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Arnd Meyer

The linear Naghdi shell equation in a coordinate free description

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References

[1] D.Chapelle, K.-J. Bathe, *The Finite Element Analysis of Shells - Fundamentals*, 2nd ed.,410p.,Comp.Fluid and Solid Mech.,Springer (2011).

[2] P.G.Ciarlet,*The Finite Element Method for Elliptic Problems*, 530p., North-Holland, Amsterdam, (1978).

[3] P.G.Ciarlet, L.Gratie, *A new approach to linear shell theory*, Preprint Ser.No.17, Liu Bie Ju Centre for Math.Sc.(2005).

[4] A.Meyer,*The Koiter shell equation in a coordinate free description - extended*, CSC13-01, TU Chemnitz (2013).

[5] P.M.Naghdi, *Foundations of elastic shell theory*, *Progress in Solid Mechanics* vol.4,3-90, North-Holland Publ.Amsterdam (1963).

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Abstract

We give an alternate description of the usual shell equation that does not depend on the special mid surface coordinates, but uses differential operators defined on the mid surface.

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Then, using $\boldsymbol{u} = u_k \boldsymbol{A}^k$

$$Grad_{\mathcal{S}}\boldsymbol{u} = \boldsymbol{A}^i \boldsymbol{u}_{,i} = \boldsymbol{A}^i (u_{k,i} - u_j \Gamma^j_{ki}) \boldsymbol{A}^k + \boldsymbol{A}^i (u_j B^j_i) \boldsymbol{A}_3$$

we end up with

$$\boldsymbol{\theta} = -Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3 = (-u_j B^j_i - u_{3,i}) \boldsymbol{A}^i.$$

Remark: This formula $Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3$ can be understood as a generalization of the *Weingarten map*. Indeed, if $\boldsymbol{U}$ belongs to the tangential plane only (i.e. $\boldsymbol{U} = \boldsymbol{u}$ or $u_3 \equiv 0$), then we have

$$Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3 = (u_j B^j_i) \boldsymbol{A}^i = \mathcal{B} \cdot \boldsymbol{U}.$$

5.2 The Koiter shell equation from substituting $\boldsymbol{\theta}$

Now we may insert the vector $\boldsymbol{\theta}$ as $-Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3$ in the linearized strain tensor for finding an expression that should coincide with the coordinate free Koiter shell equations in [4]. We recall $\mathcal{E}^{Naghdi}$ as

$$2\mathcal{E}^{Naghdi} = \mathcal{E}_1 + \mathcal{E}_1^T$$

and use due to (8)

$$\mathcal{E}_1 = Grad_{\mathcal{S}}\boldsymbol{U} + \boldsymbol{\theta}\boldsymbol{A}_3 + \tau h (Grad_{\mathcal{S}}\boldsymbol{\theta} \cdot \mathcal{A}) - \tau h (\mathcal{B} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T).$$

With $\boldsymbol{\theta} = -Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3$ this leads to

$$\begin{aligned}
\mathcal{E}_1 &= Grad_{\mathcal{S}}\boldsymbol{U} - Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3 \boldsymbol{A}_3 \\
&\quad - \tau h (Grad_{\mathcal{S}} (Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3) \cdot \mathcal{A} - \tau h (\mathcal{B} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T) \\
&= Grad_{\mathcal{S}}\boldsymbol{U} \cdot \mathcal{A} - \tau h ([Grad_{\mathcal{S}} Grad_{\mathcal{S}}\boldsymbol{U}] \cdot \boldsymbol{A}_3) \cdot \mathcal{A} \\
&\quad - \tau h (\boldsymbol{A}^i (Grad_{\mathcal{S}}\boldsymbol{U}) \cdot \boldsymbol{A}_{3,i}) \cdot \mathcal{A} - \tau h (\mathcal{B} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T).
\end{aligned}$$

The last line disappears, due to

$$\begin{aligned}
&(\boldsymbol{A}^i (Grad_{\mathcal{S}}\boldsymbol{U}) \cdot \boldsymbol{A}_{3,i}) \cdot \mathcal{A} \\
&= (\boldsymbol{A}^i (-Grad_{\mathcal{S}}\boldsymbol{U} \cdot B_{ik} \boldsymbol{A}^k) \cdot \mathcal{A} \\
&= -B_{ik} \boldsymbol{A}^i (\boldsymbol{A}^k \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T) \cdot \mathcal{A} \\
&= -\mathcal{B} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T.
\end{aligned}$$

This result

$$\begin{aligned}
2\mathcal{E}^{Koiter} &= Grad_{\mathcal{S}}\boldsymbol{U} \cdot \mathcal{A} + \mathcal{A} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T \\
&\quad - \tau h (\ ([Grad_{\mathcal{S}} Grad_{\mathcal{S}}\boldsymbol{U}] \cdot \boldsymbol{A}_3) \cdot \mathcal{A} + \mathcal{A} \cdot ([Grad_{\mathcal{S}} Grad_{\mathcal{S}}\boldsymbol{U}] \cdot \boldsymbol{A}_3)^T\ )
\end{aligned}$$

coincides perfectly with the direct result in [4].

5.1 The linearization of $a_3(U) - A_3$

We start with

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\alpha}\beta,$$

where

$$\alpha = |\boldsymbol{A}_1 \times \boldsymbol{A}_2| \qquad \text{and} \qquad \beta = \frac{\alpha}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}$$

and linearize both factors separately:

$$\frac{1}{\alpha}(\boldsymbol{a}_1 \times \boldsymbol{a}_2) = \frac{1}{\alpha}(\ (\boldsymbol{A}_1 + \boldsymbol{U}_{,1}) \times (\boldsymbol{A}_2 + \boldsymbol{U}_{,2})\ ) = \boldsymbol{A}_3 + \boldsymbol{V} + \text{ nonlinear terms},$$

with the abbreviation

$$\boldsymbol{V} = \frac{1}{\alpha}(\boldsymbol{A}_1 \times \boldsymbol{U}_{,2}) + \frac{1}{\alpha}(\boldsymbol{U}_{,1} \times \boldsymbol{A}_2).$$

Now,

$$\beta^{-2} = 1 + 2\,(\boldsymbol{V} \cdot \boldsymbol{A}_3) + \text{ h.o.t.}$$

is obtained as $1 + 2Div_{\mathcal{S}}\boldsymbol{U} +$ h.o.t. (see [4]), hence

$$\beta = 1 - Div_{\mathcal{S}}\boldsymbol{U} + \text{ h.o.t.}$$

The product of both parts leads to

$$\boldsymbol{a}_3 - \boldsymbol{A}_3 = \boldsymbol{V} - (Div_{\mathcal{S}}\boldsymbol{U})\boldsymbol{A}_3 + \text{ h.o.t.}$$

Now we may write

$$\boldsymbol{V} = (\boldsymbol{A}^i\boldsymbol{A}_i + \boldsymbol{A}_3\boldsymbol{A}_3) \cdot \boldsymbol{V} = (\boldsymbol{A}^i\boldsymbol{A}_i) \cdot \boldsymbol{V} + (Div_{\mathcal{S}}\boldsymbol{U})\boldsymbol{A}_3$$

and the end result is obtained from $(\boldsymbol{A}^i\boldsymbol{A}_i) \cdot \boldsymbol{V} = -Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3$:

$$\begin{aligned} \boldsymbol{A}_1 \cdot \boldsymbol{V} &= \frac{1}{\alpha}\,[\boldsymbol{U}_{,1}\,,\ \boldsymbol{A}_2\,,\ \boldsymbol{A}_1] = -\boldsymbol{A}_3 \cdot \boldsymbol{U}_{,1} \\ \boldsymbol{A}_2 \cdot \boldsymbol{V} &= \frac{1}{\alpha}\,[\boldsymbol{A}_1\,,\ \boldsymbol{U}_{,2}\,,\ \boldsymbol{A}_2] = -\boldsymbol{A}_3 \cdot \boldsymbol{U}_{,2}\,, \end{aligned}$$

so,

$$(\boldsymbol{A}^i\boldsymbol{A}_i) \cdot \boldsymbol{V} = -\boldsymbol{A}^i(\boldsymbol{U}_{,i} \cdot \boldsymbol{A}_3) = -(\boldsymbol{A}^i\boldsymbol{U}_{,i}) \cdot \boldsymbol{A}_3 = -Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3. \tag{16}$$

Now, the setting $\boldsymbol{\theta} = -Grad_{\mathcal{S}}\boldsymbol{U} \cdot \boldsymbol{A}_3$ is exactly the same as (15), which can be seen from the splitting $\boldsymbol{U} = \boldsymbol{u} + u_3\,\boldsymbol{A}_3$

$$\begin{aligned} Grad_{\mathcal{S}}\boldsymbol{U} &= Grad_{\mathcal{S}}\boldsymbol{u} + (Grad_{\mathcal{S}}u_3)\boldsymbol{A}_3 + u_3 Grad_{\mathcal{S}}\boldsymbol{A}_3 \\ &= Grad_{\mathcal{S}}\boldsymbol{u} + (Grad_{\mathcal{S}}u_3)\boldsymbol{A}_3 - u_3\,\mathcal{B}. \end{aligned}$$

1 Introduction

We consider the deformation of a thin shell of constant thickness h under mechanical loads.

If a usual linear elastic material behavior is proposed, then consequently a linearized strain tensor has to be considered. In this case, the well established linear shell equation is obtained after some additional simplifications.

We consider these simplifications from the initial large strain equation to the linearized shell equation. Based on this, we are able to find a coordinate free description, which means that differential operators (defined on the mid surface of the shell) are used instead of derivatives with respect to the surface parameters (coordinates) (η^1, η^2).

2 Basic differential geometry

2.1 The initial mid surface

We start with the description of the basic differential geometry on both the undeformed shell (initial domain) and the shell after deformation. All vectors and matrices belonging to the initial configuration (mainly the co- and contravariant basis vectors and the matrices of first and second fundamental forms) are written as capital letters. All these quantities belonging to the deformed structure are the same lower case letters. Let

$$\mathcal{S}_0 = \left\{\boldsymbol{Y}\left(\eta^1, \eta^2\right) : \left(\eta^1, \eta^2\right) \in \Omega \subset \mathbb{R}^2\right\}$$

be the mid surface of the undeformed shell, where $\boldsymbol{Y}$ denote the points of the surface in the 3-dimensional space and (η^1, η^2) run through a parameter domain Ω. Then we have

$$\begin{aligned} \boldsymbol{A}_i &= \frac{\partial}{\partial \eta^i}\boldsymbol{Y} \quad \text{the tangential vectors } i = 1, 2 \\ \boldsymbol{A}_3 &= \boldsymbol{A}^3 = (\boldsymbol{A}_1 \times \boldsymbol{A}_2) \,/\, |\boldsymbol{A}_1 \times \boldsymbol{A}_2| \quad \text{surface normal vector.} \end{aligned}$$

This defines the first metrical fundamental forms $A_{ij} = \boldsymbol{A}_i \cdot \boldsymbol{A}_j$ written as the (2×2)-matrix

$$\underline{A} = (A_{ij})_{ij=1}^2\,.$$

The surface element is

$$d\mathcal{S} = |\boldsymbol{A}_1 \times \boldsymbol{A}_2|\ d\eta^1 d\eta^2 = (det\underline{A})^{1/2}\ d\eta^1 d\eta^2$$

and the contravariant basis is
$\boldsymbol{A}^j = A^{jk}\boldsymbol{A}_k$ with $\boldsymbol{A}^j \cdot \boldsymbol{A}_k = \delta^j_k$ and A^{jk} the entries of $\underline{A}^{-1}$.

The second fundamental forms are

$$B_{ij} = \left(\frac{\partial^2}{\partial\eta^i\partial\eta^j}\boldsymbol{Y}\right)\cdot\boldsymbol{A}_3 = \boldsymbol{A}_{i,j}\cdot\boldsymbol{A}_3 = -\boldsymbol{A}_i\cdot\boldsymbol{A}_{3,j}$$

forming the matrix $\underline{B} = (B_{ij})^2_{ij=1}$.

We recall the Gauss– and Weingarten–equations

$$\begin{aligned}\boldsymbol{A}_{i,j} &= \Gamma^k_{ij}\boldsymbol{A}_k + B_{ij}\boldsymbol{A}_3,\\ \boldsymbol{A}^i_{,j} &= -\,\Gamma^i_{jk}\boldsymbol{A}^k + B^i_j\boldsymbol{A}_3,\\ \boldsymbol{A}_{3,i} &= -\,B_{ij}A^{jk}\boldsymbol{A}_k = -B^k_i\boldsymbol{A}_k = -B_{ij}\boldsymbol{A}^j\end{aligned}$$

with the Christoffel symbols $2\Gamma^k_{ij} = A^{kl}(A_{il,j} + A_{jl,i} - A_{ij,l})$.

Throughout this paper we use Einstein's summation convention, where consequently all indices run from 1 to 2 only.

Later on, we will need the two second order tensors $\mathcal{A} = A_{ij}\boldsymbol{A}^i\boldsymbol{A}^j$ and $\mathcal{B} = B_{ij}\boldsymbol{A}^i\boldsymbol{A}^j$ often referred as metric tensor and curvature tensor of the surface $\mathcal{S}_0$.

Throughout this paper a pair of vectors (first order tensors) as $\boldsymbol{A}^1\boldsymbol{A}^2$ (or $\boldsymbol{A}_1\boldsymbol{A}_2$ or similar) is understood as second order tensor. A second order tensor in general is any linear combination of such pairs. The main meaning of a second order tensor is its action as a map of the (3-dimensional) vector functions onto itself via the dot product:

$$\begin{aligned}(\boldsymbol{A}^1\boldsymbol{A}^2)\cdot\boldsymbol{U} &= \boldsymbol{A}^1\ (\boldsymbol{A}^2\cdot\boldsymbol{U})\\ \boldsymbol{U}\cdot(\boldsymbol{A}^1\boldsymbol{A}^2) &= \boldsymbol{A}^2\ (\boldsymbol{U}\cdot\boldsymbol{A}^1)\end{aligned}$$

consequently the second order tensor $\boldsymbol{A}^1\boldsymbol{A}^2$ has a trace $tr(\boldsymbol{A}^1\boldsymbol{A}^2) = \boldsymbol{A}^1\cdot\boldsymbol{A}^2$ and the transposed tensor of $\boldsymbol{A}^1\boldsymbol{A}^2$ is $(\boldsymbol{A}^1\boldsymbol{A}^2)^T = \boldsymbol{A}^2\boldsymbol{A}^1$.

The double dot product between two second order tensors such as

$$(\boldsymbol{A}^1\boldsymbol{A}^2) : (\boldsymbol{A}^3\boldsymbol{A}^4) = (\boldsymbol{A}^2\cdot\boldsymbol{A}^3)(\boldsymbol{A}^1\cdot\boldsymbol{A}^4)$$

is a scalar function on (η^1,η^2). Later on, we use 4th order tensors in the same manner, as a 4-tuple of vectors ($\boldsymbol{A}^1\boldsymbol{A}^2\boldsymbol{A}^3\boldsymbol{A}^4$ and an arbitrary linear combination of those) or as a pair of second order tensors. Here, the main operation is the double dot product as a map of second order tensors onto second order tensors.

From this definition both tensors $\mathcal{A}$ and $\mathcal{B}$ are rank-2-tensors mapping each vector into the tangential space $span(\boldsymbol{A}_1,\boldsymbol{A}_2) = span(\boldsymbol{A}^1,\boldsymbol{A}^2)$. Especially $\mathcal{A}$ is the orthogonal projector onto this 2-dimensional space, due to:

$$\mathcal{A} = A_{ij}\boldsymbol{A}^i\boldsymbol{A}^j = \boldsymbol{A}_j\boldsymbol{A}^j = I - \boldsymbol{A}_3\boldsymbol{A}_3.$$

we obtain the well–known Naghdi shell energy

$$W(\boldsymbol{U},\boldsymbol{\theta}) = h\ W_{membr}(\boldsymbol{U}) + h\ W_{shear}(\boldsymbol{U},\boldsymbol{\theta}) + \frac{h^3}{12}W_{bend}(\boldsymbol{U},\boldsymbol{\theta})$$

with

$$\begin{aligned}W_{membr}(\boldsymbol{U}) &= \frac{1}{2}\int\limits_{\mathcal{S}_0}\gamma_{ij}(\underline{u})\ c^{ijkl}\ \gamma_{kl}(\underline{u})\ d\mathcal{S}\\ W_{shear}(\boldsymbol{U},\boldsymbol{\theta}) &= \frac{1}{2}\int\limits_{\mathcal{S}_0}\zeta_i(\underline{u},\underline{\theta})\ d^{ik}\ \zeta_k(\underline{u},\underline{\theta})\ d\mathcal{S}\\ W_{bend}(\boldsymbol{U},\boldsymbol{\theta}) &= \frac{1}{2}\int\limits_{\mathcal{S}_0}\chi_{ij}(\underline{u},\underline{\theta})\ c^{ijkl}\ \chi_{kl}(\underline{u},\underline{\theta})\ d\mathcal{S}.\end{aligned}$$

Note that the splitting $\mathcal{E}^{Naghdi} = \mathcal{E}^a + \tau h\,\mathcal{E}^b$ leads to

$$W(\boldsymbol{U},\boldsymbol{\theta}) = h\ W^a(\boldsymbol{U},\boldsymbol{\theta}) + \frac{h^3}{12}W^b(\boldsymbol{U},\boldsymbol{\theta})$$

in any case (from integrating τ^1 over $(-1/2,+1/2)$). But the splitting $\mathcal{E}^a = \mathcal{E}^{membr} + \mathcal{E}^{shear}$ does not necessarily imply the same splitting for the energy W^a. Here, we have used the special St.Venant-Kirchhoff material (14), where

$$\mathcal{E}^{membr} : \mathfrak{C} : \mathcal{E}^{shear} = 0.$$

This is not given for more general materials.

5 Introducing the Kirchhoff-Hypothesis towards Koiter–shell

In [1], the Koiter shell equation called the "membrane–bending model" is obtained from the "shear-membrane-bending model" by substituting

$$\theta_i = -u_{3,i} - u_jB^j_i, \qquad (15)$$

which leads to

$$\zeta_i(\underline{u},\underline{\theta}) = 0.$$

The basic setting for introducing Kirchhoff's hypothesis in [4] is the non-linear definition of $\boldsymbol{\theta} = \boldsymbol{a}_3(\boldsymbol{U}) - \boldsymbol{A}_3$. This cannot be used here, we have to consider only linear dependencies on $\boldsymbol{U}$. An interesting linearization of $\boldsymbol{a}_3 - \boldsymbol{A}_3$ leads to the same result (15), but can be given in a coordinate free form as well.

with the Lamé constants

$$2\mu = \frac{E}{1+\nu} \qquad \text{and} \qquad \lambda = 2\mu\frac{\nu}{1-\nu}$$

(for the plane stress assumption). Here, $\mathfrak{I}$ is the 4th order identity map ($\mathfrak{I} : \mathcal{X} = \mathcal{X}$ for each 2nd order tensor $\mathcal{X}$) and $(I\ I) : \mathcal{X} = I\ (I : \mathcal{X}) = I\ tr\mathcal{X}$.

Now, we end up with different representations of the shell energy, depending on which strain formulation is inserted into (13). If the material tensor $\mathfrak{C}$ is constant over the thickness (independent on τ), we integrate over $\tau \in [-1/2, +1/2]$ and end up with the 2 parts:

$$W(\boldsymbol{U},\boldsymbol{\theta}) = h\ W^a(\boldsymbol{U},\boldsymbol{\theta}) + \frac{h^3}{12}W^b(\boldsymbol{U},\boldsymbol{\theta})$$

with

$$W^a(\boldsymbol{U},\boldsymbol{\theta}) = \frac{1}{2}\int\limits_{\mathcal{S}_0} \mathcal{E}^a : \mathfrak{C} : \mathcal{E}^a\ d\mathcal{S}$$
$$W^b(\boldsymbol{U},\boldsymbol{\theta}) = \frac{1}{2}\int\limits_{\mathcal{S}_0} \mathcal{E}^b : \mathfrak{C} : \mathcal{E}^b\ d\mathcal{S}.$$

Here, $\mathcal{E}^a$ and $\mathcal{E}^b$ are the two parts of $\mathcal{E}^{Naghdi}$ with

$$2\mathcal{E}^a = Grad_{\mathcal{S}}\boldsymbol{U} + (Grad_{\mathcal{S}}\boldsymbol{U})^T + \ \boldsymbol{\theta}\boldsymbol{A}_3 + \boldsymbol{A}_3\boldsymbol{\theta},$$
and
$$2\mathcal{E}^b = Grad_{\mathcal{S}}\boldsymbol{\theta}\cdot\mathcal{A} + \mathcal{A}\cdot(Grad_{\mathcal{S}}\boldsymbol{\theta})^T - \ (Grad_{\mathcal{S}}\boldsymbol{U}\cdot\mathcal{B} + \mathcal{B}\cdot(Grad_{\mathcal{S}}\boldsymbol{U})^T)$$

This is a coordinate free form of the Naghdi shell energy.

The coordinate dependent representation of the shell energy arises from the substitution of

$$\mathcal{E}^{Naghdi} = \gamma_{ij}\boldsymbol{A}^i\boldsymbol{A}^j + \zeta_i(\boldsymbol{A}^i\boldsymbol{A}_3 + \boldsymbol{A}_3\boldsymbol{A}^i) + \tau h\chi_{ij}\boldsymbol{A}^i\boldsymbol{A}^j$$

into (13). From the definition of

$$\begin{aligned} c^{ijkl} &= (\boldsymbol{A}^i\boldsymbol{A}^j) : \mathfrak{C} : (\boldsymbol{A}^k\boldsymbol{A}^l) = 2\mu A^{il}A^{jk} + \lambda\, A^{ij}A^{kl} \\ &= \frac{E}{1+\nu}(\,A^{il}A^{jk} + \frac{\nu}{1-\nu}\,A^{ij}A^{kl}), \\ d^{ik} &= (\boldsymbol{A}^i\boldsymbol{A}_3 + \boldsymbol{A}_3\boldsymbol{A}^i) : \mathfrak{C} : (\boldsymbol{A}^k\boldsymbol{A}_3 + \boldsymbol{A}_3\boldsymbol{A}^k) \\ &= \frac{2E}{1+\nu}A^{ik} \end{aligned}$$

(Here, I denotes the identity tensor mapping each vector $\boldsymbol{U}$ onto itself).
It should be stressed that the two vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are dependent on the parametrization (η^1, η^2) chosen to define $\mathcal{S}_0$ but $\boldsymbol{A}_3$ not, hence $\mathcal{A}$ and $\mathcal{B}$ are independent on the special coordinates (η^1, η^2) but functions on the given point $\boldsymbol{Y}$ of $\mathcal{S}_0$ only. So, $(\boldsymbol{Y} \to \boldsymbol{A}_3)$ is called the Gaussian map and $\mathcal{B}$ the Weingarten map. Furthermore the surface gradient as gradient operator on the tangential space also is independent on the special parametrization (η^1, η^2), obviously

$$Grad_{\mathcal{S}} = \boldsymbol{A}^i\frac{\partial}{\partial\eta^i}.$$

The matrix $\underline{A}^{-1}\underline{B}$ has two eigenvalues λ_1 and λ_2 as main curvatures at $\boldsymbol{Y}(\eta^1, \eta^2)$, as well as the tensor $\mathcal{B}$ has these eigenvalues (together with a 0 as rank-2 tensor), so

$$\begin{aligned} &H = (\lambda_1+\lambda_2)/2 = tr\mathcal{B}/2 = tr(\underline{A}^{-1}\underline{B})/2 \text{ is the mean curvature and} \\ &K = \lambda_1\cdot\lambda_2 = det(\underline{A}^{-1}\underline{B}) \qquad\qquad \text{the Gaussian curvature at } \boldsymbol{Y}. \end{aligned}$$

2.2 The initial shell

The initial shell is the 3-dimensional manifold

$$\mathcal{H}_0 = \left\{\boldsymbol{X}\left(\eta^1,\eta^2,\tau=\eta^3\right) = \boldsymbol{Y}\left(\eta^1,\eta^2\right) + \tau h\boldsymbol{A}_3,\ \left(\eta^1,\eta^2\right)\in\Omega,\ |\tau| \le \frac{1}{2}\right\} \tag{1}$$

with the constant thickness h and $\boldsymbol{A}_3$ from 2.1. For an easy description of the following let $\tau = \eta^3$ be a synonym for the (dimensionless) thickness coordinate. We may use η^1 and η^2 dimensionless as well, then $\boldsymbol{A}_i$ have length dimension (in m) and $\boldsymbol{A}^i$ in $1/m$ while $\boldsymbol{A}_3 = \boldsymbol{A}^3$ is dimensionless in any case. In 3D we have to consider the covariant basis

$$\boldsymbol{G}_i = \frac{\partial}{\partial\eta^i}\boldsymbol{X} = \boldsymbol{A}_i + \tau h\boldsymbol{A}_{3,i}\ ,\quad i = 1,2$$

and $\boldsymbol{G}_3 = h\boldsymbol{A}_3$ as well as the contravariant tensor basis $\boldsymbol{G}^i$ $(i = 1,2)$ and $\boldsymbol{G}^3 = h^{-1}\boldsymbol{A}_3$.

The volume element of the shell is

$$dV = [\boldsymbol{G}_1\ ,\ \boldsymbol{G}_2\ ,\ \boldsymbol{G}_3]\ d\eta^1 d\eta^2 d\tau = h\ det(\underline{G})^{1/2}\ d\eta^1 d\eta^2 d\tau$$

with the (2×2)-matrix $\underline{G} = (G_{ij})^2_{i,j=1}$, $G_{ij} = \boldsymbol{G}_i\cdot\boldsymbol{G}_j$, which is simply calculated as

$$\underline{G} = \underline{A}\left(\underline{I} - \tau h\underline{A}^{-1}\underline{B}\right)^2 = \left(\underline{A} - \tau h\underline{B}\right)\underline{A}^{-1}\left(\underline{A} - \tau h\underline{B}\right). \tag{2}$$

From this, the volume element is well-known as

$$dV = h\,(1 - 2\tau hH + (\tau h)^2 K)\ d\tau\ d\mathcal{S} = (1 - \tau h\lambda_1)(1 - \tau h\lambda_2)\ h\, d\tau\ d\mathcal{S}$$

Here, the necessary condition

$$\epsilon_{\mathcal{H}} := (h/2) \max_{(\eta^1,\eta^2)\in\Omega} (\max(\lambda_1, \lambda_2)) \ < \ 1$$

guarantees the admissibility of the parametrization of the initial shell. Consistent with the historic literature, we strengthen this inequality in the following considerations to the case of thin shells as

$$\epsilon_{\mathcal{H}} << 1. \tag{3}$$

This allows the approximation of the volume element by $h\ d\tau\ d\mathcal{S}$ as well as the approximation of the matrix $(\underline{I} - \tau h \underline{A}^{-1}\underline{B})$ by $\underline{I}$ without significant errors.

2.3 Special case: the plate

Here we have a simplification on $\mathcal{S}_0$ such as $\boldsymbol{Y} = L_1 \boldsymbol{e}_1 \cdot \eta^1 + L_2 \boldsymbol{e}_2 \cdot \eta^2$, yielding $\boldsymbol{A}_3 = \boldsymbol{e}_3$ independent on (η^1, η^2). From this a lot of simplifications arise:

$$\boldsymbol{G}_i = \boldsymbol{A}_i, \qquad \underline{B} = \mathbb{O}, \qquad \mathcal{B} = 0$$

2.4 The deformed shell

The basic assumption of the simple shell models consists in keeping a straight line of the points

$$\left\{ \boldsymbol{Y}(\eta^1, \eta^2) + \tau h \boldsymbol{A}_3(\eta^1, \eta^2) : |\tau| \le \frac{1}{2} \right\}$$

after the deformation also, i. e. the mid surface is deformed as

$$\mathcal{S}_t = \left\{ \boldsymbol{y}(\eta^1, \eta^2) = \boldsymbol{Y}(\eta^1, \eta^2) + \boldsymbol{U}(\eta^1, \eta^2)\ :\ (\eta^1, \eta^2) \in \Omega \right\}$$

with an unknown displacement vector $\boldsymbol{U}$ (a function of (η^1, η^2) as well as of $\boldsymbol{Y}$). The weaker assumption defines the deformed shell as

$$\mathcal{H}_t = \left\{ \boldsymbol{x}(\eta^1, \eta^2, \tau) = \boldsymbol{y}(\eta^1, \eta^2) + \tau h\ \boldsymbol{d}(\eta^1, \eta^2) \right\} \tag{4}$$

with an additional vector field $\boldsymbol{d}(\eta^1, \eta^2)$ (the so called director vector).

and

$$\zeta_i(\underline{u}, \underline{\theta}) = \frac{1}{2}(u_{3,i} + \theta_i + u_j B_i^j).$$

Here, $\underline{u} = (u_1, u_2, u_3)^T$ and $\underline{\theta} = (\theta_1, \theta_2)^T$ contain the single 5 unknowns.

The remaining bending part (second and third line in (8)) follows from the same considerations and leads to

$$\chi_{ij}(\underline{u}, \underline{\theta}) = \frac{1}{2}(\theta_{i|j} + \theta_{j|i} - B_j^k u_{k|i} - B_i^k u_{k|j}) + C_{ij} u_3.$$

(The coefficients C_{ij} belong to the tensor $\mathcal{B}^2 = B_{ik} A^{kl} B_{lj} \boldsymbol{A}^i \boldsymbol{A}^j$.)

If we try to split the Naghdi-strain tensor into its three parts in a coordinate free manner as

$$\mathcal{E}^{Naghdi} \ = \ \mathcal{E}^{membr} + \mathcal{E}^{shear} + \tau h\, \mathcal{E}^{bend},$$

we only have the third part directly from (8)

$$\begin{aligned} \mathcal{E}^{bend} &= Grad_{\mathcal{S}}\boldsymbol{\theta} \cdot \mathcal{A} + \mathcal{A} \cdot (Grad_{\mathcal{S}}\boldsymbol{\theta})^T - (Grad_{\mathcal{S}}\boldsymbol{U} \cdot \mathcal{B} + \mathcal{B} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T) \\ &= \chi_{ij}(\underline{u}, \underline{\theta})\ \boldsymbol{A}^i \boldsymbol{A}^j, \end{aligned}$$

while for the other two parts a projection onto $\mathcal{A}$ resp. $\boldsymbol{A}_3\boldsymbol{A}_3$ has to be used (from the calculations in (11) and (12)).

$$\begin{aligned} \mathcal{E}^{membr} &= Grad_{\mathcal{S}}\boldsymbol{U} \cdot \mathcal{A} + \mathcal{A} \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T \\ &= \gamma_{ij}(\underline{u})\ \boldsymbol{A}^i \boldsymbol{A}^j \\ \mathcal{E}^{shear} &= Grad_{\mathcal{S}}\boldsymbol{U} \cdot (\boldsymbol{A}_3\boldsymbol{A}_3) + (\boldsymbol{A}_3\boldsymbol{A}_3) \cdot (Grad_{\mathcal{S}}\boldsymbol{U})^T + \boldsymbol{\theta}\boldsymbol{A}_3 + \boldsymbol{A}_3\boldsymbol{\theta} \\ &= \zeta_i(\underline{u}, \underline{\theta})\ (\boldsymbol{A}^i \boldsymbol{A}_3 + \boldsymbol{A}_3 \boldsymbol{A}^i) \end{aligned}$$

4 The resulting shell energy

We complete the resulting deformation energy of the shell by inserting $\mathcal{E}^{Naghdi}$ into the energy functional. Due to the desired small strain assumption in $\mathcal{E}^{Naghdi}$, we use a linear material law, such as

$$W(\boldsymbol{U}, \boldsymbol{\theta}) = \frac{1}{2} \int_{\mathcal{H}_0} \mathcal{E} : \mathfrak{C} : \mathcal{E}\ dV \tag{13}$$

with a (possibly space dependent) 4th order material tensor $\mathfrak{C}$. The most simple case, the St.Venant-Kirchhoff material, is considered to be

$$\mathfrak{C} = 2\mu\mathfrak{I} + \lambda(I\ I) \tag{14}$$

So,

$$\begin{aligned}2\mathcal{E}_{13}^{lin} =& \left[\boldsymbol{A}_3\cdot\boldsymbol{U}_{,i}+\boldsymbol{\theta}\cdot\boldsymbol{A}_i\right]\left(\boldsymbol{A}^i\boldsymbol{A}_3+\boldsymbol{A}_3\boldsymbol{A}^i\right)\\ =& Grad_{\mathcal{S}}\boldsymbol{U}\cdot\boldsymbol{A}_3\boldsymbol{A}_3+\boldsymbol{A}_3\boldsymbol{A}_3\cdot(Grad_{\mathcal{S}}\boldsymbol{U})^T\\ &+\boldsymbol{\theta}\boldsymbol{A}_3+\boldsymbol{A}_3\boldsymbol{\theta}.\end{aligned}$$

3.3 Linearization - the resulting strain tensor

From the addition of both parts and $\mathcal{A}+\boldsymbol{A}_3\boldsymbol{A}_3 = I$ we obtain

$$\begin{aligned}2\mathcal{E}^{Naghdi} &= Grad_{\mathcal{S}}\boldsymbol{U}+(Grad_{\mathcal{S}}\boldsymbol{U})^T+\boldsymbol{\theta}\boldsymbol{A}_3+\boldsymbol{A}_3\boldsymbol{\theta}\\ &\quad+\tau h(Grad_{\mathcal{S}}\boldsymbol{\theta}\cdot\mathcal{A}+\mathcal{A}\cdot(Grad_{\mathcal{S}}\boldsymbol{\theta})^T)\\ &\quad-\tau h(Grad_{\mathcal{S}}\boldsymbol{U}\cdot\mathcal{B}+\mathcal{B}\cdot(Grad_{\mathcal{S}}\boldsymbol{U})^T) \qquad (8)\\ &= Grad_{\mathcal{S}}\boldsymbol{U}\cdot(I-\tau h\mathcal{B})+(I-\tau h\mathcal{B})\cdot(Grad_{\mathcal{S}}\boldsymbol{U})^T\\ &\quad+\tau h\left(Grad_{\mathcal{S}}\boldsymbol{\theta}\cdot\mathcal{A}+\mathcal{A}\cdot(Grad_{\mathcal{S}}\boldsymbol{\theta})^T\right)+\boldsymbol{\theta}\boldsymbol{A}_3+\boldsymbol{A}_3\boldsymbol{\theta} \qquad (9)\end{aligned}$$

The first line in equation (8) contains the so-called "membrane" and "shear" part of the linearized strain, while the other two lines are the "bending" part. This can be seen, from inserting the expansion of $\boldsymbol{U}$ and $\boldsymbol{\theta}$ into these expressions and collecting the respective terms together. This leads to the formulas given in [1] called the "shear-membrane-bending" model.

$$\begin{aligned}\mathcal{E}^{Naghdi} =& \gamma_{ij}\boldsymbol{A}^i\boldsymbol{A}^j+\zeta_i(\boldsymbol{A}^i\boldsymbol{A}_3+\boldsymbol{A}_3\boldsymbol{A}^i)+\tau h\,\chi_{ij}\boldsymbol{A}^i\boldsymbol{A}^j \qquad (10)\\ & \text{"membrane"}+\text{"shear"}+\text{"bending part"}\end{aligned}$$

For proving the equivalence of (8) with [1], we start with the split

$$\boldsymbol{U}=\boldsymbol{u}+u_3\boldsymbol{A}_3, \qquad \boldsymbol{u}=u_i\boldsymbol{A}^i, \qquad \boldsymbol{\theta}=\theta_i\boldsymbol{A}^i,$$

which leads to

$$\begin{aligned}Grad_{\mathcal{S}}\boldsymbol{U} &= Grad_{\mathcal{S}}\boldsymbol{u}+(Grad_{\mathcal{S}}u_3)\boldsymbol{A}_3+u_3\,Grad_{\mathcal{S}}\boldsymbol{A}_3\\ &= Grad_{\mathcal{S}}\boldsymbol{u}+u_{3,i}\boldsymbol{A}^i\boldsymbol{A}_3-u_3\mathcal{B} \qquad (11)\\ Grad_{\mathcal{S}}\boldsymbol{u} &= \boldsymbol{A}^j(u_{i,j}\boldsymbol{A}^i-u_i\Gamma^i_{jk}\boldsymbol{A}^k+u_iB^i_j\boldsymbol{A}_3)\\ &= u_{i|j}\boldsymbol{A}^j\boldsymbol{A}^i+u_iB^i_j\boldsymbol{A}^j\boldsymbol{A}_3. \qquad (12)\end{aligned}$$

Now, all terms on $\boldsymbol{A}^i\boldsymbol{A}^j$ and on $\boldsymbol{A}^i\boldsymbol{A}_3+\boldsymbol{A}_3\boldsymbol{A}^i$ are collected together obtaining

$$\gamma_{ij}(\underline{u})=\frac{1}{2}(u_{i|j}+u_{j|i})-u_3B_{ij}$$

Here, we use the setting

$$\boldsymbol{d}=\boldsymbol{A}_3+\boldsymbol{\theta}$$

with a new vector function $\boldsymbol{\theta}=\boldsymbol{\theta}(\eta^1,\eta^2)$ independent of $\boldsymbol{U}$

Let

$$\boldsymbol{a}_i=\frac{\partial}{\partial\eta^i}\boldsymbol{y}=\boldsymbol{A}_i+\boldsymbol{U}_{,i}$$

the tangential vectors after deformation, then analogously we have

$$\boldsymbol{a}_3=(\boldsymbol{a}_1\times\boldsymbol{a}_2)\big/|\boldsymbol{a}_1\times\boldsymbol{a}_2|$$

as surface normal vector of $\mathcal{S}_t$. Again we have

$$\begin{aligned}\underline{a}&=(a_{ij})_{i,j=1}^2 \quad \text{with} \quad a_{ij}=\boldsymbol{a}_i\cdot\boldsymbol{a}_j\\ \underline{b}&=(b_{ij})_{i,j=1}^2 \quad \text{with} \quad b_{ij}=\boldsymbol{a}_{i,j}\cdot\boldsymbol{a}_3\end{aligned}$$

as new first and second fundamental forms.

With

$$\boldsymbol{d}=\boldsymbol{A}_3+\boldsymbol{\theta}, \qquad (5)$$

the 3D covariant basis is now

$$\boldsymbol{g}_i=\frac{\partial}{\partial\eta^i}\boldsymbol{x}=\boldsymbol{a}_i+\tau h(\boldsymbol{A}_{3,i}+\boldsymbol{\theta}_{,i})=\boldsymbol{G}_i+\boldsymbol{U}_{,i}+\tau h\boldsymbol{\theta}_{,i}$$

and $\boldsymbol{g}_3=h(\boldsymbol{A}_3+\boldsymbol{\theta})$. Hence, we can define the (2×2)-matrix $\underline{g}=(g_{ij})$ from $g_{ij}=\boldsymbol{g}_i\cdot\boldsymbol{g}_j$, which can be simplified in the following manner:

$$\begin{aligned}g_{ij} =& G_{ij}+\boldsymbol{G}_i\cdot(\boldsymbol{U}_{,j}+\tau h\boldsymbol{\theta}_{,j})+\boldsymbol{G}_j\cdot(\boldsymbol{U}_{,i}+\tau h\boldsymbol{\theta}_{,i})+\text{nonlinear terms}\\ =& G_{ij}+\boldsymbol{A}_i\cdot(\boldsymbol{U}_{,j}+\tau h\boldsymbol{\theta}_{,j})+\boldsymbol{A}_j\cdot(\boldsymbol{U}_{,i}+\tau h\boldsymbol{\theta}_{,i})\\ &+\tau h(\boldsymbol{A}_{3,i}\cdot\boldsymbol{U}_{,j}+\boldsymbol{A}_{3,j}\cdot\boldsymbol{U}_{,i})+\text{nonlinear terms}+\mathcal{O}(\tau h)^2.\end{aligned}$$

For sake of easy abbreviation of the terms above, we introduce the (2×2)-matrix $\underline{M}(\boldsymbol{U})$ with the entries $m_{ij}=\boldsymbol{A}_i\cdot\boldsymbol{U}_{,j}$ and $\underline{M}(\boldsymbol{\theta})$ analogously. Then we have an easy equation for the linearized matrix $\underline{g}$ from

$$\boldsymbol{A}_{3,i}\cdot\boldsymbol{U}_{,j}=-B_{ik}A^{kl}\boldsymbol{A}_l\cdot\boldsymbol{U}_{,j}$$

hence

$$(\boldsymbol{A}_{3,i}\cdot\boldsymbol{U}_{,j})_{i,j=1}^2=-\underline{BA}^{-1}\underline{M}(\boldsymbol{U})$$

and

$$\begin{aligned}\underline{g}=\underline{G}\;&+\underline{M}(\boldsymbol{U})+\underline{M}(\boldsymbol{U})^T+\tau h(\underline{M}(\boldsymbol{\theta})+\underline{M}(\boldsymbol{\theta})^T) \qquad (6)\\ &-\tau h(\underline{BA}^{-1}\underline{M}(\boldsymbol{U})+\underline{M}(\boldsymbol{U})^T\underline{A}^{-1}\underline{B})+\mathcal{O}(\tau h)^2.\end{aligned}$$

3 The strain tensor and its simplifications

From the definition of the deformed shell as (4,5), we may deduce the 3D-deformation gradient

$$\mathcal{F} = \boldsymbol{g}_i \boldsymbol{G}^i + \boldsymbol{g}_3 \boldsymbol{G}^3 = \boldsymbol{g}_i \boldsymbol{G}^i + (\boldsymbol{A}_3 + \boldsymbol{\theta})\boldsymbol{A}^3,$$

the right Cauchy Green tensor $\mathcal{C} = \mathcal{F}^T \cdot \mathcal{F}$ and the strain tensor $\mathcal{E} = \frac{1}{2}(\mathcal{C} - I)$. Obviously, $\mathcal{E}$ consists of 3 parts

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_{13} + \mathcal{E}_{33}$$

with

$$\begin{aligned} 2\mathcal{E}_1 &= (g_{ij} - G_{ij})\, \boldsymbol{G}^i \boldsymbol{G}^j \\ 2\mathcal{E}_{13} &= \epsilon_i(\boldsymbol{G}^i \boldsymbol{A}_3 + \boldsymbol{A}_3 \boldsymbol{G}^i)\ , \quad \epsilon_i = \boldsymbol{g}_i \cdot (\boldsymbol{A}_3 + \boldsymbol{\theta}) \\ 2\mathcal{E}_{33} &= (2\boldsymbol{A}_3 \cdot \boldsymbol{\theta} + |\boldsymbol{\theta}|^2)\boldsymbol{A}_3 \boldsymbol{A}_3 \end{aligned}$$

In general, the so–called 5-unknown-variant is considered, where $\boldsymbol{U}$ contains three unknowns as $\boldsymbol{U} = U_i \boldsymbol{A}^i + U_3 \boldsymbol{A}_3$ but $\boldsymbol{\theta}$ only the two unknowns θ_i and $\boldsymbol{\theta} \cdot \boldsymbol{A}_3 = 0$. Note that this implies $\boldsymbol{A}_3 \cdot \boldsymbol{\theta}_{,i} + \boldsymbol{A}_{3,i} \cdot \boldsymbol{\theta} = 0$ and simplifies the coefficient ϵ_i in $\mathcal{E}_{13}$ and $\mathcal{E}_{33}$ vanishes after linearization.

The usual linearization of these tensors will lead to the so called Naghdi shell equation. When $\mathcal{E}$ is approximated by $\mathcal{E}^{Naghdi}$, which is now some linear differential operator with respect to $\boldsymbol{U}$ and $\boldsymbol{\theta}$, then the Naghdi shell energy is considered as

$$W^{Naghdi} = \frac{1}{2} \int_{-1/2}^{1/2} \int_{\mathcal{S}_0} \mathcal{E}^{Naghdi} : \mathfrak{C} : \mathcal{E}^{Naghdi}\ d\mathcal{S}\ h\, d\tau.$$

The integration over the thickness is carried out explicitly due to the dependence of $\mathcal{E}^{Naghdi}$ linearly on τh.

Now let us consider these linearization processes. Here, we usually do both, we neglect all terms of order $\mathcal{O}(\tau h)^2$ or higher and we omit all terms which are nonlinear with respect to $\boldsymbol{U}$ or $\boldsymbol{\theta}$.

We start with the ideas given in [1]. For linearizing the shell equation we state:

- linearize $(g_{ij} - G_{ij})$ to $\epsilon_{ij}(\boldsymbol{U}, \boldsymbol{\theta})$ and ϵ_i analogously
- replace the volume element by $h\, d\tau\ d\mathcal{S}$
- and substitute all vectors $\boldsymbol{G}^i$ of the tensor bases by $\boldsymbol{A}^i$.

This last setting seems to be crucial, because we neglect terms of order $\mathcal{O}(\tau h)$, but is accepted at least in [1]. In [4] the expansion of $\boldsymbol{G}^i$ with respect to $\boldsymbol{A}^i$ is considered and this substitution is equivalent to the approximation of the matrix $(\underline{I} - \tau h \underline{A}^{-1} \underline{B})$ by $\underline{I}$ from the thin shell assumption (3).

3.1 Linearization - the part $\mathcal{E}_1$

Following (6), the first part of $\mathcal{E}$ is approximated by $\epsilon_{ij}(\boldsymbol{U}, \boldsymbol{\theta})\boldsymbol{A}^i \boldsymbol{A}^j$ with the matrix of the coefficients as

$$\begin{aligned} &\underline{M}(\boldsymbol{U}) + \underline{M}(\boldsymbol{U})^T + \tau h(\underline{M}(\boldsymbol{\theta}) + \underline{M}(\boldsymbol{\theta})^T) \\ &-\tau h(\underline{B}\underline{A}^{-1}\underline{M}(\boldsymbol{U}) + \underline{M}(\boldsymbol{U})^T \underline{A}^{-1}\underline{B}). \end{aligned} \tag{7}$$

Hence,

$$\begin{aligned} 2\epsilon_{ij} = {} & \boldsymbol{A}_i \cdot \boldsymbol{U}_{,j} + \boldsymbol{A}_j \cdot \boldsymbol{U}_{,i} + \tau h(\boldsymbol{A}_i \cdot \boldsymbol{\theta}_{,j} + \boldsymbol{A}_j \cdot \boldsymbol{\theta}_{,i}) \\ & - \tau h(B_{ik} A^{kl} \boldsymbol{A}_l \cdot \boldsymbol{U}_{,j} + \boldsymbol{U}_{,i} \cdot \boldsymbol{A}_l A^{lk} B_{kj}) \end{aligned}$$

From

$$(\boldsymbol{A}_j \cdot \boldsymbol{U}_{,i})\boldsymbol{A}^i \boldsymbol{A}^j = \boldsymbol{A}^i \boldsymbol{U}_{,i} \cdot \boldsymbol{A}_j \boldsymbol{A}^j = Grad_{\mathcal{S}} \boldsymbol{U} \cdot \mathcal{A}$$

and

$$\begin{aligned} &(B_{ik} A^{kl} \boldsymbol{A}_l \cdot \boldsymbol{U}_{,j})\boldsymbol{A}^i \boldsymbol{A}^j = \\ &= \boldsymbol{A}^i B_{ik}(\boldsymbol{A}^k \cdot \boldsymbol{A}^l)(\boldsymbol{A}_l \cdot \boldsymbol{U}_{,j})\boldsymbol{A}^j = \\ &= \mathcal{B} \cdot \mathcal{A} \cdot (Grad_{\mathcal{S}} \boldsymbol{U})^T = \mathcal{B} \cdot (Grad_{\mathcal{S}} \boldsymbol{U})^T \end{aligned}$$

we obtain for $\mathcal{E}_1$ the expression

$$\begin{aligned} 2\mathcal{E}_1^{lin} = {} & Grad_{\mathcal{S}} \boldsymbol{U} \cdot \mathcal{A} + \mathcal{A} \cdot (Grad_{\mathcal{S}} \boldsymbol{U})^T \\ & + \tau h(Grad_{\mathcal{S}} \boldsymbol{\theta} \cdot \mathcal{A} + \mathcal{A} \cdot (Grad_{\mathcal{S}} \boldsymbol{\theta})^T) \\ & - \tau h(Grad_{\mathcal{S}} \boldsymbol{U} \cdot \mathcal{B} + \mathcal{B} \cdot (Grad_{\mathcal{S}} \boldsymbol{U})^T) \end{aligned}$$

3.2 Linearization - the part $\mathcal{E}_{13}$

Here, we have

$$2\mathcal{E}_{13} = \epsilon_i(\boldsymbol{G}^i \boldsymbol{A}_3 + \boldsymbol{A}_3 \boldsymbol{G}^i)\ \ ,$$

with

$$\begin{aligned} \epsilon_i &= \boldsymbol{g}_i \cdot (\boldsymbol{A}_3 + \boldsymbol{\theta}) \\ &= (\boldsymbol{A}_i \cdot \boldsymbol{\theta}) + (\boldsymbol{U}_{,i} \cdot \boldsymbol{A}_3) + \text{ nonlinear terms.} \end{aligned}$$