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Fast simulation of (nearly) incompressible nonlinear elastic material at large strain via adaptive mixed FEM

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Abstract

The main focus of this work lies on the simulation of the deformation of mechanical components that consist of nonlinear elastic, incompressible material and that are subject to large deformations. Starting from a nonlinear formulation a discrete problem can be derived by using linearisation techniques and an adaptive mixed finite element method. It turns out to be a saddle point problem that can be solved via a Bramble-Pasciak conjugate gradient method. With some modifications the simulation can be improved.

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1 Introduction
2 Basics
3 Mixed variational formulation
4 Solution method
5 Error estimation
6 LBB conditions
7 Improvement suggestions

References

Because of $Q = Q_\zeta$ one can replace $Q_\zeta$ by $Q$ and this leads to the desired saddle point problem for all $Q \in \mathcal{Q}$ and $V \in \mathcal{V}_0$:

$$
\begin{align*}
t f_0(V) - a_D(U; V) - a_V(U; P_{\infty}; V) &= a(U, P_{\infty}; \delta U; V) + a_V(U, \delta P_{\infty}; V) \\
 c_\zeta(U; P_{\infty}, Q) - b_\zeta(U; Q) &= a_V(U; \delta U, Q) - c_\zeta(U; \delta P_{\infty}, Q)
\end{align*}
$$

(7.27)

(7.28)

\section{Introduction}

\subsection{Content}

The object of this work is the numerical simulation of mechanical components, that consist of nonlinear elastic and (nearly) incompressible material and which are subject to a large deformation. As a special case that also includes linear elastic material behaviour with small deformations. The special property of incompressible material, namely the constant volume during any shape changing deformation, needs special treatment in the mathematical formulation. Therefore a mixed ansatz plays an important role.

\subsection{Notation}

In order to describe the considered problem of deformation, we need several mechanical quantities and the corresponding operators. These operators mostly are defined as tensors of order $n$ and the space of these tensors is denoted with $\mathcal{T}_n$. By choosing a fixed basis the notation of Voigt can be used. That allows the representation of the tensors as matrices or vectors. For distinction we use different typefaces for different types of values. This is shown in table 1.1, only few exceptions can occur.

<table>
<thead>
<tr>
<th>$Q, \phi$</th>
<th>scalar function, tensor of order zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V, v$</td>
<td>vector field, tensor of order one</td>
</tr>
<tr>
<td>$\mathcal{T}, \sigma$</td>
<td>tensor of order two</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>tensor of order four</td>
</tr>
<tr>
<td>$a, \vec{A}$</td>
<td>$n$-vector of coefficients w.r.t. a basis (of a function space)</td>
</tr>
<tr>
<td>$A$</td>
<td>matrix</td>
</tr>
</tbody>
</table>

Table 1.1: types of notation

Throughout this paper, pairs of vectors, such as $UV$, form a 2\textsuperscript{nd} order tensor and in general a 2\textsuperscript{nd} order tensor is any linear combination of such pairs. In the same way, a pair of 2\textsuperscript{nd} order tensors, such as $EF$, defines a 4\textsuperscript{th} order tensor. Obviously any 2\textsuperscript{nd} order tensor $E$ is a linear operator with $E: \mathcal{T}_1 \rightarrow \mathcal{T}_1$, as well as any 4\textsuperscript{th} order tensor $M$ is a linear operator with $M: \mathcal{T}_2 \rightarrow \mathcal{T}_2$.\n
$$
E : U \rightarrow E \cdot U = (CD) \cdot U = (D \cdot U)C
$$

(1.1)

As usual we define a dot product $V \cdot U \in \mathbb{R}$ for all first order tensors $U, V \in \mathcal{T}_1$. 


t f_0(V) - a_D(U; V) - a_V(U; P_{\infty}; V) = a(U, P_{\infty}; \delta U; V) + a_V(U, \delta P_{\infty}; V)

c_\zeta(U; P_{\infty}, Q) - b_\zeta(U; Q) = a_V(U; \delta U, Q) - c_\zeta(U; \delta P_{\infty}, Q)

(7.27)

(7.28)
The double dot products are defined as follows.

\[
CD : UV = (D \cdot U)(C \cdot V) \in \mathbb{R} \quad \forall U, V, C, D \in T_1 \tag{1.2}
\]

\[
ABCD : UV = (D \cdot U)(C \cdot V) \quad A \in T_2 \quad \forall U, V, C, D, A, B \in T_1 \tag{1.3}
\]

For later use we also define the symmetric part of a second order tensor.

\[
2 \text{Sym}(\mathbf{Y}) = \mathbf{Y} + \mathbf{Y}^T \tag{1.4}
\]

With \([\cdot, \cdot, \cdot] \) we denote the usual scalar triple product and with \(\langle \cdot, \cdot \rangle \) we denote the usual outer vector product. Furthermore we use the \(L^2\)-scalar products (1.5)-(1.7) and norms (1.8)-(1.11) for all \(Q, P \in \mathbb{T}_0, V, U \in T_1 \) and \(E, F \in T_2\).

\[
\langle Q, P \rangle_{0, \Omega} := \int_\Omega Q P \, d\Omega \tag{1.5}
\]

\[
\langle V, U \rangle_{0, \Omega} := \int_\Omega V \cdot U \, d\Omega \tag{1.6}
\]

\[
\langle E, F \rangle_{0, \Omega} := \int_\Omega E : F \, d\Omega \tag{1.7}
\]

\[
\|Q\|_{0, \Omega}^2 := \langle Q, Q \rangle_{0, \Omega} \tag{1.8}
\]

\[
\|V\|_{0, \Omega}^2 := \langle V, V \rangle_{0, \Omega} \tag{1.9}
\]

\[
\|V\|_1^2 := \int_\Omega |\text{Grad} \, V|^2 \, d\Omega \tag{1.10}
\]

\[
\|V\|_{1, \Omega}^2 := \|V\|_{0, \Omega}^2 + |V|_1^2 \tag{1.11}
\]

Next to the tensors itself, its derivatives are also of importance, especially the derivatives of scalar tensors or tensors of second order. The derivative of a scalar tensor \(\zeta(\mathbf{Y}) : T_2 \rightarrow \mathbb{R}\) is denoted with \(\partial \zeta = \frac{\partial \zeta(\mathbf{Y})}{\partial \mathbf{Y}}\). This is a tensor of second order that fulfills equation (1.12) for all applied directions \(\delta \mathbf{Y} \in T_2\).

\[
\zeta(\mathbf{Y} + \delta \mathbf{Y}) = \zeta(\mathbf{Y}) + \langle \partial \zeta(\mathbf{Y}) \rangle \delta \mathbf{Y} + o(\|\delta \mathbf{Y}\|^2) \tag{1.12}
\]

The derivative \(\mathbf{T} = \frac{\partial \mathbf{T}(\mathbf{Y})}{\partial \mathbf{Y}}\) of a tensor \(\mathbf{T}(\mathbf{Y}) \in T_2\) with \(\mathbf{Y} \in T_2\) is of order four and is defined via (1.13).

\[
\mathbf{T}(\mathbf{Y} + \delta \mathbf{Y}) = \mathbf{T}(\mathbf{Y}) + \langle \partial \mathbf{T}(\mathbf{Y}) \rangle \delta \mathbf{Y} + o(\|\delta \mathbf{Y}\|^2) \tag{1.13}
\]

We adapt our notation from (3.46) - (3.49) and define a new bilinear form.

\[
b_c(U, \delta U, Q) := \langle \mathcal{Z}(U) \cdot S_c(U), \mathcal{E}(U, \delta U), Q \rangle_{L^2(\Omega)} \tag{7.17}
\]

\[
\mathcal{Z}(U) := \frac{1}{\zeta(I_3(U))^2} \left( I - \frac{\partial \zeta(I_3)}{\zeta(I_3)} \right) \left( \frac{\partial \zeta(I_3)}{\zeta(I_3)} \right) \tag{7.18}
\]

Then we can recompute the linearisation, which results in a modified linear problem for all \(Q \in \mathcal{Q}\) and \(\mathbf{V} \in \mathcal{V}_0\).

\[
t_f(\mathbf{V}) - a_D(U, \mathbf{V}) = a_U(U, P_{\mathcal{Q}_\infty}, \mathbf{V}) + a_V(U, \delta P_{\mathcal{Q}_\infty}, \mathbf{V}) \tag{7.19}
\]

\[
c(P_{\mathcal{Q}_\infty}, Q) - b_0(U, Q) = b_U(U, \delta U, Q) - c(\delta P_{\mathcal{Q}_\infty}, Q) \tag{7.20}
\]

with a slightly changed stress and material tensor in \(a(\ldots)\).

\[
\partial \mathcal{T}(U, P_{\mathcal{Q}_\infty}) = \left( \partial \mathcal{T}(U) + \kappa_D \phi_V(I_3) \cdot \mathbf{S}_c(U) \right) + P_{\mathcal{Q}_\infty} \cdot (\zeta(I_3) \mathbf{S}_c(U)) \tag{7.21}
\]

\[
\partial \mathcal{M}(U, P_{\mathcal{Q}_\infty}) = \frac{1}{\zeta(I_3)^2} \left[ \partial \mathcal{T}(U) + \kappa_D \phi_V(I_3) \cdot \mathbf{S}_c(U) + \kappa_D \phi^2(I_3) \right] \frac{\partial \zeta(I_3)}{\partial \zeta(I_3)} \tag{7.22}
\]

In order to achieve a saddle point problem again we have to match the bilinear forms \(b(U, \delta U, Q)\) and \(a(U, \delta P_{\mathcal{Q}_\infty}, \mathbf{V})\). In (7.20) we rewrite \(Q = Z \cdot Q_c\) with \(Q_c \in \mathcal{Q}_c := \{Q_c = Z : Q \in \mathcal{Q}\}\) (in brief it is \(Z = Z(U)\)) and reformulate the scalar products with equivalent equations (7.23)-(7.25). In the last step (7.26) we define new bilinear forms.

\[
c(P_{\mathcal{Q}_\infty}, Z^{-1} \cdot Q_c) - b_0(U, Z^{-1} \cdot Q_c) = b_U(U, \delta U, Z^{-1} \cdot Q_c) - c(\delta P_{\mathcal{Q}_\infty}, Z^{-1} \cdot Q_c) \tag{7.23}
\]

\[
\left( \zeta(\mathbf{S}_c) \mathbf{S}_c(U) \cdot \mathcal{E}(U, \delta U), Z^{-1} \cdot \mathbf{Q}_c \right)_{L^2(\Omega)} = \left( \mathcal{Z}(U) \cdot \mathbf{S}_c(U) \cdot \mathbf{S}_c(U), Z^{-1} \cdot \mathbf{Q}_c \right)_{L^2(\Omega)} \tag{7.24}
\]

\[
\mathbf{c}(U, P_{\mathcal{Q}_\infty}, Q_c) - b_c(U, Q_c) = a_U(U, \delta U, Q_c) - c(\delta P_{\mathcal{Q}_\infty}, Q_c) \tag{7.25}
\]

\[
c(U, P_{\mathcal{Q}_\infty}, Q_c) - b_0(U, Q_c) = a_U(U, \delta U, Q_c) - c(\delta P_{\mathcal{Q}_\infty}, Q_c) \tag{7.26}
\]
With this ansatz we receive a new formulation of the stress tensor and the material
tensor in the initial configuration.

\[
\hat{T} = T_D(U) + \kappa_D \zeta(I_3(U)) \cdot S_V + P_\infty \cdot S_V
\]

(7.8)

\[
\Rightarrow P_\infty \cdot S_V = \frac{\kappa_D \zeta(I_3)}{\zeta(I_3)} \cdot 2 \zeta(I_3) \frac{\partial \zeta(I_3)}{\partial C}
\]

(7.10)

7.3 Scaling function

We change the volumetric part of the stress tensor \( \hat{T} \) by introducing a suitable
scaling function \( \zeta(I_3) \in \mathbb{R} \) for all \( I_3 = I_3(C(U)) \) which leads to (7.10).

\[
P_\infty \cdot S_V = \frac{\kappa_D \zeta(I_3)}{\zeta(I_3)} \cdot 2 \zeta(I_3) \frac{\partial \zeta(I_3)}{\partial C}
\]

(7.11)

A reasonable property of \( \zeta(I_3) \) seems to be given by \( \zeta(1) = 1 \) and that is why
(7.12) could be a good choice.

\[
\zeta(I_3) = (I_3)^g, \quad g \neq 0
\]

(7.12)

Now the substitution with the hydrostatic pressure \( P_\infty \) requires a slightly changed
side condition (7.13).

\[
\kappa P_\infty = \zeta(I_3)^{-1} \zeta(I_3)
\]

(7.13)

or \( \kappa_\infty P_\infty = \zeta(I_3)^{-1} \zeta(I_3) \)

(7.14)

Note: Formula (7.10) yields \( S_V = \zeta(I_3) \cdot C^{-1} \).

Instead of (3.51) we now have to consider a modified system of equations.

\[
\left( \begin{array}{c}
t_f\theta(V) \\
\theta
\end{array} \right) = \left( \begin{array}{c}
a_D(U; V) + \alpha(U; P_\infty; V) \\
b_D(U; Q) - c(P_\infty; Q)
\end{array} \right)
\]

(7.15)

\[
= \left( \begin{array}{c}
\zeta(U, \mathcal{E}(U, V))_{0, \Omega} + (P_\infty \cdot S_V(U), \mathcal{E}(U, V))_{0, \Omega}
\end{array} \right)
\]

(7.16)

2 Basics

The following subsections deal with the basic terms, that are needed to describe
an elastic deformation ([1] (chap. 6) and [2]). In doing so geometrical and
kinematic terms are introduced first and the basic equations of motion are shown
later on.

2.1 Geometry

We consider an arbitrary elastic body \( K \) as a representation of any given me-
chanical component. We assume that \( K \) has the initial configuration (2.1) prior
to the deformation.

\[
\Omega = \Omega_0 = \left\{ X(\eta) \in \mathbb{R}^3 : \eta \in \mathcal{P} \subset \mathbb{R}^3 \right\}
\]

(2.1)

Via the influences of various external loads a deformation process occurs that transfers \( K \) into a current configuration.

\[
\Omega_\tau = \left\{ X(\eta) \in \mathbb{R}^3 : \eta \in \mathcal{P} \subset \mathbb{R}^3 \right\} \text{ with } \tau \geq 0
\]

(2.2)

We note that for all \( \tau \) the parametrising set \( \mathcal{P} \) remains the same. Hence the
material point \( X(\eta) \in \Omega \) turns over to the spacial point \( x(\eta) \) during the
deformation.

Because of the fixed parametrisation it is possible to define a covariant and con-
travariant tensor basis in \( \Omega \) that is denoted with \( G_i \) and \( G^i \) respectively.

\[
G_i = \frac{\partial}{\partial x^i} X(\eta),
\]

(2.3)

\[
G^i \cdot G_i = \delta_i^i
\]

(2.4)

Analogously one can define these bases in \( \Omega_\tau \).

\[
g_i = \frac{\partial}{\partial \eta^i} x(\eta),
\]

(2.5)

\[
g^i \cdot g_i = \delta_i^i
\]

(2.6)

Note: With these tensors of order one and by using the summation convention
of Einstein the unit tensor of second order \( \mathcal{I} \in \mathbb{T}_2 \) can be written as shown in
(2.7).

\[
\sum_{i=1}^{3} G_i G^i = G \cdot G^i = \mathcal{I} = g_i g^i
\]

(2.7)
We use the summation convention of Einstein and introduce two differential operators in each configuration, i.e. the gradient or the divergence.

\[
\text{Grad} = G^i \frac{\partial}{\partial \eta^i} \quad \text{in } \Omega \quad (2.8)
\]

\[
\text{grad} = g^i \frac{\partial}{\partial \eta^i} \quad \text{in } \Omega, \quad (2.9)
\]

\[
\text{Div} = \text{Grad} \cdot = \left( G^i \frac{\partial}{\partial \eta^i} \right) \quad \text{in } \Omega \quad (2.10)
\]

\[
\text{div} = \text{grad} \cdot = \left( g^i \frac{\partial}{\partial \eta^i} \right) \quad \text{in } \Omega, \quad (2.11)
\]

**Note:** For vector fields \( U \) we have \( (\text{Grad} \ U) = G^i \frac{\partial}{\partial \eta^i} U = G^i U_j \) which implies a correct Taylor expansion \( U(X + V) = U(X) + V \cdot \text{Grad } U + O(\|V\|^2) \).

### 2.2 Kinematics

The deformation of the body \( K \) shall be described by a sufficiently smooth function \( \Phi \) such that (2.12) holds.

\[
\Phi : \Omega \to \Omega, \quad X(\eta) \mapsto x(\eta) \quad (2.12)
\]

Introducing the first order displacement tensor \( U \) one can rewrite \( \Phi \) as a sum.

\[
\Phi(X) = x = X + U(X) \quad (2.13)
\]

Using the derivatives of \( U \) the covariant tensor basis of \( \Omega \), decomposes with

\[
g_i = \frac{\partial}{\partial \eta^i}(X + U) = G_i + \frac{\partial}{\partial \eta^i} U = G_i + U, \quad (2.14)
\]

We need some important tensors to measure the deformation. First of all we get the deformation gradient \( F \) that is defined via (2.15).

\[
F \cdot \, dX = dx \quad (2.15)
\]

With a few transformation steps (use the deformation, the displacement and the tensor basis) the explicit representation (2.16) follows.

\[
F = (\text{Grad } \Phi)^T = I + (\text{Grad } U)^T = g_i G^i \quad (2.16)
\]

### 7 Improvement suggestions

Especially in cases of very large deformations we notice that the numerical realisation sometimes exhibits a bad behaviour. One problem is the decreasing minimum of \( J(U_k) \). It tends to zero because sometimes given rectangular angles are deformed into a nearly straight line. On the other hand after a certain adaptive refinement we witness a jump in the estimated error and a negative residual \( \|w\|^2 \cdot A \cdot w = (A_\phi \cdot w)^2 \cdot A \cdot (A_\phi \cdot w) \) or even a negative determinant \( J \) occurs. May be the problem is that we use all terms of \( a(\ldots) \) to build the matrix system \( A \) although \( a(\ldots) \) is only coercive on the kernel of \( b(\ldots) \). Or maybe in that case the initial value \( (U_h, P_h) \) contains a high error and that influences \( A \). The next sections show some improvements.

#### 7.1 Specific energy density function

Instead of (3.27) we can use (7.1) to formulate the deviatoric part of \( \phi(C) \).

\[
\phi(C) = \phi_D(I_1, I_3) = c_{10} \left( a_1 - 3 - \ln(I_3) \right) \quad (7.1)
\]

This leads to an easier formulation of \( T_D \) and \( M_D \).

\[
T_D(U) = 2c_{10} w - 2c_{10} C_{-1}(U) \quad (7.2)
\]

\[
\frac{\partial}{\partial C} T_D(U) = 2c_{10} \left( \left( C^{-1} C^{-1} \right) - I_3^{-1} \left( a_1 \|C\| - (\|C\| - a_1 \|J\| + C) \right) \right) \quad (7.3)
\]

\[
= 2c_{10} C \quad (7.4)
\]

#### 7.2 Simple split of the bulk modulus

The bulk modulus can be decomposed into two parts.

\[
K_D + K_\infty := K \leq \infty \quad \text{such that } \quad 0 \leq K_D < K_\infty \leq \infty \quad (7.5)
\]

Thereby we predefine a new structure of \( \phi(C) \).

\[
\phi(C) = \phi_D(I_1, I_3) + \frac{1}{2} K_D (\phi_I(I_3))^2 + \frac{1}{2} K_\infty (\phi_I(I_3))^2 \quad (7.6)
\]

\[
= \phi_I(I_3, I_3) \quad (7.7)
\]

The hydrostatic pressure comes in with the substitution (7.7) but in the case of incompressibility with \( K = K_\infty = \infty \) the new variable \( P_\infty \) equals \( P \).

\[
P_\infty = K_\infty \phi_I(C) \quad (7.7)
\]
and in general it follows (6.20) or (6.21) as a pull back onto $\Omega^\tau$.

\[ a(U, P; \delta U, V) = -4 \langle I_3, P \rangle \text{Sym}(\text{Grad } V)^T \cdot F^{-1} \rangle_{\partial \Omega}, \]

\[ + 2 \langle \pi(I_3, P), (\text{Grad } V)^T \cdot F^{-1}, (\text{Grad } \delta U)^T \cdot F^{-1} \rangle_{\partial \Omega}, \]

\[ = \langle J^{-1} \pi(I_3, p), 2 \text{Sym}(\text{Grad } v), 2 \text{Sym}(\text{Grad } \delta u) \rangle_{\partial \Omega}, \]

\[ + 2 \langle J^{-1} \pi(I_3, p) \rangle \text{Grad } v, \text{Grad } \delta u \rangle_{\partial \Omega}, \]

\[ + 2 \langle \text{Grad } v, \frac{\partial \Phi}{\partial a} \text{Grad } \delta u \rangle_{\partial \Omega}. \]  

(6.20)

(6.21)

We could sum up.

\[ a(U, P; \delta U, V) = 2 \int_\Omega \frac{\partial \Phi}{\partial a} (\text{Grad } V)^T \cdot \text{Grad } \delta U \, d\Omega \]

\[ - 2 \int_\Omega \pi(I_3, P) (F^{-T} \cdot \text{Grad } V) : (F^{-T} \cdot \text{Grad } \delta U) \, d\Omega. \]  

(6.22)

To show (6.1) we use (6.20) with the argument $(V, V)$. Then, the term \[ \langle \text{Grad } V, \frac{\partial \Phi}{\partial a} \text{Grad } V \rangle_{\partial \Omega} \] would give us the needed norm estimation if the rest vanishes. We get the feeling that Korn’s inequality (see [12], [13] thm. 3 or 4) is applicable with (6.23) for $v|_{x_0, \partial} = 0$.

\[ \left\| 2 \text{Sym}(\text{Grad } v) \right\|_{\partial \Omega}^2 \]

\[ \leq \left\| \text{Grad } v + (\text{Grad } v)^T \right\|^2_{\partial \Omega} + c^+ \left\| \text{Grad } v \right\|^2_{\partial \Omega}. \]  

(6.23)

Note: It is $c^+ = 2$ if $v|_{x_0, \partial} = 0$.

A positive coefficient $-J\pi(I_3, p)$ would not destroy this inequality, but yet we have no knowledge about its sign in (6.21). Furthermore if $c^+$ is smaller than 2 then a remainder is left.

To guarantee that $\Phi(X)$ is a feasible deformation the determinant of $F$, denoted by $J$, has to fulfill condition (2.17).

\[ \det F = J > 0 \]  

(2.17)

Note: There exists a unique polar decomposition of $F$ with $F = P \cdot V = U \cdot P$, where $V$ (and $U$ resp.) is a unitary tensor of rotation and $P$ is a symmetric stretch tensor. Due to $F$ being invertible $P$ is even positive definite.

Furthermore we can derive the (right) Cauchy-Green strain tensor $C$ which describes the local change of length and the Green-Lagrange strain tensor $E$.

\[ C := F^T \cdot F \]

\[ E := \frac{1}{2}(C - I) \]

\[ 2E = \text{Grad } U + (\text{Grad } U)^T + \text{Grad } V \cdot (\text{Grad } U)^T \]  

(2.18)

(2.19)

(2.20)

For later use we additionally need the directional derivative $E(U; V)$ of the strain.

\[ 2E(U; V) = \text{Grad } V + (\text{Grad } V)^T \]

\[ + \text{Grad } U \cdot (\text{Grad } V)^T + \text{Grad } V \cdot (\text{Grad } U)^T \]

\[ = \text{Grad } V \cdot F(U) + (F(U))^T \cdot (\text{Grad } V)^T \]  

(2.21)

2.3 Equilibrium of forces

We assume that the deformation $\Phi$ of the body $K$ is caused by the influences of external loads. These loads can be of different types: a given displacement (2.22) on the Dirichlet boundary, a deformation independent force density per unit mass (2.24), i.e. an acceleration field, or a force density per unit surface (2.23) on the Neumann boundary, a so called surface force.

\[ U_0 \in T_1 \text{ on } \Gamma_D \subset \partial \Omega \]

\[ \rho \in \mathbb{R}^3 \text{ on } \Gamma_N \subset \partial \Omega \]

\[ f \in \mathbb{R}^3 \text{ with } f(X(\eta)) = f(x(\eta)) \forall \eta \in \mathcal{P} \]  

(2.22)

(2.23)

(2.24)

After the deformation, $K$ shall be in a state of equilibrium of forces. With these assumptions we can derive integral equilibriums and by using the theorem of Cauchy (see [1], p. 275) we can prove the existence of a displacement $U(X(\eta))$ in $\Omega$ and even the existence of a symmetric, second order stress tensor $\sigma(U, x(\eta)) \in C^1(\Omega)$ with the properties (2.25) - (2.27).

\[ \text{div}(\sigma) + \rho, f = 0 \text{ in } \Omega, \]

\[ n \cdot \sigma = g_r \text{ on } \Gamma_r N \subset \partial \Omega, \]

\[ U = U_0 \text{ on } \Gamma_D, \Gamma_r D = \Gamma_D + U_0 \]  

(2.25)

(2.26)

(2.27)
For a certain parameter τ the scalar material density in Ωτ is given by ρτ and nτ ∈ T1 describes the outer normal vector in a boundary point x ∈ ∂Ωτ. The boundary conditions can also be stated component wise.

Note: On parts of the boundary without any given loads it is nτ · σ = 0.

Our goal is now to simulate the displacement U.

2.4 Incompressibility

Incompressibility means that there is no change in volume by any deformation and change of shape. To ensure this J has to fulfill condition (2.28).

\[ \det \mathbf{F} = J \equiv 1. \quad (2.28) \]

Because J gives the ratio of the volume of K before and after the deformation (see [3]), the condition above yields a constant volume. In case of almost incompressibility the condition (2.28) needs to be fulfilled only approximately.

Furthermore one has a restriction to the material parameter K, which is called the bulk modulus. This number describes how much pressure needs to be applied to produce a relative change of volume and to compress a body K. Because we want to consider nearly or complete incompressible material we have to include the limit

\[ K \rightarrow \infty \quad (2.29) \]

in our calculations.

6.3 Coercivity

We plug in M(U, P) from (4.28) and the stress tensor from (3.43).

\[
\begin{align*}
\langle & \mathcal{E}(U; V), M(U, P) ; \mathcal{E}(U; \delta U) \rangle_{\Omega, \partial \Omega} \\
= & -\langle \mathcal{E}(U; V) , 4I_\delta \left( \frac{\partial \phi_\delta}{\partial \mathbf{F}^{-1} \mathbf{I}} + P \frac{\partial \phi}{\partial \mathbf{F}^{-1} \mathbf{I}} \right) \hat{\mathbf{C}} : \mathcal{E}(U; \delta U) \rangle_{\Omega, \partial \Omega} \\
& + \langle \mathcal{E}(U; V) , 4I_\delta \left( \frac{\partial \phi_\delta}{\partial \mathbf{F}^{-1} \mathbf{I}} + \frac{\partial \phi}{\partial \mathbf{F}^{-1} \mathbf{I}} \right) (\mathbf{C}^{-1} : \mathcal{E}(U; \delta U)) \rangle_{\Omega, \partial \Omega} \\
& + \langle \mathcal{E}(U; V) , 4I_\delta \left( \frac{\partial \phi_\delta}{\partial \mathbf{F}^{-1} \mathbf{I}} + \frac{\partial \phi}{\partial \mathbf{F}^{-1} \mathbf{I}} \right) (\mathbf{C}^{-1} \mathbf{I} + \mathbf{Z} \mathbf{C}^{-1} : \mathcal{E}(U; \delta U)) \rangle_{\Omega, \partial \Omega} \\
& \quad (6.15)
\end{align*}
\]

Since δU and V should be in the kernel of B we are allowed to omit the two last terms because it is \( \langle Q , \mathbf{C}^{-1} : \mathcal{E}(U; V) \rangle_{\Omega, \partial \Omega} = 0 \) for \( Q \in L^2(\Omega) \). Furthermore it is (6.16).

\[
\begin{align*}
\langle & \text{Grad } V , \hat{\mathbf{T}}(U, P) : \text{Grad } \delta U \rangle_{\Omega, \partial \Omega} \\
= & \langle \text{Grad } V , 2\frac{\partial \phi_\delta}{\partial \mathbf{F}^{-1}} \text{Grad } \delta U \rangle_{\Omega, \partial \Omega} \\
& + \langle \mathcal{F}^{-T} : \text{Grad } V , 2I_\delta \left( \frac{\partial \phi_\delta}{\partial \mathbf{F}^{-1} \mathbf{I}} + P \frac{\partial \phi}{\partial \mathbf{F}^{-1} \mathbf{I}} \right) \mathcal{F}^{-T} : \text{Grad } \delta U \rangle_{\Omega, \partial \Omega} \\
& \quad (6.16)
\end{align*}
\]

After direct computing from (3.27) we define \( \pi(I_3, P) \).

\[
\pi(I_3, P) := I_3 \left( \frac{\partial \phi}{\partial \mathbf{F}^{-1} \mathbf{I}} + \frac{\partial \phi}{\partial \mathbf{F}^{-1} \mathbf{I}} \right) = - c_\phi \frac{a_1}{3} I_3^{-1/3} + \frac{1}{2} P \quad (6.17)
\]

\[
2 \frac{\partial \phi}{\partial \mathbf{F}^{-1} \mathbf{I}} = 2 c_\phi a_1 I_3^{-1/3} > 0 \quad (6.18)
\]

Note: In \(-\pi(I_3, P)\) it clearly is \( c_\phi \frac{a_1}{3} I_3^{-1/3} > 0 \), but the second addend depends on the problem.

By a short computation with interchanging tensor factors it is

\[
4 \mathcal{E}(U; \delta U) : \hat{\mathbf{C}} : \mathcal{E}(U; V) = \mathcal{E}(U; \delta U) \cdot \mathbf{C}^{-1} : \mathcal{E}(U; V) \cdot \mathbf{C}^{-1}
\]

\[
= 2 \text{Sym} \left( \text{Grad } \delta U \cdot \mathcal{F}^{-1} \right) : 2 \text{Sym} \left( \text{Grad } V \cdot \mathcal{F}^{-1} \right)
\]

\[
= \left( \text{grad } \delta U + (\text{grad } \delta U)^T \right) : \left( \text{grad } V + (\text{grad } V)^T \right)
\]

\[
= 4 \mathcal{E}(\delta U) : \mathcal{E}(V) \quad (6.19)
\]
The scalar product gives
\[
\langle Q \cdot C^{-1} : (F^T \cdot (\text{Grad } V)^T) \rangle_{0, \Omega} = \langle Q \cdot F^{-1} : (\text{grad } V)^T \rangle_{0, \Omega} = \int_\Omega Q(X) \cdot \text{div}(V(X)) \, dX = \int_{\Omega_x} q(x) \cdot \text{div}(v(x)) \frac{1}{J(x)} \, dx .
\] (6.9)

We can apply the theorem of the surjectivity of the divergence on \(L^2\): For all \(q \in L^2(\Omega_x)\) there is a \(v \in \mathcal{H}^1(\Omega_x)^3\) with the properties
\[
q = \text{div } v , \\
\|v\|_{1, \Omega_x} \leq c_1 \|q\|_{0, \Omega_x} .
\] (6.10)

We can choose such a pair \((q, v)\) and use it in (6.9).
\[
\langle Q \cdot C^{-1} \cdot \text{Grad } V, F \rangle_{0, \Omega} = \langle J^{-1} \cdot q, q \rangle_{0, \Omega_x} = \langle Q, Q \rangle_{0, \Omega} = \|Q\|^2_{0, \Omega} .
\] (6.12)

With (6.11) we estimate the norms.
\[
c_0 \|V\|_V \leq \|v\|_{1, \Omega_x} \leq c_1 \|q\|_{0, \Omega_x} \leq c_2 \|Q\|_{0, \Omega} .
\] (6.13)

Hence the inf-sup condition follows.
\[
\sup_{V \in \mathcal{V}_0} \frac{\langle Q, \text{div } V \rangle_{0, \Omega}}{\|V\|_{\mathcal{V}_0}} = \sup_{V \in \mathcal{V}_0} \frac{\langle Q, \text{div } V \rangle_{0, \Omega}}{\|V\|_{\mathcal{V}_0}} \geq \frac{\|Q\|^2_{0, \Omega}}{\|V\|_{\mathcal{V}_0}} \geq \frac{\|Q\|^2_{0, \Omega}}{c_2 c_0^{-1} \|Q\|_{0, \Omega}} = c_0 \|Q\|_{0, \Omega} ,
\] (6.14)

3 Mixed variational formulation

In this chapter we formulate the nonlinear problem of deformation for incompressible material. To that it is necessary to introduce a new process variable. Additionally the representation of \(\widetilde{T}\) as a derivative of the specific energy density function \(\phi(C)\) becomes important, which is discussed in a later subsection.

3.1 Variational formulation of nonlinear elasticity

To solve the problem (2.25) - (2.27) by means of the finite element method the weak formulation is needed. As usual we multiply with test functions \(V \in \mathcal{V}_0\)
\[
\mathcal{V}_D := \{ V \in (\mathcal{H}^1(\Omega))^3 : \quad V_{\mid_{\Gamma_D}} = U_0 \} ,
\] (3.1)
\[
\mathcal{V}_0 := \{ V \in (\mathcal{H}^1(\Gamma))^3 : \quad V_{\mid_{\Gamma_D}} = 0 \} \quad \forall \psi \in \mathcal{V}_0 .
\] (3.2)

and integrate over the domain \(\Omega_x\). This yields the integral equation
\[
(\text{div}(\sigma) \cdot v)_{0, \Omega_x} + (\rho, f, v)_{0, \Omega_x} = 0 \quad \forall v \in \mathcal{V}_0 .
\] (3.3)

After the application of the integral theorem of Gauss we get (3.4).
\[
(\sigma, \text{grad } v)_{0, \Omega_x} + (\rho, f, v)_{0, \Omega_x} + (n, \sigma, \psi)_{\Gamma_D} = 0 \quad \forall v \in \mathcal{V}_0 .
\] (3.4)

Contrary to the case of small deformations we need to distinguish between the values on \(\Omega_x\) and \(\Omega_x\). But there are quite some handy transformation rules available.
\[
d\Omega_x = \left[ g_1, g_2, g_3 \right] \, dy^1 dy^2 dy^3 = \left[ F \cdot G_1, F \cdot G_2, F \cdot G_3 \right] \, dy = \text{det } F \left[ G_1, G_2, G_3 \right] \, dy = J \, d\Omega
\] (3.5)
\[
\rho = J^{-1} \rho_0
\] (3.6)

With the introduction of two new second order tensors, namely the first and second Piola-Kirchhoff stress tensor \(\sigma\) and \(\Sigma\) respectively we can also generate a pull back of the stress \(\sigma\) from \(\Omega_x\) to \(\Omega_x\).
\[
\sigma = J^{-1} F \cdot \Sigma
\] (3.7)
\[
\Sigma = J^{-1} \cdot F \cdot \widetilde{T} \cdot F^T
\] (3.8)
Differently from $\sigma$ these new terms are living on the initial configuration and at least $\dot{T}$ keeps the symmetry property of $\sigma$.

With (3.8) and the symmetry of $\dot{T}$ we transform the left hand side of (3.4).

\[
\langle \sigma, \text{grad } v \rangle_{0,\Omega} = \left\langle J \cdot J^{-1} F \cdot \dot{T} \cdot \text{grad } V \right\rangle_{0,\Omega} \\
= \left\langle \dot{T}, \text{Grad } V \right\rangle_{0,\Omega} \\
= \int_{\Omega} \dot{T} : (F^T \cdot (\text{Grad } V)^T) \, d\Omega \\
= \left\langle \dot{T}, \mathcal{E}(U; V) \right\rangle_{0,\Omega}
\]  

(3.9)

W.l.o.g. we assume $\rho_0 = 1$. Applying $\rho_0(x) = J^{-1}\rho_0(X)$ from (3.6) on $\langle \rho_0 f, v \rangle_{0,\Omega}$, from (3.4) we obtain

\[
\langle \rho_0 f, v \rangle_{0,\Omega} = \langle f, v \rangle_{0,\Omega}.
\]  

(3.10)

Again w.l.o.g., we assume that

\[
\Gamma_N, := \left\{ \{x', \eta', \gamma'\} : \eta' = \eta_0 \text{ constant}, (y', \gamma) \in \mathcal{P}_N \right\}.
\]  

(3.11)

Then it is $d_s = \|g_1 \times g_2\| \, dy \, dz$ and the boundary integral in (3.4) can be transformed.

\[
\langle n_s \cdot \sigma, v \rangle_{0,\Gamma_N} = \int_{\Gamma_N} \langle n_s \cdot \sigma, v \rangle \, ds_s \\
= \int_{\Gamma_0} \left\| g_1 \times g_2 \right\| \| g_1 \times g_2 \| \, dy \, dz \\
= \int_{\Gamma_0} \left\| g_1 \times g_2 \, \sigma \cdot v \right\| \, dy \, dz \\
= \int_{\Gamma_0} \left\| g_1 \times g_2 \right\| \left\| g_1 \times g_2 \, \sigma \cdot v \right\| \, dy \, dz \\
= \int_{\Gamma_0} \left\| g_1 \times g_2 \right\| \left\| g_1 \times g_2 \right\| \left\| g_1 \times g_2 \, \sigma \cdot v \right\| \, dy \, dz \\
= \left\langle n_0 \cdot \dot{T}, V \right\rangle_{0,\Omega}
\]  

(3.12)

Setting $g_0 := n_0 \cdot \dot{T}$ we finally get the weak formulation on $\Omega$.

6 LBB conditions

It is known that a saddle point problem is uniquely solvable, if the three bilinear forms $a(U, P; b(U, Q, V))$, $b(U, Q, V)$ and $c(P, Q)$ are continuous and fulfill the conditions (6.4) - (6.3) (see [1] sec. III.4 theorem 4.11 or [5] sec. 3.2).

\[
a(U, P; v, V) \geq \alpha \| V \|_{V0}^2 \forall V \in \mathcal{N}_b(B); \alpha > 0
\]  

(6.1)

\[
\sup_{V \in \mathcal{V}_0} \frac{b(U, Q, V)}{\| V \|_{V0}} \geq \beta \| Q \|_0 \forall Q \in \mathcal{Q}; \beta > 0
\]  

(6.2)

\[
c(Q, Q) \geq 0 \forall Q \in \mathcal{Q}
\]  

(6.3)

Here we use the kernel of $B$, which is an associate operator to $b(\ldots)$.

\[
\mathcal{N}_b(B) = \left\{ V \in \mathcal{V}_0 : b(U, Q, V) = \langle Q, B(U) \cdot V \rangle = 0 \forall Q \in \mathcal{Q} \right\}
\]

Instead of showing the coercivity (6.1) it suffices to show the stability on the kernel.

\[
\sup_{V \in \mathcal{V}_0} \frac{a(U, P; \delta U, V)}{\| V \|_{V0}} \geq \alpha \| \delta U \|_{V0} \forall \delta U \in \mathcal{N}_b(B); \alpha > 0
\]  

(6.4)

\[
\sup_{V \in \mathcal{V}_0} \frac{a(U, P; \delta U, V)}{\| V \|_{V0}} \geq \alpha \| V \|_{V0} \forall V \in \mathcal{N}_b(B); \alpha > 0
\]  

(6.5)

6.1 First steps

Condition (6.3) can be shown easily with (4.13).

\[
c(Q, Q) = \int_{\Omega} \kappa \cdot Q^2 \, d\Omega \geq 0 \tag{6.6}
\]

\[
c(Q, Q) \leq \int_{\Omega} \kappa \cdot Q^2 \, d\Omega \leq \max \kappa \cdot \| Q \|_{0,\Omega}^2 \tag{6.7}
\]

6.2 Inf-sup condition

Due to the symmetry of $\mathcal{S}_b(U) = C^{-1}(U)$ (6.8) holds.

\[
\sup_{V \in \mathcal{V}_0} \frac{b(U, Q, V)}{\| V \|_{V0}} = \sup_{V \in \mathcal{V}_0} \frac{1}{\| V \|_{V0}} \langle Q, \mathcal{S}_b(U) \cdot \mathcal{E}(U; V) \rangle_{0,\Omega}
\]  

(6.8)
With \( d(eU,eP) := |\gamma(T)eU|_{0,\Omega}^2 + \|eP\|_{0,\Omega}^2 \) we get the element wise error indicator 
\( \eta_{\text{comb}}^2 \) with (5.21).

\[
\eta_{\text{comb}}^2 := \|e_P\|_{0,T}^2 + h_T^2 \|R_P\|_{0,T}^2 + h_T^2 \sum_{F \subset \partial T} \|R_{PF}(T)\|_{0,F}^2
\]

(5.20)
\[
\mathcal{J}(eU,eP) \leq c_2 \left\{ \sum_{T \in \mathcal{T}_h} \eta_{T,\text{comb}}^2 \right\}^{1/2}
\]

However, numerical tests indicate that the \( r_P \) part can be left out because it is dominated by the other ones. There are also reasonable arguments why it is possible to leave out the element residual too.

To prevent the case that there is no mesh refinement if the estimated error is zero (e.g. if \( \partial \Omega = \Gamma_D \) without any inner faces) we set the refinement condition as follows.

\[
\text{refine element } T_i \text{ if } \eta_{T,i} \geq p \cdot \max_{T_i} \{ \eta_{T,i} \}, \quad p \in (0,1)
\]

(5.22)

3.2 Stress tensor of nonlinear elasticity

The tensor \( \hat{T} \) from (3.8) can be derived from the Clausius-Duhem inequality (see [4] sec. 2-5 or [3] p. 5).

\[
\frac{1}{2} \hat{T} : C - \rho_0 \dot{\phi}(C) \geq 0
\]

(3.14)

Here \( \dot{\phi}(C) \) stands for the free Helmholtz energy density per mass unit and the dot symbolises the usual time derivative. For simplification we define the specific strain energy density function per volume unit \( \phi(C) \).

\[
\phi(C) := \rho_0 \dot{\phi}(C)
\]

(3.15)

As it is shown in [4], sec. 5 the inequality (3.14) yields the so called law of hyperelasticity.

\[
\hat{T} = 2 \frac{\partial \phi(C)}{\partial C}
\]

(3.16)

3.3 Decomposition ansatz of Flory

We consider the so called Flory split of the deformation gradient \( \mathcal{F} \) (and later on of the Cauchy-Green strain tensor \( C \)) into a deviatoric and a volumetric (isochoric) part (see [3], [5] or [6]).

\[
\mathcal{F} = \mathcal{F}_D \cdot \mathcal{F}_V
\]

(3.17)

The part \( \mathcal{F}_D \) shall describe the change of shape whereas \( \mathcal{F}_V \) shall describe the change of volume during the deformation \( \Phi \). That is why one can postulate

\[
\det(\mathcal{F}_D) = 1 \quad \text{and} \quad \det(\mathcal{F}_V) = J^{\gamma=0} = 1.
\]

(3.18)

(3.19)
These conditions can be fulfilled easily by setting $\mathcal{F}_D$ to $J^{-1/3}F$ and $\mathcal{F}_V$ to $J^{1/3}I$. Analogously one can multiplicatively decompose $\mathcal{C}$ with $\mathcal{C} = \mathcal{C}_D \cdot \mathcal{C}_V$. Both quantities have to fulfill (3.20) and (3.21).

$$\mathcal{C}_D = \mathcal{F}_D^T \cdot \mathcal{F}_D = \left( J^{-1/3}F \right)^T \cdot \left( J^{-1/3}F \right) = J^{-2/3} \mathcal{C} \quad (3.20)$$

$$\mathcal{C}_V = \mathcal{F}_V^T \cdot \mathcal{F}_V = \left( J^{1/3}I \right)^T \cdot \left( J^{1/3}I \right) = J^{2/3} \mathcal{I} \quad (3.21)$$

This decomposition also affects the specific strain energy density, in such a way that $\phi(\mathcal{C})$ splits into two additive parts. Again we get a deviatoric part $\phi_D(\mathcal{C})$ which is dedicated to the energy of the changing shape and a volumetric part $\phi_V(\mathcal{C})$ which only describes the energy of the changing volume during a deformation.

$$\phi(\mathcal{C}) = \phi_D(\mathcal{C}_D) + U(\mathcal{C}_V) = \phi_D(\mathcal{C}_D) + \frac{K}{2} \phi_V(\mathcal{C}_V)^2 \quad (3.22)$$

As we consider isotropic and objective material we can choose an energy density that only depends on the three tensor invariants (3.25) instead of the tensor itself.

$$\phi(\mathcal{C}) = \phi_D(I_1(\mathcal{C}_D), I_2(\mathcal{C}_D), I_3(\mathcal{C}_D)) + \frac{K}{2} \phi_V(I_1(\mathcal{C}_V), I_2(\mathcal{C}_V), I_3(\mathcal{C}_V))^2 \quad (3.23)$$

with

$$I_1(\mathcal{Y}) = tr(\mathcal{Y})$$

$$I_2(\mathcal{Y}) = \frac{1}{2} \left( tr(\mathcal{Y})^2 - tr(\mathcal{Y}^2) \right) \quad (3.24)$$

$$I_3(\mathcal{Y}) = \text{det}(\mathcal{Y}) \quad \forall \mathcal{Y} \in \mathcal{T}_2$$

With (3.20) and (3.21) the identity $I_1(\mathcal{C}_D) = 1$ follows. In addition we can show that $I_k(\mathcal{C}_V)$ depends only on $J$ for all $k = 1, 2, 3$. Therefore we can omit the dependencies on $I_1(\mathcal{C}_D)$ and replace $I_1(\mathcal{C}_V)$ by $J^2$. Additionally we also decide to omit all terms depending on $I_2(\mathcal{C}_D)$.

$$\phi(\mathcal{C}) = \phi_D(I_1(\mathcal{C}_D)) + \frac{K}{2} \phi_V(J^2)^2 \quad (3.26)$$

We also want to achieve that $\phi_V(J^2)$ is convex and is zero in $J = 1$. In this case a suitable material function is the Neo-Hooke material (see [5]). Plugging in our values this yields (3.27).

$$\phi(\mathcal{C}) = c_0(I_1(\mathcal{C}_D)^3 - 3) + \frac{K}{2} \left( \ln(J) \right)^2$$

$$= c_0 \left( I_1(\mathcal{C}) I_1(\mathcal{C})^{-1/3} - 3 \right) + \frac{K}{2} \left( \frac{1}{3} \ln(I_1(\mathcal{C})) \right)^2 \quad (3.27)$$

Note: As an abbreviation we use $I_k := I_k(\mathcal{C})$ from now on.

The sum of all error indicators gives an upper bound on our error functional.

$$\mathcal{J}(\mathcal{C}) = \frac{1}{|\gamma(T)eU|_{1,\Omega}} \left( a'(\mathcal{U}, P; eU) - a'(\mathcal{U}_h, P_h; eU) \right) \quad (5.13)$$

$$\leq c_3 \left( \sum_{r \in \mathcal{T}_h} \eta_r^2 \right)^{1/2} \quad (5.14)$$

Following the derivation for the case of linear elasticity, where we usually choose $\gamma(T)^2$ to represent the smallest eigenvalue of $\mathcal{M}$, we take $\gamma(T)^2 = c_{10}$.

Note: Analogously one can treat the second equation of (5.1). We denote the error with $\epsilon_p := P - P_h$ in 0, $\Omega$ and define a test function $Q := (I - I_h)eP$ in 0. $\Omega$. Subsequent we consider the error functional (5.17) and the corresponding denominator $\mathcal{D}$.

$$\mathcal{J}(\epsilon_p) = \frac{b_0(U; \epsilon_p) - c(P; \epsilon_P) - b_0(U_h; \epsilon_P) + c(P_h; \epsilon_P)}{\|\epsilon_p\|_{0,\Omega}} \quad (5.15)$$

$$\mathcal{D} = b_0(U; \epsilon_p) - c(P; \epsilon_P) - b_0(U_h; \epsilon_p) + c(P_h; \epsilon_P)$$

$$= -b_0(U_h, Q) + c(P_h, Q) = (\kappa P_h - \phi_3(U_h), Q)_{0,\Omega} \quad (5.16)$$

With $(\kappa P_h - \phi_3(U_h)) = r_p \in 0, T$ we get the following estimate.

$$\mathcal{D} \leq C \left( \sum_{\mathcal{T}} \|r_p\|_{0,\mathcal{T}}^2 \right)^{1/2} \cdot \|\epsilon_p\|_{0,\Omega} \quad (5.17)$$

We should set (5.18).

$$\|\epsilon_p\|_{0,\Omega}^{1/2} = \|\epsilon_p\|_{0,\Omega} \quad (5.18)$$

If we combine these two results, we would get the following.

$$d(\mathcal{C}) \cdot \mathcal{J}(\mathcal{C})^{1/2} \cdot \mathcal{D}(\mathcal{C})^{1/2}$$

$$= a'(\mathcal{U}, P; eU) - a'(\mathcal{U}_h, P_h; eU)$$

$$- b_0(U; \epsilon_P) + b_0(U_h; \epsilon_P) + c(P; \epsilon_P) - c(P_h; \epsilon_P)$$

$$\leq c_2 \left( \sum_{\mathcal{T}} \frac{b_0^2}{\gamma(T)^2} \|R_{T}\|_{0,\mathcal{T}}^2 + \sum_{\mathcal{T}, P \in \mathcal{T}_T} \frac{b_0^2}{\gamma(T)^2} \|R_{P(T)}\|_{0,\mathcal{T}}^2 + \sum_{\mathcal{T}} \|r_p\|_{0,\mathcal{T}}^2 \right)^{1/2}$$

$$\cdot \left( \|\gamma(T)eU\|_{1,\Omega} + \|\epsilon_p\|_{0,\Omega}^2 \right)^{1/2} \quad (5.19)$$
We introduce the abbreviation $T_h = \hat{T}(U_h, P_h)$ and the operator $\Pi_{F,T}$ that is the identity if $F \in \Gamma_N$ and zero elsewhere. Again we set $\rho_0$ equal to one, w.l.o.g. The integral theorem of Gauss yields (5.7).

$$D = \sum_{T \in \mathcal{T}_h} \langle f, V \rangle_{0,T} + \sum_{F \subset \partial T} \langle g \cdot n, V \rangle_{0,F}$$

$$+ \sum_{T \in \mathcal{T}_h} \left( \langle \text{Div}(\hat{T}_h), V \rangle_{0,T} - \sum_{F \subset \partial T} \langle n \cdot \hat{T}_h, V \rangle_{0,F} \right)$$

$$= \sum_{T \in \mathcal{T}_h} \left( f + \text{Div}(\hat{T}_h), V \right)_{0,T} + \sum_{F \subset \partial T} \langle \Pi_{F,T} g - n \cdot \hat{T}_h, V \rangle_{0,F} \right) \right)$$

(5.7)

We define the residuals $R_T$ and $R_{F(T)}$.

$$R_T := \left\{ \begin{array}{ll}
\frac{1}{2} \left( n \cdot \hat{T}_h \right)_{|T} & \text{if } T = T_1 \cap T_2 \\
0 & \text{else}
\end{array} \right. \right.$$

(5.8)

$$R_{F(T)} := \left\{ \begin{array}{ll}
\frac{1}{2} \left( n \cdot \hat{T}_h \right)_{|F} & \text{if } F \subset \Gamma_N \\
0 & \text{else}
\end{array} \right. \right.$$

(5.9)

Note: $n_{|F} = -n_{|T}$ \Rightarrow $R_{F(T)} = \frac{1}{2} n_{|T} \cdot \left( \hat{T}_h_{|T} - \hat{T}_h_{|F} \right)$

With the new notation, (5.7) can be reformulated as follows.

$$D = \sum_{T \in \mathcal{T}_h} \left( R_T, V \right)_{0,T} + \sum_{F \subset \partial T} \left( R_{F(T)}, V \right)_{0,F}$$

(5.10)

This can be estimated from above by using Cauchy-Schwarz inequalities, interpolation estimates and the Clément interpolation operator as well as the patches $\hat{T}$ and $\hat{F}$ around $T$ and $F$ respectively and a scalar material function $\gamma(T) = \gamma_T |T$ that is constant on every element $T$. Finally this approach yields (5.11).

$$D \leq C \left\{ \sum_{T \in \mathcal{T}_h} \frac{h_T^2}{\gamma_T^2} \| R_T \|_{0,T}^2 + \sum_{T \subset \partial \Omega} \frac{h_T}{\gamma_T} \left\| R_{F(T)} \right\|_{0,F}^2 \right\}^{1/2} \cdot |\gamma(T) eU|_{1,\Omega}$$

(5.11)

From (5.11) we can define the element wise error indicator $\eta_T$ as shown in (5.12) and $d_U(eU, eU)$ with $d_U(eU, eU)^{1/2} := |\gamma(T) eU|_{1,\Omega}$.

$$\eta_T^2 := \frac{h_T^2}{\gamma_T^2} \| R_T \|_{0,T}^2 + \sum_{F \subset \partial T} \frac{h_T}{\gamma_T} \left\| R_{F(T)} \right\|_{0,F}^2$$

(5.12)

### 3.4 Hydrostatic pressure

With (3.16) and (3.27) we can define the second Piola-Kirchhoff stress tensor more precisely.

$$\hat{T}(C) = 2 \frac{\partial \phi(C)}{\partial C} = \frac{\partial \phi_C}{\partial C} + K \phi_V \frac{\partial \phi_V}{\partial C}$$

(3.28)

However the product term $\left( \Phi \phi_V \right)$ cannot be determined explicitly, because $K$ diverges according to (2.29) and $\phi_V$ vanishes for $J = 1$. One way out is given by the substitution (3.29), which eliminates the problematic term.

$$P := K \phi_V(I_1) = \frac{1}{\kappa} \phi_V(I_1)$$

(3.29)

The new material parameter $\kappa$ in (3.29) is defined as the reciprocal of the bulk modulus and it is called the compressibility of the material. Obviously, we have $\kappa = 0$ for incompressible material and $\kappa \geq 0$ for nearly incompressible material.

After plugging in the substitution (3.29) in (3.28) we obtain a new formulation of the stress

$$\hat{T} = \hat{T}(U, P) = \hat{T}(C(U), P) = T_C(C) + P \cdot S_C(C)$$

(3.30)

together with the side condition (3.31), which is later used in a weak sense.

$$\phi_V(I_1) - \kappa P = 0$$

(3.31)

### 3.5 Piola-Kirchhoff stress

The derivative $\frac{\partial \phi(C)}{\partial C}$ in (3.28) can be build with the help of the pseudo invariants $a_i$ (see [7], [8]).

$$a_i(Y) := \frac{1}{i} \text{tr}(Y^T)$$

\begin{align*}
a_1(Y) &:= \frac{1}{2} \text{tr}(Y^T) & i = 1, 2, 3, \forall Y \in T_2 \\
a_2 &:= \left( a_1 \ a_2 \ a_3 \right)^T & \text{with } a_i := a_i(C) \end{align*}

(3.32)

(3.33)

They permit the reformulation of the tensor invariants

$$I_1(Y) = a_1(Y)$$

$$I_2(Y) = \frac{1}{2} \left( a_1(Y)^2 - 2a_2(Y) \right)$$

(3.34)

$$I_3(Y) = \frac{1}{6} \left( a_1(Y)^3 - 6a_1(Y)a_2(Y) + 6a_3(Y) \right)$$

(3.35)
and they have a simple derivative w.r.t. any second order tensor $Y$ (see [8], use Taylor expansion).
\[
\frac{\partial a_i(Y)}{\partial Y} = Y^{i-1}
\]
(3.35)

With (3.34) the specific energy density function can be reformulated in terms of $\omega$, instead of (3.27).
\[
\phi(C) = \phi(I(\omega)) + \frac{K}{2} \phi^2(I(\omega))
\]
(3.36)
\[
= \phi(I(\omega)) + \frac{K}{2} \phi^2(\omega)
\]
(3.37)

Note: We keep the notation $\phi_D$ and $\phi_V$ although the dependencies of the functions has been changed.

By applying the chain rule to (3.37) the derivatives of $\phi_D(C)$ and $\phi_V(C)$ with respect to $C$ can be determined as the second order tensors
\[
T_D = 2 \frac{\partial \phi_D(C)}{\partial C} = 2 \sum_{i=1}^{3} \left( \frac{\partial \phi_D(a)}{\partial a_i} \frac{\partial a_i}{\partial C} \right),
\]
(3.38)
\[
S_V = 2 \frac{\partial \phi_V(C)}{\partial C} = 2 \sum_{i=1}^{3} \left( \frac{\partial \phi_V(a)}{\partial a_i} \frac{\partial a_i}{\partial C} \right).
\]
(3.39)

In general we get
\[
T_D = 2 \frac{\partial \phi_D}{\partial a_1} I + 2 \frac{\partial \phi_D}{\partial a_2} C + 2 \frac{\partial \phi_D}{\partial a_3} C^2,
\]
(3.40)
\[
S_V = 2 \frac{\partial \phi_V}{\partial a_1} I + 2 \frac{\partial \phi_V}{\partial a_2} C + 2 \frac{\partial \phi_V}{\partial a_3} C^2.
\]
(3.41)

If the dependency on $I_1$ and $I_3$ (see (3.27)) is used directly we obtain in the same way
\[
T_D = 2 c_D I_2^{-1/3} \left( I - \frac{a_1}{3} I^{-1} \right),
\]
(3.42)
\[
S_V = C^{-1}.
\]
(3.43)

The equivalence of (3.42), (3.43) with (3.40), (3.41) would follow from the theorem of Cayley-Hamilton as well.

### 3.6 Mixed formulation

With the decomposition (3.30) of $\hat{T}$ we get the equation (3.44) instead of (3.13).
\[
\langle T_D(U), E(U; V) \rangle_{0,T} + \langle P \cdot S_V(U), E(U; V) \rangle_{0,T} = f_0(V)
\]
(3.44)
\[
\forall \ V \in V_0
\]

### 5 Error estimation

In order to control the adaptive mesh refinement we need an appropriate error estimator of the approximated solution $(U_h, P_h)$ of (5.1).

\[
a^*(U_h, P_h; V) = f_0(V) \ \forall V \in V_{h,0}
\]
(5.1)
\[
b_0(U_h; Q) - c(P_h; Q) = 0 \ \forall Q \in Q_h
\]
(5.2)

\[
\text{s.t. } a^*(U, P; V) := a_D(U; V) + a_V(U, P; V) = \left( T(U, P), \text{Grad} \ V \right)_{0,\Omega}
\]

#### 5.1 Reliability

We use $e_U \in V_0$ with $e_U := U - U_h$ to denote the error and define a test function $V \in V_0$ with $V = I e_U - I_h e_U$ and the projection $I_h : V_0 \rightarrow V_{h,0}$.

As it is shown in [2] the form $a^*(U, P; V)$ does not lead to a proper energy norm (in a sense of $\| T(U, P) \|_{a^*}$) and the Galerkin orthogonality becomes like (5.3).
\[
\Delta a^*(U) := a^*(U, P; V) - a^*(U_h, P_h; V) = a_D(U; V) - f_0(V) = 0 \ \forall V \in V_0
\]
(5.3)

Now we consider (5.4) as a functional to measure the error, whereas $d_U(V, V)$ denotes some kind of norm.
\[
\mathcal{J}(e_U) = \left( a^*(U, P; e_U) - a^*(U_h, P_h; e_U) \right) d_U(e_U, e_U)^{-1/2}
\]
(5.4)

After exploiting the Galerkin orthogonality the denominator $D := \Delta a^*(e_U)$ yields (5.5).
\[
D = a^*(U, P; V) - a^*(U_h, P_h; V) = \left( \hat{T}(U, P), \xi(U, V) \right)_{\Omega} - \left( \hat{T}(U_h, P_h), \xi(U_h, P_h; V) \right)_{\Omega}
\]
(5.5)

\[
= \left( \rho_0 f, V \right)_{\Omega} + \left( g, V \right)_{\Gamma_{\text{in}}} - \left( \hat{T}(U_h, P_h), \text{Grad} V \right)_{0,\Omega}
\]
(5.6)

Using the discretisation $T_h$ we get (5.6).
\[
D = \sum_{T \in \mathcal{T}_h} \left( \rho_0 f, V \right)_{h,T} - \left( \hat{T}(U_h, P_h), \text{Grad} V \right)_{0,T}
\]
+ \sum_{F \in \Gamma_{\text{in}}} \left( g, V \right)_{h,F}
(5.6)
Since the matrices are self adjoint
\[ \langle A_0^{-1} \cdot A \cdot x_0, y_0 \rangle_u = \langle x_0, A_0^{-1} \cdot A \cdot y_0 \rangle_u \] (4.37)
\[ \langle C_0^{-1} \cdot K \cdot x_0, y_0 \rangle_u = \langle x_0, C_0^{-1} \cdot K \cdot y_0 \rangle_u \] (4.38)
we can estimate the eigenvalues in (4.36) with the Rayleigh quotient \( \Theta_n(x_0) \).
\[ \Theta_{n+1}(x_0) := \frac{\langle A_0^{-1} \cdot A \cdot x_0, x_0 \rangle_u}{\langle x_0, x_0 \rangle_u} \leq \lambda_{\text{max}}(A_0^{-1} \cdot A) \] (4.39)
\[ \lambda_{\text{min}}(C_0^{-1} \cdot K) \leq \frac{\langle C_0^{-1} \cdot K \cdot x_0, x_0 \rangle_u}{\langle x_0, x_0 \rangle_u} =: \Theta_{n+1}(C_0^{-1} \cdot K(x_0)) \] (4.40)
We choose
\[ \delta = \frac{\Theta_{n+1}(x_0)}{\Theta_{n+1}(C_0^{-1} \cdot K(x_0))} \leq \frac{\lambda_{\text{max}}(A_0^{-1} \cdot A)}{\lambda_{\text{min}}(C_0^{-1} \cdot K)} \] (4.41)
which yields the estimate (4.42) after applying this choice to (4.36).
\[ \kappa(S_{n+1} \cdot S_n) \leq c_n \kappa(M) \leq c_n \kappa(A_0^{-1} \cdot A) \cdot \kappa(C_0^{-1} \cdot K) \] (4.42)

Note: With the exact choice of \( \delta \) we would get
\[ \kappa(M) = \max \left\{ \kappa(A_0^{-1} \cdot A), \kappa(C_0^{-1} \cdot K) \right\} \quad \text{for} \quad \delta = \frac{\lambda_{\text{max}}(A_0^{-1} \cdot A)}{\lambda_{\text{min}}(C_0^{-1} \cdot K)} \]
\[ \kappa(M) = \kappa(A_0^{-1} \cdot A) \cdot \kappa(C_0^{-1} \cdot K) \quad \text{for} \quad \delta = \frac{\lambda_{\text{max}}(A_0^{-1} \cdot A)}{\lambda_{\text{min}}(C_0^{-1} \cdot K)} \]

Additionally we have to guarantee condition (3.31) in a weak sense. Therefor we choose suitable test functions \( Q \in Q := L^2(\Omega) \) and get
\[ \langle \phi_V(I_3(U)), Q \rangle_{\text{H}} - \langle \kappa \cdot P, Q \rangle_{\text{H}} = 0 \quad \forall \ Q \in Q \]. (3.45)

We introduce a new notation
\[ a_D(U, V) = \langle T_D(U), \mathcal{E}(U, V) \rangle_{\Omega} \] (3.46)
\[ a_V(U, P, V) = \langle P \cdot S(U), \mathcal{E}(U, V) \rangle_{\Omega} \] (3.47)
\[ b_0(U, Q) = \langle \phi_V(I_3(U)), Q \rangle_{\Omega} \] (3.48)
\[ c(P, Q) = \langle \kappa \cdot P, Q \rangle_{\Omega} \] (3.49)
\[ f_0(V) = \langle f, V \rangle_{\Omega} + \langle g, V \rangle_{\Gamma_N} \] (3.50)
that allows us to formulate the mixed boundary value problem of (nearly) incompressible nonlinear elasticity with large deformations via an nonlinear system of equations. This yields the nonlinear saddle-point like problem.

Find the displacement \( U \in V_D \) and the hydrostatic pressure \( P \in Q \) such that
\[ a_D(U, V) + a_V(U, P, V) = f_0(V) \]
\[ b_0(U, Q) - c(P, Q) = 0 \] (3.51)
holds for all test functions \( (V, Q) \in V_0 \times Q \).
4 Solution method

To solve problem (3.51) we use Newton’s method to linearise the equations and mixed finite elements to discretise them afterwards. The resulting system can be solved with a method of conjugate gradients due to Bramble and Pasciak (see [9] or [10]). Altogether this leads to a nested iteration method.

4.1 Newton’s method

First of all we transform (3.51) into a homogeneous problem.

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
aD(U; V) + aV(U; P; V) - f_0(V) \\
b_0(U; Q) - c(P; Q)
\end{pmatrix} =: S(U; P; V; Q) \tag{4.1}
\]

Even though (4.1) seems to be nonlinear in \( U \) only, we have to perform a linearisation in both \( U \) and \( P \). This can be done with Newton’s method. We choose a initial solution \((U_0, P_0)\), solve equation (4.2) several times for \( t \geq 0 \)

\[
S'(U_t; P_t; V, Q, \delta U, \delta P) = -S(U_t; P_t; V, Q) \quad \forall (V, Q) \in \mathbb{V}_0 \times \mathbb{Q} \tag{4.2}
\]

and update the current initial solution \((U_t, P_t)\) with the increment \((\delta U, \delta P)\) in each step.

\[
\begin{pmatrix}
U_{t+1} \\
P_{t+1}
\end{pmatrix} = \begin{pmatrix}
U_t \\
P_t
\end{pmatrix} + \begin{pmatrix}
\delta U \\
\delta P
\end{pmatrix} \tag{4.3}
\]

Because this method has only local convergence we additionally use incremental load steps. Instead of (4.2) and (4.1) we consider the operator (4.4) with \( t \in (0, 1] : t \to 1 \) and equation (4.5) for all test functions \( V \in \mathbb{V}_0 \) and \( Q \in \mathbb{Q} \).

\[
\begin{pmatrix}
aD(U; V) + aV(U; P; V) - t f_0(V) \\
b_0(U; Q) - c(P; Q)
\end{pmatrix} = S(U; P; t; V; Q) \tag{4.4}
\]

\[
S'(U; P; V; Q, \delta U, \delta P) = -S(U; P; t; V; Q) \tag{4.5}
\]

For any fixed \( t \) we choose \( \frac{1}{t_{\text{tol}}} < \varepsilon_{\text{tol}} \) as the stopping criteria. For \( t = 1 \), this yields the approximate solution of the problem (3.51).

Note: We abbreviate \( S := S(U; P; t; V; Q) \) and \( S' := S'(U; P; V; Q, \delta U, \delta P) \).

4.2 Newton’s equation

We need the representation of the linear operator \( S' \) as the first derivative of \( S \) applied to \((\delta U, \delta P)\). This follows from the Taylor expansion (1.6) of \( S \) with omitting the sufficient small nonlinear terms \( \mathcal{O}( ||\delta P||^2 ) \), \( \mathcal{O}( ||\delta U||^2 ) \) and \( \mathcal{O}( ||U|| \cdot ||\delta P||) \).

Find \( \frac{\partial S}{\partial P} \) such that (4.31) holds.

\[
S_0 \left( \frac{\partial S}{\partial P} \right) := \begin{pmatrix} A & -B \end{pmatrix} \left( \frac{\partial S}{\partial P} \right) = \begin{pmatrix} B & -C \end{pmatrix} \tag{4.31}
\]

At first glance this system (4.31) is indefinite, however it can be solved with the conjugate gradient method by Bramble and Pasciak. The idea is to use the matrix \( S_{0,0}^{-1} \) in (4.32) as a preconditioner on the system and to define a new suitable duality product (4.33).

\[
S_{0,0}^{-1} := \begin{pmatrix} A_0^{-1} & 0 \\
0 & C_0^{-1} \end{pmatrix} \tag{4.32}
\]

\[
\langle x, y \rangle := \langle x, y \rangle + \langle x, y - y \rangle \tag{4.33}
\]

Here we choose \( A_0 \) to be a preconditioner of \( A \) (e.g. with BPX) and \( C_0 \) to be a preconditioner of the Schur complement \( K := B^T A_0^{-1} B + \gamma C \) (e.g. by taking the diagonal of \( C \)). The positive parameters \( \delta \) and \( \gamma \) shall guarantee \( A - \gamma A_0 \succ 0 \) and decrease the condition number of the given system (see next subsection). To the resulting system we apply the usual conjugate gradient method. But we use a matrix free method, i.e. the stiffness matrix is never assembled completely but used element wise. According to that the occurring matrix-vector products are determined on an element wise basis.

4.5 Optimal parameter

The described conjugate gradient method by Bramble and Pasciak is especially good if the “right” parameters \( \gamma \) and \( \delta \) are used.

To determine \( \gamma \) we use a rather simple method. We consider the value of \( \langle w^T A w - \gamma w^T A_0 w, w \rangle \). If it is negative we reduce \( \gamma \) by a fixed factor until the condition is fulfilled again.

To determine \( \delta \) we can use the eigenvalues of the matrices (see [11], section 4.2). First we consider the diagonal of \( (S_{0,0}^{-1} \cdot S_0) \) that is denoted by \( M \).

\[
M = \begin{pmatrix} A_0^{-1} & 0 \\
0 & C_0^{-1} \end{pmatrix} \tag{4.34}
\]

The corresponding condition number fulfills (4.36).

\[
\kappa(S_{0,0}^{-1} \cdot S_0) \leq c_{\Delta}(\gamma, A, A_0) \kappa(M) \tag{4.35}
\]

\[
c_{\Delta} \max \left\{ \lambda_{\text{max}}(A_0^{-1} \cdot A), \delta \lambda_{\text{max}}(C_0^{-1} \cdot K) \right\} \leq \min \left\{ \lambda_{\text{min}}(A_0^{-1} \cdot A), \delta \lambda_{\text{min}}(C_0^{-1} \cdot K) \right\} \tag{4.36}
\]
a more detailed representation.

\[
\mathfrak{M}(U, P) = 4I_3 \left( \frac{\partial^2 \phi_D}{\partial t^2} + P \frac{\partial^2 \phi_V}{\partial t^2} \right) \mathbf{C}^{-1} - 1
\]

\[
+ 4 \left( \frac{\partial \phi_D}{\partial t} + P \frac{\partial \phi_V}{\partial t} \right) \left( \alpha I_2 - \mathbf{C} \mathbf{I} - \alpha I + \mathbf{C} \right)
\]

\[
+ 4I_3 \left( \frac{\partial^2 \phi_D}{\partial t \partial I_3} + 3 \frac{\partial \phi_D}{\partial I_3} \right) \mathbf{C}^{-1} - 1 \mathbf{I}
\]

\[
\mathfrak{M}(U, P) = -4I_3 \left( \frac{\partial \phi_D}{\partial t} + P \frac{\partial \phi_V}{\partial t} \right) \hat{\mathbf{C}} + \mathbf{C}^{-1} \mathbf{C}^{-1} - 1 \mathbf{I}
\]

(4.27)

\[
\mathfrak{M}(U, P) = 4I_3 \left( \frac{\partial \phi_D}{\partial t} + P \frac{\partial \phi_V}{\partial t} \right) \mathbf{C}^{-1} - 1 \mathbf{I}
\]

(4.28)

### 4.4 Mixed finite element method

The given linear saddle-point problem (4.18) shall be solved by means of a mixed FE method. Hence we assume that there is a regular triangulation \( \mathcal{T}_h \) of the domain \( \Omega \) available with \( n_T \) hexahedral elements \( T_i \) and \( n_N \) nodal points. We choose the stable Taylor-Hood element which implies the ansatz of an element wise triquadratic displacement and an element wise trilinear pressure. The computations can be made with the help of the reference element \( \mathcal{T} = [-1,1]^3 \) and the associated transformation map (4.29).

\[
B_i : \mathcal{T} \rightarrow T_i, \quad \hat{X} \mapsto X
\]

(4.29)

As suitable function spaces we take (4.30).

\[
V_a = \left\{ \mathbf{V} \in C(\Omega)^3 \cap H^1(\Omega)^3 : \mathbf{V}|_{T \cap \partial B_i} \in Q_2(\tilde{T})^3 \quad \forall \, T_i \in \mathcal{T}_h \right\}
\]

\[
V_{h,0} = \left\{ \mathbf{V} \in V_a : \mathbf{V}|_{T \cap \partial} = 0 \right\}
\]

\[
V_{h,D} = \left\{ \mathbf{V} \in V_a : \mathbf{V}|_{T \cap \partial} = \mathbf{U}_h \right\}
\]

\[
Q_h = \left\{ Q \in C(\Omega) \cap L^3(\Omega) : Q|_{T \cap \partial B_i} \in Q_2(\tilde{T}) \quad \forall \, T_i \in \mathcal{T}_h \right\}
\]

(4.30)

We choose suitable triquadratic and trilinear nodal ansatz functions \( \Phi^{(i)} \) and \( \Psi^{(i)} \) resp. to represent the functions of \( V_{h,a} \) and \( Q_h \) as a linear combination of these ansatz functions with certain coefficients \( \mathfrak{A}^{(i)} \) and \( \mathfrak{B}^{(i)} \). That leads to a discrete saddle point problem which can be rewritten in matrix-vector form.

as well as the reordering of the remaining terms w.r.t. \( S' \) and \( S \).

\[
S(U + \delta U, P + \delta P; t, \mathbf{V}, Q) = \left( a_D(U + \delta U; V) + a_V(U + \delta U, P + \delta P; V) - t f_0(V) \right)
\]

(4.6)

\[
= \left( S(U, P; t, V, Q) + S'(U, P, V, Q, \delta U, \delta P) \right)
\]

(4.7)

Therefor we consider the Taylor expansions of the integral forms (3.46) - (3.49) and of the integrands \( \mathfrak{E}(U), \mathfrak{E}(U; V), \mathfrak{T}_D(U), \mathfrak{S}(U) \) and \( \phi_V(U) \). Altogether this yields (4.8) and (4.9).

\[
a_D(U + \delta U; V) + a_V(U + \delta U, P + \delta P; V) - t f_0(V)
\]

\[
= a_D(U; V) + a_V(U, P; V) - t f_0(V)
\]

\[
+ a_V(U, \delta P; V)
\]

\[
+ \left< \mathfrak{E}(U; V), \left( P \cdot 2 \frac{\partial \mathcal{S}_h(C)}{\partial C} + 2 \frac{\partial \mathcal{T}_D(C)}{\partial C} \right) \mathfrak{E}(U; \delta U) \right>_{0,\Omega}
\]

\[
+ \left< \text{Grad } V, (P \cdot \mathcal{S}_h(U) + \mathcal{T}_D(U)) \cdot \text{Grad } \delta U \right>_{0,\Omega}
\]

\[
+ \mathcal{O}\left( \|\delta U\|^2 \right) + \mathcal{O}\left( \|\delta P\|^2 \right)
\]

(4.8)

\[
b_h(U + \delta U; Q) - c(\delta P, Q)
\]

\[
= b_h(U; Q) - c(\delta P, Q)
\]

\[
+ \left< Q, \mathcal{S}_h(U), \mathfrak{E}(U; \delta U) \right>_{0,\Omega} - c(\delta P, Q) + \mathcal{O}\left( \|\delta U\|^2 \right)
\]

(4.9)

Using the material tensor

\[
\mathfrak{M}(U, P) := 2 \frac{\partial \mathcal{T}_D(C)}{\partial C} + P \cdot 2 \frac{\partial \mathcal{S}_h(C)}{\partial C}
\]

(4.10)
we introduce a new notation.
\[
a(U, P; \delta U, V) = \left\langle E(U; V), \mathcal{M}(U, P) : E(U; \delta U) \right\rangle_{\Omega} = \left\langle \text{Grad } V, T(U, P) \right\rangle_{\Omega}
\]
\[
0, \Omega
\]
\[
(4.11)
\]
Finally by omitting the nonlinear terms in (4.8) and (4.9) resp. the operators \( S' \) and \( S \) are built.
\[
S'(U, P; V, Q, \delta U, \delta P) = \left\langle a(U, P; \delta U, V) + b(U; \delta P), V \right\rangle, \quad (4.16)
\]
\[
- S(U, P; V, Q) = \left\langle f(t, U, P; V), g(U, P, Q) \right\rangle. \quad (4.17)
\]
This formulation together with the linear \textit{Newton's equation} leads to the linear \textit{saddle-point problem} in each \textit{Newton's step}.

Let \((t, U, P)\) be a given tripel in \((0, 1] \times \Omega \times Q\). Find the solution pair \((\delta U, \delta P) \in \Omega \times Q\) that fulfills the system (4.18)
\[
a(U, P; \delta U, V) + b(U; \delta P, V) = f(t, U, P; V) \quad \text{for all test functions } (V, Q) \in \Omega \times Q.
\]

\[\text{Reformulating both parts of the material tensor from (4.20)}\]
\[
\mathcal{M}_c = 4 \sum_{i=1}^{3} \left( \frac{\partial^2 \phi_{i}}{\partial a_i \partial a_j} - \frac{\partial \phi_i}{\partial a_k} \right) + 4 \left( \frac{\partial \phi_i}{\partial a_k} \right) \mathcal{C} \quad (4.25)
\]
yields (4.26).
\[
\mathcal{M}(U, P) = \left\langle \frac{\partial \phi D}{\partial a_i}, \mathcal{C} \right\rangle \quad (4.26)
\]

\text{By means of the material function (3.27) that is under consideration we can find}