{1}{9\mu^2}\Big(- x_{k-1,k+1}y_{k-1,k+1}c^k + (x_{k-1,k+1}y_{k,k+1} + x_{k,k+1}y_{k-1,k+1})b^k_{k+1}\\
&\qquad + (x_{k-1,k+1}y_{k-1,k} + x_{k-1,k}y_{k-1,k+1})b^k_{k-1}\Big)^\mathsf{T},\\
B_{2:} &= \Big(\underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{b})\underline{H}_{k+1} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b^k_{k+1})^\mathsf{T}\\
&\qquad + \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_{k+1}\underline{M}_k\Big)_{2:}\\
&= -\tfrac{1}{\mu^2}x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}}(c^k)^\mathsf{T} + \tfrac{1}{\mu^2}(x_{k-1,\mathrm{b}}y_{k,\mathrm{b}} + x_{k,\mathrm{b}}y_{k-1,\mathrm{b}})(b^k_{k+1})^\mathsf{T}\\
&\quad + \tfrac{1}{54\mu^2}\Big(6x_{k,k-1}y_{k,k-1}(c^k + 2b^k_{k+1} + 2b^k_{k-1})\\
&\qquad + 6x_{k,k-1}(-y_{k-1,k+1}c^k + y_{k,k-1}b^k_{k+1} + y_{k+1,k}b^k_{k-1})\\
&\qquad + 6y_{k,k-1}(-x_{k-1,k+1}c^k + x_{k,k-1}b^k_{k+1} + x_{k+1,k}b^k_{k-1})\Big)^\mathsf{T}\\
&= \tfrac{1}{9\mu^2}\Big((-9x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}} + x_{k,k-1}y_{k,k-1} - x_{k,k-1}y_{k-1,k+1} - x_{k-1,k+1}y_{k,k-1})c^k\\
&\qquad + (9x_{k-1,\mathrm{b}}y_{k,\mathrm{b}} + 9x_{k,\mathrm{b}}y_{k-1,\mathrm{b}} + 4x_{k,k-1}y_{k,k-1})b^k_{k+1}\\
&\qquad + (2x_{k,k-1}y_{k,k-1} + x_{k,k-1}y_{k+1,k} + x_{k+1,k}y_{k,k-1})b^k_{k-1}\Big)^\mathsf{T},\\
C_{2:} &= \Big(\underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{b})\underline{H}_{k-1} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b^k_{k-1})^\mathsf{T}\\
&\qquad + \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_{k-1}\underline{M}_k\Big)_{2:}\\
&= -\tfrac{1}{\mu^2}x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}}(c^k)^\mathsf{T} + \tfrac{1}{\mu^2}(x_{k+1,\mathrm{b}}y_{k,\mathrm{b}} + x_{k,\mathrm{b}}y_{k+1,\mathrm{b}})(b^k_{k-1})^\mathsf{T}\\
&\quad + \tfrac{1}{54\mu^2}\Big(6x_{k+1,k}y_{k+1,k}(c^k + 2b^k_{k+1} + 2b^k_{k-1})\\
&\qquad + 6x_{k+1,k}(-y_{k-1,k+1}c^k + y_{k,k-1}b^k_{k+1} + y_{k+1,k}b^k_{k-1})\\
&\qquad + 6y_{k+1,k}(-x_{k-1,k+1}c^k + x_{k,k-1}b^k_{k+1} + x_{k+1,k}b^k_{k-1})\Big)^\mathsf{T}\\
&= \tfrac{1}{9\mu^2}\Big((-9x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}} + x_{k+1,k}y_{k+1,k} - x_{k+1,k}y_{k-1,k+1} - x_{k-1,k+1}y_{k+1,k})c^k\\
&\qquad + (2x_{k+1,k}y_{k+1,k} + x_{k+1,k}y_{k,k-1} + x_{k,k-1}y_{k+1,k})b^k_{k+1}\\
&\qquad + (9x_{k+1,\mathrm{b}}y_{k,\mathrm{b}} + 9x_{k,\mathrm{b}}y_{k+1,\mathrm{b}} + 4x_{k+1,k}y_{k+1,k})b^k_{k-1}\Big)^\mathsf{T}.
\end{aligned}$$

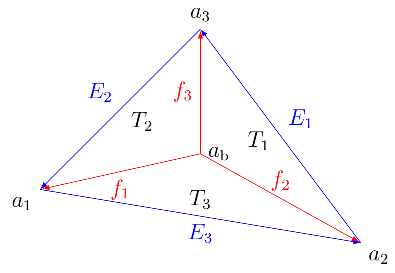

Figure 1: Triangle T with splitting at the barycenter

Their determinants

$$\mu = \det \underline{J}_k = x_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}} - x_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}}$$

are equal due to the use of the barycenter as the splitting point.

The final shape functions are constructed to fulfil three propositions.

1. The functions Ψ are cubic polynomials in each subtriangle, are continuous within T, and fulfil

$$\begin{aligned}
\Psi_j(a_i) &= [1,0,0]\,\delta_{ij}\\
\nabla\Psi_j(a_i) &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\delta_{ij} \quad \forall\, i,j = 1,2,3
\end{aligned}$$

   with the Kronecker delta

$$\delta_{ij} = \begin{cases}1 & i = j,\\ 0 & i \neq j.\end{cases}$$

2. The normal derivatives of all functions are linear along outer element edges with respect to the local line coordinate.
3. The functions are C^1-continuous inside T.

The final shape functions are defined with the help of basic functions and some transformations in order to assure the above propositions. We shortly repeat the results here, the whole derivation can be found in [2].

The formulas for all shape functions on subtriangle T_k read

$$\begin{aligned}
\Psi_k|_{T_k} &= \hat{\Phi}_0\,\underline{H}_k\,\underline{M}_k,\\
\Psi_{k+1}|_{T_k} &= \hat{\Phi}_1\,\underline{H}_k + \hat{\beta}\,(b^{k+1}_k)^\mathsf{T} + \hat{\Phi}_0\,\underline{H}_k\,\underline{M}_{k+1},\\
\Psi_{k-1}|_{T_k} &= \hat{\Phi}_2\,\underline{H}_k + \hat{\beta}\,(b^{k-1}_k)^\mathsf{T} + \hat{\Phi}_0\,\underline{H}_k\,\underline{M}_{k-1}
\end{aligned} \tag{1}$$

with the basic functions

$$\begin{aligned}\hat{\Phi}_0(\hat{a}) &= (1-\hat{x}-\hat{y})^2\,[1+2\hat{x}+2\hat{y},\ \hat{x},\ \hat{y}],\\ \hat{\Phi}_1(\hat{a}) &= \hat{x}^2\,[3-2\hat{x},\ \hat{x}-1,\ \hat{y}],\\ \hat{\Phi}_2(\hat{a}) &= \hat{y}^2\,[3-2\hat{y},\ \hat{x},\ \hat{y}-1],\\ \hat{\beta}(\hat{a}) &= \hat{x}\hat{y}(1-\hat{x}-\hat{y})\end{aligned} \tag{2}$$

given on the reference triangle

$$\hat{T} = \{[\hat{x},\hat{y}]^\mathsf{T} \in \mathbb{R}^2 : \hat{x}\ge 0, \hat{y} \ge 0, \hat{x}+\hat{y}\le 1\} \tag{3}$$

and the auxiliary terms

$$\begin{aligned}
\underline{H}_k &= \begin{bmatrix} 1 & 0 & 0\\ 0 & & \\ 0 & \multicolumn{2}{c}{J_k^\mathsf{T}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & x_{k+1,\mathrm{b}} & y_{k+1,\mathrm{b}}\\ 0 & x_{k-1,\mathrm{b}} & y_{k-1,\mathrm{b}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & f_{k+1}^\mathsf{T} \\ 0 & f_{k-1}^\mathsf{T} \end{bmatrix},\\
b_k^{k+1} &= \frac{1}{|E_k|^2}\begin{bmatrix} 6E_k^\mathsf{T} f_{k-1}\\ 3\mu N_k + 2|E_k|^2 f_{k-1}\end{bmatrix},\\
b_k^{k-1} &= \frac{1}{|E_k|^2}\begin{bmatrix} -6E_k^\mathsf{T} f_{k+1}\\ 3\mu N_k + 2|E_k|^2 f_{k+1}\end{bmatrix},\\
c^k &= \begin{bmatrix} 6\\ -2f_k\end{bmatrix},\\
\underline{S} &= -6\begin{bmatrix} 3 & f_1^\mathsf{T}\\ 3 & f_2^\mathsf{T}\\ 3 & f_3^\mathsf{T}\end{bmatrix} = -6\begin{bmatrix} 3 & x_{1,\mathrm{b}} & y_{1,\mathrm{b}}\\ 3 & x_{2,\mathrm{b}} & y_{2,\mathrm{b}}\\ 3 & x_{3,\mathrm{b}} & y_{3,\mathrm{b}}\end{bmatrix},\\
\underline{S}^{-1} &= -\frac{1}{54\mu}\begin{bmatrix} \mu & \mu & \mu\\ 3y_{2,3} & 3y_{3,1} & 3y_{1,2}\\ 3x_{3,2} & 3x_{1,3} & 3x_{2,1}\end{bmatrix} = -\frac{1}{54}\begin{bmatrix} 1 & 1 & 1\\ \frac{3}{\mu}N_1 & \frac{3}{\mu}N_2 & \frac{3}{\mu}N_3\end{bmatrix},\\
\underline{T}_k &= e_{k-1}(b_{k+1}^k)^\mathsf{T} + e_{k+1}(b_{k-1}^k)^\mathsf{T} + e_k(b_{k+1}^k + b_{k-1}^k + c^k)^\mathsf{T},\\
\underline{M}_k &= -\underline{S}^{-1}\underline{T}_k.
\end{aligned}$$

The e_j in the formula for $\underline{T}_k$ denote the j-th unit vectors with $(e_j)_i = \delta_{ij}$.

3 Transformation of second derivatives

The shape functions (1) are formulated with the help of the basic functions (2), which are given on the reference triangle (3). Each of the three subtriangles is mapped to the reference triangle by an affine linear mapping like illustrated in

$$\begin{aligned}
&= \tfrac{1}{9\mu^2}\Big((9y_{k+1,\mathrm{b}}^2 - y_{k+1,k}^2 + 2y_{k+1,k}y_{k-1,k+1})c^k\\
&\quad + (-2y_{k+1,k}^2 - 2y_{k+1,k}y_{k,k-1})b_{k+1}^k\\
&\quad + (-18y_{k,\mathrm{b}}y_{k+1,\mathrm{b}} - 4y_{k+1,k}^2)b_{k-1}^k\Big)^\mathsf{T}.
\end{aligned}$$

The definition of $\hat{a}_\mathrm{b}$ as barycenter yields

$$\begin{aligned}
y_{k,\mathrm{b}} &= y_k - y_\mathrm{b} = y_k - \tfrac{1}{3}(y_k + y_{k+1} + y_{k-1}) = \tfrac{2}{3}y_k - \tfrac{1}{3}y_{k+1} - \tfrac{1}{3}y_{k-1}\\
&= \tfrac{1}{3}(y_{k,k+1} + y_{k,k-1}).
\end{aligned}$$

We consider the coefficients of $(c^k)^\mathsf{T}$, $(b_{k+1}^k)^\mathsf{T}$, and $(b_{k-1}^k)^\mathsf{T}$ in $B_{1:}$ without the common factor of $1/(9\mu^2)$ with the help of the above formula and get

$$\begin{aligned}
&9y_{k-1,\mathrm{b}}^2 - y_{k,k-1}^2 + 2y_{k,k-1}y_{k-1,k+1}\\
&\quad = (y_{k-1,k} + y_{k-1,k+1})^2 - y_{k,k-1}^2 + 2y_{k,k-1}y_{k-1,k+1}\\
&\quad = y_{k-1,k}^2 + 2y_{k-1,k}y_{k-1,k+1} + y_{k-1,k+1}^2 - y_{k,k-1}^2 + 2y_{k,k-1}y_{k-1,k+1}\\
&\quad = y_{k-1,k+1}^2,\\
&-18y_{k-1,\mathrm{b}}y_{k,\mathrm{b}} - 4y_{k,k-1}^2\\
&\quad = -2(2y_{k-1} - y_k - y_{k+1})(2y_k - y_{k+1} - y_{k-1}) - 4(y_k - y_{k-1})^2\\
&\quad = -8y_{k-1}y_k + 4y_{k-1}y_{k+1} + 4y_{k-1}^2 + 4y_k^2 - 2y_ky_{k+1} - 2y_ky_{k-1}\\
&\qquad + 4y_{k+1}y_k - 2y_{k+1}^2 - 2y_{k+1}y_{k-1} - 4y_k^2 + 8y_ky_{k-1} - 4y_{k-1}^2\\
&\quad = -2y_ky_{k-1} + 2y_{k+1}y_{k-1} + 2y_ky_{k+1} - 2y_{k+1}^2 = 2y_{k-1,k+1}y_{k+1,k},\\
&-2y_{k,k-1}^2 - 2y_{k,k-1}y_{k+1,k}\\
&\quad = -2y_{k,k-1}(y_{k,k-1} + y_{k+1,k}) = -2y_{k+1,k-1}y_{k,k-1} = 2y_{k,k-1}y_{k-1,k+1},
\end{aligned}$$

which are the same coefficients as in $A_{1:}$. Therefore, it holds $B_{1:} = A_{1:}$. Similarly, $C_{1:} = A_{1:}$ follows from

$$\begin{aligned}
&9y_{k+1,\mathrm{b}}^2 - y_{k+1,k}^2 + 2y_{k+1,k}y_{k-1,k+1}\\
&\quad = (y_{k+1,k} + y_{k+1,k-1})^2 - y_{k+1,k}^2 + 2y_{k+1,k}y_{k-1,k+1}\\
&\quad = y_{k+1,k}^2 + 2y_{k+1,k}y_{k+1,k-1} + y_{k+1,k-1}^2 - y_{k+1,k}^2 + 2y_{k+1,k}y_{k-1,k+1}\\
&\quad = y_{k-1,k+1}^2,\\
&-2y_{k+1,k}^2 - 2y_{k+1,k}y_{k,k-1}\\
&\quad = -2y_{k+1,k}(y_{k+1,k} + y_{k,k-1}) = -2y_{k+1,k}y_{k+1,k-1} = 2y_{k-1,k+1}y_{k+1,k},\\
&-18y_{k,\mathrm{b}}y_{k+1,\mathrm{b}} - 4y_{k+1,k}^2\\
&\quad = -2(2y_k - y_{k+1} - y_{k-1})(2y_{k+1} - y_{k-1} - y_k) - 4(y_{k+1} - y_k)^2\\
&\quad = -8y_ky_{k+1} + 4y_ky_{k-1} + 4y_k^2 + 4y_{k+1}^2 - 2y_{k+1}y_{k-1} - 2y_{k+1}y_k\\
&\qquad + 4y_{k-1}y_{k+1} - 2y_{k-1}^2 - 2y_{k-1}y_k - 4y_{k+1}^2 + 8y_{k+1}y_k - 4y_k^2\\
&\quad = 2y_ky_{k-1} - 2y_ky_{k+1} - 2y_{k-1}^2 + 2y_{k-1}y_{k+1} = 2y_{k-1,k+1}y_{k,k-1}.
\end{aligned}$$

Next we use $N_k + N_{k+1} + N_{k-1} = 0$ to reformulate $\underline{M}_k$ as

$$
\begin{aligned}
\underline{M}_k &= -\underline{S}^{-1}\underline{T}_k \\
&= \tfrac{1}{54}\begin{bmatrix} 1 & 1 & 1 \\ \tfrac{3}{\mu}N_1 & \tfrac{3}{\mu}N_2 & \tfrac{3}{\mu}N_3 \end{bmatrix}\Big(e_{k-1}(b_{k+1}^k)^\intercal + e_{k+1}(b_{k-1}^k)^\intercal + e_k(b_{k+1}^k + b_{k-1}^k + c^k)^\intercal\Big) \\
&= \tfrac{1}{54}\begin{bmatrix} (c^k + 2b_{k+1}^k + 2b_{k-1}^k)^\intercal \\ \tfrac{3}{\mu}\Big(N_k(c^k + b_{k+1}^k + b_{k-1}^k)^\intercal + N_{k+1}(b_{k-1}^k)^\intercal + N_{k-1}(b_{k+1}^k)^\intercal\Big) \end{bmatrix} \\
&= \tfrac{1}{54}\begin{bmatrix} (c^k + 2b_{k+1}^k + 2b_{k-1}^k)^\intercal \\ \tfrac{3}{\mu}\Big(N_k(c^k)^\intercal - N_{k+1}(b_{k+1}^k)^\intercal - N_{k-1}(b_{k-1}^k)^\intercal\Big) \end{bmatrix}.
\end{aligned}
$$

With these intermediate results, we are now ready to formulate the rows of $\underline{A}$, $\underline{B}$, and $\underline{C}$ as linear combinations of the row vectors $(c^k)^\intercal$, $(b_{k+1}^k)^\intercal$, and $(b_{k-1}^k)^\intercal$. Denote the i-th row of $\underline{A}$ by $A_{i:}$, for $\underline{B}$ and $\underline{C}$ respectively.

We first consider $A_{1:}$, $B_{1:}$, and $C_{1:}$ and get

$$
\begin{aligned}
A_{1:} &= \Big(\underline{F}_k\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_k\,\underline{M}_k\Big)_{1:} \\
&= \tfrac{1}{54\mu^2}\Big(- 6y_{k-1,k+1}^2(c^k + 2b_{k+1}^k + 2b_{k-1}^k)^\intercal \\
&\qquad - 12y_{k-1,k+1}(-y_{k-1,k+1}c^k + y_{k,k-1}b_{k+1}^k + y_{k+1,k}b_{k-1}^k)^\intercal\Big) \\
&= \tfrac{1}{9\mu^2}y_{k-1,k+1}\Big(y_{k-1,k+1}c^k + 2y_{k+1,k}b_{k+1}^k + 2y_{k,k-1}b_{k-1}^k\Big)^\intercal, \\
B_{1:} &= \Big(\underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{b})\underline{H}_{k+1} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b_{k+1}^k)^\intercal \\
&\qquad + \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_{k+1}\underline{M}_k\Big)_{1:} \\
&= \tfrac{1}{\mu^2}y_{k-1,\mathrm{b}}^2(c^k)^\intercal - \tfrac{2}{\mu^2}y_{k-1,\mathrm{b}}y_{k,\mathrm{b}}(b_{k+1}^k)^\intercal \\
&\quad + \tfrac{1}{54\mu^2}\Big(- 6y_{k,k-1}^2(c^k + 2b_{k+1}^k + 2b_{k-1}^k) \\
&\qquad - 12y_{k,k-1}(-y_{k-1,k+1}c^k + y_{k,k-1}b_{k+1}^k + y_{k+1,k}b_{k-1}^k)\Big)^\intercal \\
&= \tfrac{1}{9\mu^2}\Big((9y_{k-1,\mathrm{b}}^2 - y_{k,k-1}^2 + 2y_{k,k-1}y_{k-1,k+1})c^k \\
&\qquad + (-18y_{k-1,\mathrm{b}}y_{k,\mathrm{b}} - 4y_{k,k-1}^2)b_{k+1}^k \\
&\qquad + (-2y_{k,k-1}^2 - 2y_{k,k-1}y_{k+1,k})b_{k-1}^k\Big)^\intercal, \\
C_{1:} &= \Big(\underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{b})\underline{H}_{k-1} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b_{k-1}^k)^\intercal \\
&\qquad + \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_{k-1}\underline{M}_k\Big)_{1:} \\
&= \tfrac{1}{\mu^2}y_{k+1,\mathrm{b}}^2(c^k)^\intercal - \tfrac{2}{\mu^2}y_{k,\mathrm{b}}y_{k+1,\mathrm{b}}(b_{k-1}^k)^\intercal \\
&\quad + \tfrac{1}{54\mu^2}\Big(- 6y_{k+1,k}^2(c^k + 2b_{k+1}^k + 2b_{k-1}^k) \\
&\qquad - 12y_{k+1,k}(-y_{k-1,k+1}c^k + y_{k,k-1}b_{k+1}^k + y_{k+1,k}b_{k-1}^k)\Big)^\intercal
\end{aligned}
$$

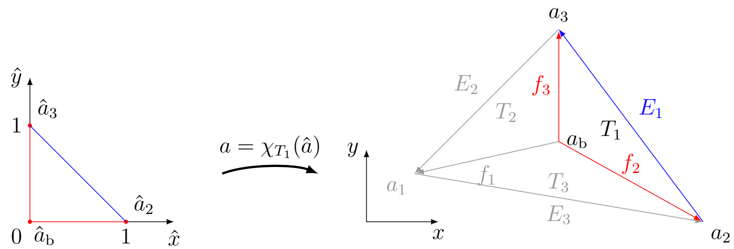

Figure 2: Mapping between the reference triangle and T_1

Figure 2. Inner edges are mapped to the axes of the reference triangle. This can be formulated as

$$
a = \chi_{T_k}(\hat{a}) = \underline{J}_k\hat{a} + a_\mathrm{b}, \quad \hat{a} = \hat{\chi}_{T_k}(a) = \chi_{T_k}^{-1}(a) = \underline{J}_k^{-1}(a - a_\mathrm{b}) \quad \text{for } a \in T_k
$$

with the Jacobian

$$
\underline{J}_k = [f_{k+1} \;\; f_{k-1}] = \begin{bmatrix} x_{k+1,\mathrm{b}} & x_{k-1,\mathrm{b}} \\ y_{k+1,\mathrm{b}} & y_{k-1,\mathrm{b}} \end{bmatrix}
$$

associated with the subtriangle T_k.

The derivatives with respect to the coordinates x and y can be obtained from the derivatives with respect to the master coordinates $\hat{x}$ and $\hat{y}$ via a simple transformation. It can be written for the second derivatives as

$$
(D_2\Psi|_{T_k})(\hat{a}) = \underline{F}_k\,(\hat{D}_2\Psi|_{T_k})(\hat{a}). \tag{4}
$$

with the matrix differential operators

$$
\hat{D}_2 = \left[\frac{\partial^2}{\partial\hat{x}^2}, \frac{\partial^2}{\partial\hat{x}\partial\hat{y}}, \frac{\partial^2}{\partial\hat{y}^2}\right]^\intercal, \quad D_2 = \left[\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x\partial y}, \frac{\partial^2}{\partial y^2}\right]^\intercal
$$

and an appropriate transformation matrix $\underline{F}_k$. The transformation matrix takes the form

$$
\begin{aligned}
\underline{F}_k &= \frac{1}{\mu^2}\begin{bmatrix} (\underline{J}_k)_{22}^2 & -2(\underline{J}_k)_{21}(\underline{J}_k)_{22} & (\underline{J}_k)_{21}^2 \\ -(\underline{J}_k)_{12}(\underline{J}_k)_{22} & (\underline{J}_k)_{12}(\underline{J}_k)_{21} + (\underline{J}_k)_{11}(\underline{J}_k)_{22} & -(\underline{J}_k)_{11}(\underline{J}_k)_{21} \\ (\underline{J}_k)_{12}^2 & -2(\underline{J}_k)_{11}(\underline{J}_k)_{12} & (\underline{J}_k)_{11}^2 \end{bmatrix} \\
&= \frac{1}{\mu^2}\begin{bmatrix} y_{k-1,\mathrm{b}}^2 & -2y_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}} & y_{k+1,\mathrm{b}}^2 \\ -x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}} & x_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}} + x_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}} & -x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}} \\ x_{k-1,\mathrm{b}}^2 & -2x_{k-1,\mathrm{b}}x_{k+1,\mathrm{b}} & x_{k+1,\mathrm{b}}^2 \end{bmatrix}
\end{aligned}
$$

as shown in section 8.3 of [3].

4 Second derivatives at the barycenter

The barycenter of the complete triangle has the master coordinates $\hat{a}_\mathrm{b} = [0,0]^\mathsf{T}$ for all three subtriangles. Our hypothesis that the second derivatives of the shape functions do not jump at the barycenter therefore reads

$$(D_2\Psi|_{T_1})(\hat{a}_\mathrm{b}) = (D_2\Psi|_{T_2})(\hat{a}_\mathrm{b}) = (D_2\Psi|_{T_3})(\hat{a}_\mathrm{b}).$$

This is a comparison of 3×9 values evaluated on 3 subelements, which gives 81 values which are to be shown as being 3 same sets of 27 values per set. After splitting the vector Ψ into the vertex related parts Ψ_1, Ψ_2, Ψ_3, one can use (1) (reformulated such that now the index of Ψ_* is constant and the index of T_* varies) and (4) to write

$$\begin{aligned}
\underline{A} &:= (D_2\Psi_k|_{T_k})(\hat{a}_\mathrm{b}) = \underline{F}_k\,(\hat{D}_2\Psi_k|_{T_k})(\hat{a}_\mathrm{b}) = \underline{F}_k\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_k\,\underline{M}_k,\\
\underline{B} &:= (D_2\Psi_k|_{T_{k+1}})(\hat{a}_\mathrm{b}) = \underline{F}_{k+1}\,(\hat{D}_2\Psi_k|_{T_{k+1}})(\hat{a}_\mathrm{b})\\
&= \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{b})\underline{H}_{k+1} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b^k_{k+1})^\mathsf{T} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_{k+1}\underline{M}_k,\\
\underline{C} &:= (D_2\Psi_k|_{T_{k-1}})(\hat{a}_\mathrm{b}) = \underline{F}_{k-1}\,(\hat{D}_2\Psi_k|_{T_{k-1}})(\hat{a}_\mathrm{b})\\
&= \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{b})\underline{H}_{k-1} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b^k_{k-1})^\mathsf{T} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_{k-1}\underline{M}_k
\end{aligned}$$

for any fixed k from 1 to 3. It remains to show $\underline{A} = \underline{B} = \underline{C}$; this is done in the following by evaluating all necessary terms.

The second master derivatives of all basic functions at the barycenter are

$$(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b}) = \begin{bmatrix} -6 & -4 & 0\\ -6 & -2 & -2\\ -6 & 0 & -4 \end{bmatrix}, \qquad (\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{b}) = \begin{bmatrix} 6 & -2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},$$

$$(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{b}) = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 6 & 0 & -2 \end{bmatrix}, \qquad (\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b}) = \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}.$$

This yields

$$\underline{F}_k(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{b}) = \frac{1}{\mu^2}\begin{bmatrix} 6y^2_{k-1,\mathrm{b}} & -2y^2_{k-1,\mathrm{b}} & 0\\ -6x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}} & 2x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}} & 0\\ 6x^2_{k-1,\mathrm{b}} & -2x^2_{k-1,\mathrm{b}} & 0 \end{bmatrix},$$

$$\underline{F}_k(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{b}) = \frac{1}{\mu^2}\begin{bmatrix} 6y^2_{k+1,\mathrm{b}} & 0 & -2y^2_{k+1,\mathrm{b}}\\ -6x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}} & 0 & 2x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}}\\ 6x^2_{k+1,\mathrm{b}} & 0 & -2x^2_{k+1,\mathrm{b}} \end{bmatrix},$$

$$\underline{F}_k(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b}) = \frac{1}{\mu^2}\begin{bmatrix} -2y_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}}\\ x_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}} + x_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}}\\ -2x_{k-1,\mathrm{b}}x_{k+1,\mathrm{b}} \end{bmatrix},$$

and

$$\begin{aligned}
&\underline{F}_k(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\\
&= \tfrac{1}{\mu^2}\left[\begin{matrix} -6(y_{k-1,\mathrm{b}}-y_{k+1,\mathrm{b}})^2 & -4y^2_{k-1,\mathrm{b}}+4y_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}} & \cdots\\ 6(x_{k-1,\mathrm{b}}-x_{k+1,\mathrm{b}})(y_{k-1,\mathrm{b}}-y_{k+1,\mathrm{b}}) & 4x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}}-2(x_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}}+x_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}}) & \cdots\\ -6(x_{k-1,\mathrm{b}}-x_{k+1,\mathrm{b}})^2 & -4x^2_{k-1,\mathrm{b}}+4x_{k+1,\mathrm{b}}x_{k-1,\mathrm{b}} & \cdots \end{matrix}\right.\\
&\qquad\left.\begin{matrix} 4y_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}}-4y^2_{k+1,\mathrm{b}}\\ 4x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}}-2(x_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}}+x_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}})\\ 4x_{k+1,\mathrm{b}}x_{k-1,\mathrm{b}}-4x^2_{k+1,\mathrm{b}} \end{matrix}\right]\\
&= \tfrac{1}{\mu^2}\begin{bmatrix} -6y^2_{k-1,k+1} & -4y_{k-1,\mathrm{b}}y_{k-1,k+1} & 4y_{k+1,\mathrm{b}}y_{k-1,k+1}\\ 6x_{k-1,k+1}y_{k-1,k+1} & 2x_{k-1,\mathrm{b}}y_{k-1,k+1}+2x_{k-1,k+1}y_{k-1,\mathrm{b}} & -2x_{k+1,\mathrm{b}}y_{k-1,k+1}-2x_{k-1,k+1}y_{k+1,\mathrm{b}}\\ -6x^2_{k-1,k+1} & -4x_{k-1,\mathrm{b}}x_{k-1,k+1} & 4x_{k+1,\mathrm{b}}x_{k-1,k+1} \end{bmatrix}.
\end{aligned}$$

For the next steps we recall

$$\mu = x_{k+1,\mathrm{b}}y_{k-1,\mathrm{b}} - x_{k-1,\mathrm{b}}y_{k+1,\mathrm{b}} \quad\text{and}\quad \underline{H}_k = \begin{bmatrix} 1 & 0 & 0\\ 0 & x_{k+1,\mathrm{b}} & y_{k+1,\mathrm{b}}\\ 0 & x_{k-1,\mathrm{b}} & y_{k-1,\mathrm{b}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & f^\mathsf{T}_{k+1} & \\ 0 & f^\mathsf{T}_{k-1} & \end{bmatrix}.$$

This yields

$$\underline{F}_k(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{b})\underline{H}_k = \tfrac{1}{\mu^2}\begin{bmatrix} -6y^2_{k-1,k+1} & -4\mu y_{k-1,k+1} & 0\\ 6x_{k-1,k+1}y_{k-1,k+1} & 2\mu x_{k-1,k+1} & -2\mu y_{k-1,k+1}\\ -6x^2_{k-1,k+1} & 0 & 4\mu x_{k-1,k+1} \end{bmatrix},$$

$$\begin{aligned}
\underline{F}_{k+1}(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{b})\underline{H}_{k+1} &= \tfrac{1}{\mu^2}\begin{bmatrix} 6y^2_{k-1,\mathrm{b}} & -2y^2_{k-1,\mathrm{b}}f^\mathsf{T}_{k,\mathrm{b}}\\ -6x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}} & 2x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}}f^\mathsf{T}_{k,\mathrm{b}}\\ 6x^2_{k-1,\mathrm{b}} & -2x^2_{k-1,\mathrm{b}}f^\mathsf{T}_{k,\mathrm{b}} \end{bmatrix}\\
&= \tfrac{1}{\mu^2}\begin{bmatrix} y^2_{k-1,\mathrm{b}}\\ -x_{k-1,\mathrm{b}}y_{k-1,\mathrm{b}}\\ x^2_{k-1,\mathrm{b}} \end{bmatrix}(c^k)^\mathsf{T},\\
\underline{F}_{k-1}(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{b})\underline{H}_{k-1} &= \tfrac{1}{\mu^2}\begin{bmatrix} 6y^2_{k+1,\mathrm{b}} & -2y^2_{k+1,\mathrm{b}}f^\mathsf{T}_{k,\mathrm{b}}\\ -6x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}} & 2x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}}f^\mathsf{T}_{k,\mathrm{b}}\\ 6x^2_{k+1,\mathrm{b}} & -2x^2_{k+1,\mathrm{b}}f^\mathsf{T}_{k,\mathrm{b}} \end{bmatrix}\\
&= \tfrac{1}{\mu^2}\begin{bmatrix} y^2_{k+1,\mathrm{b}}\\ -x_{k+1,\mathrm{b}}y_{k+1,\mathrm{b}}\\ x^2_{k+1,\mathrm{b}} \end{bmatrix}(c^k)^\mathsf{T},\\
\underline{F}_{k+1}(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b^k_{k+1})^\mathsf{T} &= \tfrac{1}{\mu^2}\begin{bmatrix} -2y_{k,\mathrm{b}}y_{k-1,\mathrm{b}}\\ x_{k-1,\mathrm{b}}y_{k,\mathrm{b}} + x_{k,\mathrm{b}}y_{k-1,\mathrm{b}}\\ -2x_{k,\mathrm{b}}x_{k-1,\mathrm{b}} \end{bmatrix}(b^k_{k+1})^\mathsf{T},\\
\underline{F}_{k-1}(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{b})(b^k_{k-1})^\mathsf{T} &= \tfrac{1}{\mu^2}\begin{bmatrix} -2y_{k+1,\mathrm{b}}y_{k,\mathrm{b}}\\ x_{k,\mathrm{b}}y_{k+1,\mathrm{b}} + x_{k+1,\mathrm{b}}y_{k,\mathrm{b}}\\ -2x_{k+1,\mathrm{b}}x_{k,\mathrm{b}} \end{bmatrix}(b^k_{k-1})^\mathsf{T}.
\end{aligned}$$