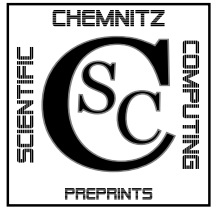
TECHNISCHE UNIVERSITÄT CHEMNITZ

Peter Benner, Mohammad-Sahadet Hossain, Tatjana Stykel

Low-rank iterative methods of periodic projected Lyapunov equations and their application in model reduction of periodic descriptor systems

CSC/11-01



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Abstract

We discuss the numerical solution of large-scale sparse projected discrete-time periodic Lyapunov equations in lifted form which arise in model reduction of periodic descriptor systems. We extend the alternating direction implicit method and the Smith method to such equations. Low-rank versions of these methods are also presented, which can be used to compute low-rank approximations to the solutions of projected periodic Lyapunov equations in lifted form with low-rank right-hand side. Moreover, we consider an application of the Lyapunov solvers to balanced truncation model reduction of periodic discrete-time descriptor systems. Numerical results are given to illustrate the efficiency and accuracy of the proposed methods.

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7 Conclusions

In this paper, we have considered the numerical computation of periodic reachability and observability Gramians as well as Hankel singular values for periodic discrete-time descriptor systems. We have suggested iterative low-rank algorithms based on the ADI and Smith iterations for computing the low-rank factors of the Gramians. These factors are then used in a balanced truncation model reduction approach to find a reduced-order model for the periodic discrete-time descriptor system. The proposed model reduction method delivers a reduced-order model that preserves the regularity and stability properties of the original system. A computable global error bound for the approximate system is also available.

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1 Introduction

We consider linear periodic discrete-time descriptor systems with time-varying dimensions in the form

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \qquad k \in \mathbb{Z}, \tag{1}$$

where $E_k \in \mathbb{R}^{\mu_{k+1} \times n_{k+1}}$, $A_k \in \mathbb{R}^{\mu_{k+1} \times n_k}$, $B_k \in \mathbb{R}^{\mu_{k+1} \times m_k}$, $C_k \in \mathbb{R}^{p_k \times n_k}$ are periodic with a period $K \geq 1$, $\sum_{k=0}^{K-1} \mu_k = \sum_{k=0}^{K-1} n_k = n$, $\sum_{k=0}^{K-1} m_k = m$ and $\sum_{k=0}^{K-1} p_k = p$. If all E_k are nonsingular, then (1) can be transformed into a periodic standard system. Such systems have received a lot of attention in the last 30 years, e.g., in satellite attitude control, helicopter design, harmonic balance methods in nonlinear circuit design, and many more, see, e.g., [6, 10, 17, 18, 30, 33]. Some control-theoretic concepts including controllability and observability, Gramians, Hankel singular values and balanced realization have been extended to periodic discrete-time descriptor systems with singular E_k, see [8, 9, 24]. Also efficient numerical methods for computing poles and zeros, $\mathbb{L}_\infty$-norm, minimal and balanced realizations have been developed for such systems [9, 33, 34, 35]. Since all these methods are based on the reduction of the periodic pairs $\{(E_k, A_k)\}_{k=0}^{K}$ to a periodic Kronecker-like form using the algorithm of [32], they are restricted to problems of small or medium size.

In this paper, we present iterative methods for solving large-scale sparse projected discrete-time periodic Lyapunov equations which arise in balanced truncation model reduction of periodic descriptor systems [3]. Recently, increasing attention has been devoted to the numerical solution of large-scale sparse Lyapunov equations using the alternating directions implicit (ADI) method [14, 20], the Smith method [11, 20, 23], and Krylov subspace methods [12, 22]. For an overview and further references, see [4]. All these methods have also been generalized to projected Lyapunov equations [27, 28]. On the other hand, an extension of the Smith method and the Krylov subspace method based on a block Arnoldi algorithm to standard periodic Lyapunov equations has been presented in [13]. Unfortunately, these methods cannot be directly applied to the projected periodic Lyapunov equations. Here, we consider the projected periodic Lyapunov equations in lifted form and propose a generalization of the ADI iteration and the Smith method for solving such equations.

The paper outline is as follows. In Section 2, we briefly review discrete-time periodic descriptor systems and their cyclic lifted representations. We also study the periodic reachability and observability Gramians from [9] using the lifted representation. In Sections 3 and 4, we discuss the numerical solution of the causal and noncausal lifted Lyapunov equations using the ADI method and the Smith method, respectively. Low-rank version of these methods are also presented that can be used to compute low-rank approximations to the solutions of

projected periodic Lyapunov equations in lifted form with low-rank right-hand side. A balanced truncation model reduction method for periodic descriptor systems is considered in Section 5. Section 6 contains a numerical example that illustrates the properties of the described iterative methods for lifted projected Lyapunov equations and its application to model reduction. Some conclusions are given in Section 7.

2 Periodic descriptor systems

Lifted representations of periodic discrete-time systems play an important role in extending many theoretical results and numerical algorithms for linear time-invariant systems to the periodic setting [6, 10, 33]. We consider here the cyclic lifted representation which was introduced first for standard periodic systems in [18].

2.1 Cyclic lifted representation of periodic systems

The *cyclic lifted representation* of the periodic descriptor system (1) is given by

$$\mathcal{E}\mathcal{X}_{k+1} = \mathcal{A}\mathcal{X}_k + \mathcal{B}\mathcal{U}_k, \quad \mathcal{Y}_k = \mathcal{C}\mathcal{X}_k, \tag{2}$$

where

$$\mathcal{E} = \operatorname{diag}(E_0, E_1, \ldots, E_{K-1}), \quad \mathcal{B} = \operatorname{diag}(B_0, B_1, \ldots, B_{K-1}),$$

$$\mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & \cdots & 0 & C_0 \\ C_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & C_{K-1} & 0 \end{bmatrix}. \tag{3}$$

The descriptor vector, system input and output of (2) are related to those of (1) via

$$\mathcal{X}_k = [x_1^T, \ldots, x_{K-1}^T, x_0^T]^T, \quad \mathcal{U}_k = [u_0^T, u_1^T, \ldots, u_{K-1}^T]^T, \quad \mathcal{Y}_k = [y_0^T, y_1^T, \ldots, y_{K-1}^T]^T,$$

respectively. The set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is said to be *regular* if $\det(M(\alpha, \beta)) \not\equiv 0$, where

$$M(\alpha, \beta) := \begin{bmatrix} \alpha_0 E_0 & 0 & \ldots & 0 & -\beta_0 A_0 \\ -\beta_1 A_1 & \alpha_1 E_1 & & & 0 \\ & \ddots & \ddots & & \\ & & & & \\ 0 & & 0 & -\beta_{K-1} A_{K-1} & \alpha_{K-1} E_{K-1} \end{bmatrix}$$

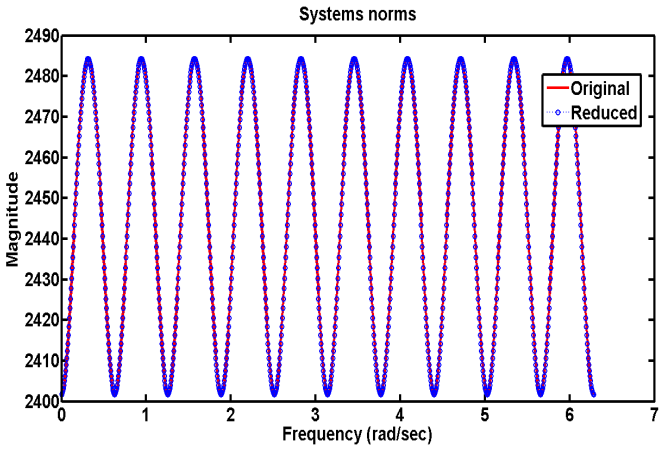


Figure 5: The frequency responses of the original and the reduced-order lifted systems.

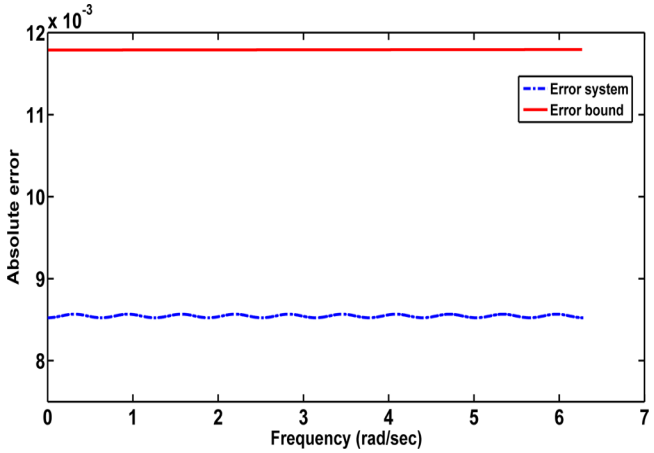
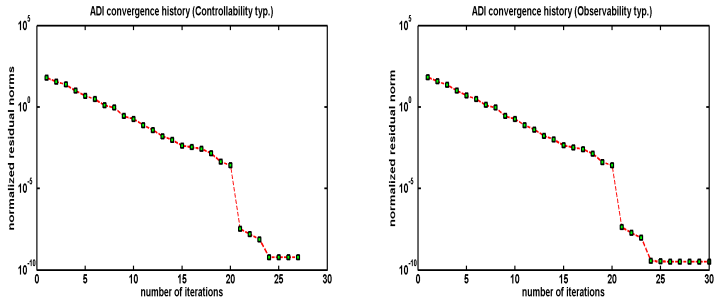


Figure 6: Absolute error and error bound.



(a) (b)

Figure 3: (a) Normalized residual norms for the causal reachability projected lifted Lyapunov equation; (b) normalized residual norms for the causal observability projected lifted Lyapunov equation.

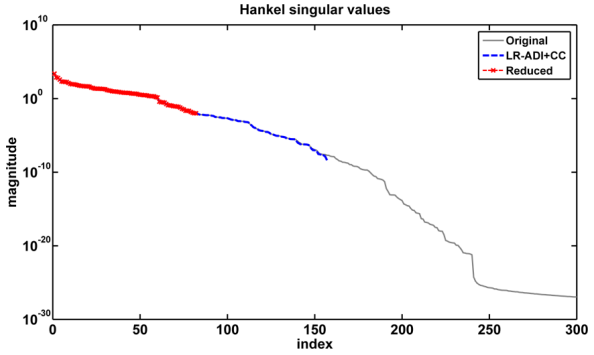


Figure 4: Causal Hankel singular values for original, computed (with column compression, CC) and reduced-order lifted systems.

with $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{K-1})$, $\beta = (\beta_0, \beta_1, \ldots, \beta_{K-1})$, and α_k, β_k are complex variables for $k = 0, 1, \ldots, K-1$.

Definition 2.1. *Let $\{(E_k, A_k)\}_{k=0}^{K-1}$ be a regular set of periodic matrix pairs. If there exist $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{K-1})$ and $\beta = (\beta_0, \beta_1, \ldots, \beta_{K-1})$ with $\alpha_k, \beta_k \in \mathbb{C}$, $k = 0, 1, \ldots, K-1$, such that $\det(M(\alpha, \beta)) = 0$, then the pair $(\pi_\alpha, \pi_\beta) = \left(\prod_{k=0}^{K-1} \alpha_k, \prod_{k=0}^{K-1} \beta_k\right) \neq (0, 0)$, is called an* eigenvalue pair *of $\{(E_k, A_k)\}_{k=0}^{K-1}$.*

Note that if $\pi_\beta \neq 0$, then $\lambda = \pi_\alpha / \pi_\beta$ is a finite eigenvalue, otherwise, $(\pi_\alpha, 0)$ represents an infinite eigenvalue of $\{(E_k, A_k)\}_{k=0}^{K-1}$. The set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is said to be *periodic stable* if it is regular and all its finite eigenvalues lie inside the unit circle. System (1) is *asymptotically stable* if the corresponding set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable.

2.2 Decomposition of periodic descriptor systems

If the descriptor matrices E_k in (1) are invertible for $k = 0, 1, \ldots, K-1$, then (1) can be transformed to the standard periodic system by multiplication from the left with E_k^{-1}. However, due to numerical instability, the explicit inversion of E_k is not advisable. Moreover, if one or more matrices E_k are singular, the transformation to the standard system is not possible any more. In this case, we can proceed by separating the causal and the noncausal parts of the periodic descriptor system. The canonical structures of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ can be used to find such a decomposition [9, 32].

Lemma 1. [9] *Suppose that the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ in system (1) is regular. For $k = 0, 1, \ldots, K-1$, there exist nonsingular matrices S_k and T_k such that*

$$S_k E_k T_{k+1} = \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & E_k^b \end{bmatrix}, \quad S_k A_k T_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I_{n_k^\infty} \end{bmatrix}, \tag{4}$$

where $T_K = T_0$, $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f = J_k$ is an $n_k^f \times n_k^f$ Jordan matrix corresponding to the finite eigenvalues, $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b = N_k$ is an $n_k^\infty \times n_k^\infty$ nilpotent matrix corresponding to the eigenvalues at infinity, $n_k = n_k^f + n_k^\infty$, and $\mu_{k+1} = n_{k+1}^f + n_k^\infty$.

Note that if ν_k is the nilpotency of the matrix N_k for $k = 0, 1, \ldots, K-1$, then $(\nu_0, \nu_1, \ldots, \nu_{K-1})$ are called the indices of a regular set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$. The index ν of the periodic descriptor system (1) is then defined as $\nu = \max(\nu_0, \nu_1, \ldots, \nu_{K-1})$.

For every $k \in \mathbb{Z}$, we let

$$x_k = T_k \begin{bmatrix} x_k^f \\ x_k^b \end{bmatrix}, \quad S_k B_k = \begin{bmatrix} B_k^f \\ B_k^b \end{bmatrix}, \quad C_k T_k = \begin{bmatrix} C_k^f & C_k^b \end{bmatrix}.$$

Then using Lemma 1, we can decompose the periodic descriptor system (1) into forward (causal) and backward (noncausal) periodic subsystems

$$\begin{array}{rcll} x_{k+1}^f & = & A_k^f x_k^f + B_k^f u_k, & y_k^f = C_k^f x_k^f, \\ E_k^b x_{k+1}^b & = & x_k^b + B_k^b u_k, & y_k^b = C_k^b x_k^b, \end{array}$$

respectively, with $y_k = y_k^f + y_k^b$. The Gramians and other systems concepts of the periodic descriptor system are now defined for these causal and noncausal subsystems independently.

2.3 Lifted representation of periodic Lyapunov equations

Consider a periodic discrete-time descriptor system (1), where the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable. It has been shown in [26] that the Gramians of discrete-time descriptor systems satisfy the projected generalized discrete-time Lyapunov equations with special right-hand sides. A similar result holds also for periodic descriptor systems, where the projected Lyapunov equations are also periodic.

Theorem 1. [8] *Consider a periodic descriptor system (1) with a peroidic stable set of matrix pairs $\{(E_k, A_k)_{k=0}^{K-1}\}$ as in (4). Let the matrices*

$$P_r(k) = T_k \begin{bmatrix} I_{n_k^f} & 0 \\ 0 & 0 \end{bmatrix} T_k^{-1}, \quad P_l(k) = S_k^{-1} \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & 0 \end{bmatrix} S_k,$$

be the spectral projectors onto the k-th right and left deflating subspaces of $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues, and let $Q_r(k) = I - P_r(k)$ and $Q_l(k) = I - P_l(k)$ be the complementary projectors.

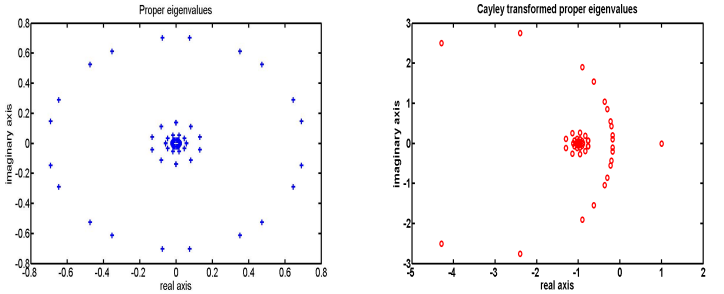
1. *The causal and noncausal reachability Gramians $\{G_k^{cr}\}_{k=0}^{K-1}$ and $\{G_k^{ncr}\}_{k=0}^{K-1}$ are the unique symmetric, positive semidefinite solutions of the projected discrete-time periodic Lyapunov equations*

$$\begin{array}{rcl} A_k G_k^{cr} A_k^T - E_k G_{k+1}^{cr} E_k^T & = & -P_l(k) B_k B_k^T P_l(k)^T, \\ G_k^{cr} & = & P_r(k) G_k^{cr} P_r(k)^T, \end{array} \tag{5}$$

and

$$\begin{array}{rcl} A_k G_k^{ncr} A_k^T - E_k G_{k+1}^{ncr} E_k^T & = & Q_l(k) B_k B_k^T Q_l(k)^T, \\ G_k^{ncr} & = & Q_r(k) G_k^{ncr} Q_r(k)^T, \end{array} \tag{6}$$

respectively, where $k = 0, 1, \ldots, K-1$, $G_K^{cr} = G_0^{cr}$ and $G_K^{ncr} = G_0^{ncr}$.



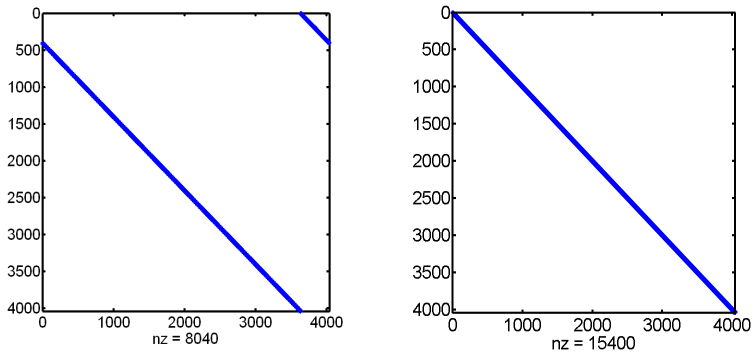
(a) (b)

Figure 2: (a) Finite eigenvalues of the pencil $\lambda\mathcal{E} - \mathcal{A}$; (b) finite eigenvalues of the Cayley-transformed pencil $\lambda\mathbf{E} - \mathbf{A}$.

step for the reachability and observability type of causal lifted Lyapunov equations. The ADI iteration is stopped as soon as the normalized Lyapunov residual exceeds $tol = 10^{-8}$.

The original lifted system (2) has 4000 causal Hankel singular values that are the combination of all the causal Hankel singular values $\sigma_{k,j}$ of the periodic descriptor system (1) given by its different subsystems for $k = 0, 1, \ldots, 9$. In Figure 6, we present the largest 300 causal Hankel singular values computed by solving the PLDALE (7) and its dual observability type PLDALE using the direct method from [3]. One can see the Hankel singular values decay fast, and, hence system (1) can be well approximated by a reduced-order model. In Figure 6, we also present 157 approximate largest causal Hankel singular values computed from the singular value decompositions of the matrices $L_k^T E_{k-1} R_k$ with the low-rank Cholesky factors R_k and L_k of the causal reachability and observability Gramians determined by Algorithm 2.

We approximate system (1) to the tolerance 10^{-2} by truncating the states corresponding to the smallest 75 causal Hankel singular values. For different subsystems, the numbers of the computed non-zero noncausal Hankel singular values are identical and given by $r_k^\infty = 2$ for $k = 0, 1, \ldots, 9$. The computed reduced-order model has subsystems of orders $(9, 10, 10, 11, 10, 9, 10, 11, 11, 11)$. Note that stability is preserved in the reduced-order system. Figure 5 shows the norms of the frequency responses $\mathcal{H}(e^{i\omega})$ and $\tilde{\mathcal{H}}(e^{i\omega})$ of the original and reduced-order lifted systems for a frequency range $[0, 2\pi]$. We observe a good match of the system norms. Finally, in Figure 6, we display the absolute error $\|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2$ and the error bound given in (29). One can see that the error bound is tight in this example.



(a) (b)

Figure 1: (a) Sparsity pattern of $\mathcal{A}$; (b) sparsity pattern of $\mathcal{E}$.

with $r_k = r_k^f + r_k^\infty$. Let $\mathcal{H}(z) = \mathcal{C}(z\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$ and $\tilde{\mathcal{H}}(z)$ be the transfer functions of the lifted system (2) and the reduced-order lifted system formed from the reduced-order subsystems in (28). Then we have the following $\mathbb{H}_\infty$-norm error bound

$$\|\mathcal{H} - \tilde{\mathcal{H}}\|_{\mathbb{H}_\infty} = \sup_{\omega\in[0,2\pi]} \|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2 \leq 2\,\mathrm{trace}\,(\mathrm{diag}(\Sigma_{0,2}, \ldots, \Sigma_{K-1,2})), \quad (29)$$

where $\|.\|_2$ denotes the spectral matrix norm and $\Sigma_{k,2}$, $k = 0, 1, \ldots, K-1$, contains the truncated causal Hankel singular values. This error bound can be obtained similarly to the standard state space case [31].

6 Numerical results

We consider an artificial periodic discrete-time descriptor system with $\mu_k = n_k = 404$, $m_k = 2$, $p_k = 3$, and period $K = 10$. The periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are periodic stable with $n_k^f = 400$ and $n_k^\infty = 4$ for every $k = 0, 1, \ldots, 9$. The original lifted system has order $n = 4040$. The sparsity patterns of $\mathcal{E}$ and $\mathcal{A}$ of the corresponding lifted system are plotted in Figure 1.

The eigenspectrum of the lifted system and the corresponding Cayley-transformed system are shown in Figure 2. We solve the causal projected lifted Lyapunov equations using low-rank ADI iteration as in Algorithm 2. The shift parameters are computed using the heuristic process discussed in Section 3.1. Figures 3(a) and 3(b) show the decay of the normalized residual norms at each ADI iteration

2. *The causal and noncausal observability Gramians* $\{G_k^{co}\}_{k=0}^{K-1}$ *and* $\{G_k^{nco}\}_{k=0}^{K-1}$ *are the unique symmetric, positive semidefinite solutions of the projected discrete-time periodic Lyapunov equations*

$$\begin{aligned} A_k^T G_{k+1}^{co} A_k - E_{k-1}^T G_k^{co} E_{k-1} &= -P_r(k)^T C_k^T C_k P_r(k), \\ G_k^{co} &= P_l(k-1)^T G_k^{co} P_l(k-1), \end{aligned}$$

and

$$\begin{aligned} A_k^T G_{k+1}^{nco} A_k - E_{k-1}^T G_k^{nco} E_{k-1} &= Q_r(k)^T C_k^T C_k Q_r(k) \\ G_k^{nco} &= Q_l(k-1)^T G_k^{nco} Q_l(k-1), \end{aligned}$$

respectively, where $k = 0, 1, \ldots, K-1$, $G_K^{co} = G_0^{co}$, $G_K^{nco} = G_0^{nco}$, $E_{-1} = E_{K-1}$ *and* $P_l(-1) = P_l(K-1)$, $Q_l(-1) = Q_l(K-1)$.

For $E_k = I$, $k = 0, 1, \ldots, K-1$, the Gramians of the periodic standard system can also be determined by solving the periodic standard Lyapunov equations in the lifted form. The solutions of these lifted equations are block diagonal matrices with the required Gramians on the diagonal [13, 29]. This result can also be extended to periodic descriptor systems.

Theorem 2. [3] *Consider a periodic discrete-time descriptor system (1) and its lifted representation (2), where the set of periodic matrix pairs* $\{(E_k, A_k)\}_{k=0}^{K-1}$ *is periodic stable. The causal and noncausal reachability Gramians* $\mathcal{G}^{cr}$ *and* $\mathcal{G}^{ncr}$ *satisfy the projected lifted discrete-time algebraic Lyapunov equations (PLDALEs)*

$$\mathcal{A}\mathcal{G}^{cr}\mathcal{A}^T - \mathcal{E}\mathcal{G}^{cr}\mathcal{E}^T = -\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T, \qquad \mathcal{G}^{cr} = \mathcal{P}_r\mathcal{G}^{cr}\mathcal{P}_r^T, \quad (7)$$

$$\mathcal{A}\mathcal{G}^{ncr}\mathcal{A}^T - \mathcal{E}\mathcal{G}^{ncr}\mathcal{E}^T = \mathcal{Q}_l\mathcal{B}\mathcal{B}^T\mathcal{Q}_l^T, \qquad \mathcal{G}^{ncr} = \mathcal{Q}_r\mathcal{G}^{ncr}\mathcal{Q}_r^T, \quad (8)$$

respectively, where $\mathcal{E}$, $\mathcal{A}$ *and* $\mathcal{B}$ *are as in (3), and*

$$\mathcal{G}^{cr} = \mathrm{diag}(G_1^{cr}, \ldots, G_{K-1}^{cr}, G_0^{cr}), \quad \mathcal{G}^{ncr} = \mathrm{diag}(G_1^{ncr}, \ldots, G_{K-1}^{ncr}, G_0^{ncr}),$$

$$\mathcal{P}_l = \mathrm{diag}(P_l(0), P_l(1), \ldots, P_l(K-1)), \qquad \mathcal{Q}_l = I - \mathcal{P}_l,$$

$$\mathcal{P}_r = \mathrm{diag}(P_r(1), \ldots, P_r(K-1), P_r(0)), \qquad \mathcal{Q}_r = I - \mathcal{P}_r.$$

A similar result can also be stated for the causal and noncausal observability Gramians [3]. In the following section, we develop iterative methods for the PLDALEs (7) and (8) with large and sparse matrix coefficients.

3 ADI method for causal lifted Lyapunov equations

The basic version of the ADI method has been proposed for linear systems in [19] and then used in [14, 15, 20, 27] to solve (projected) continuous-time Lyapunov equations. The ADI method has also be applied for solving discrete-time Lyapunov equations [2, 7].

Consider the PLDALE (7), where all the finite eigenvalues of the pencil $\lambda\mathcal{E}-\mathcal{A}$ lie inside the unit circle. If $\mathcal{E}$ is nonsingular, then the PLDALE (7) can be transformed to the standard discrete-time Lyapunov equation by the multiplication from the left and right with $\mathcal{E}^{-1}$ and $(\mathcal{E}^{-1})^T$. This equation can be solved by the ADI method [2, 7] or the Smith method [11, 20, 23]. However, for the descriptor system (1), the matrix $\mathcal{E}=\mathrm{diag}(E_0,\ldots,E_{K-1})$ is singular. If the pencil $\lambda\mathcal{E}-\mathcal{A}$ has no zero eigenvalues, then $\mathcal{A}$ is nonsingular and (7) can be transformed to the standard discrete-time Lyapunov equation by the multiplication from the left and right with $\mathcal{A}^{-1}$ and $(\mathcal{A}^{-1})^T$. In this case, however, both the ADI and Smith iterations fail to converge for the resulting Lyapunov equation. This problem can be circumvented by considering a *generalized Cayley transformation* given by

$$\mathfrak{C}(\mathcal{E},\mathcal{A})=\lambda(\mathcal{A}-\mathcal{E})-(\mathcal{A}+\mathcal{E}) \tag{9}$$

see, e.g., [16]. This transformation transfers the PLDALE (7) to an equivalent projected continuous-time algebraic Lyapunov equation (PCALE)

$$\mathbf{E}\mathcal{G}^{cr}\mathbf{A}^T+\mathbf{A}\mathcal{G}^{cr}\mathbf{E}^T=-2\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T,\quad \mathcal{G}^{cr}=\mathcal{P}_r\mathcal{G}^{cr}\mathcal{P}_r^T, \tag{10}$$

where $\lambda\mathbf{E}-\mathbf{A}=\lambda(\mathcal{A}-\mathcal{E})-(\mathcal{A}+\mathcal{E})$ is the Cayley-transformed pencil. The finite eigenvalues of $\lambda\mathcal{E}-\mathcal{A}$ lying inside the unit circle are mapped to the finite eigenvalues of $\lambda\mathbf{E}-\mathbf{A}$ in the open left half-plane, and the eigenvalue of $\lambda\mathcal{E}-\mathcal{A}$ at infinity is mapped to $\lambda=1$. Moreover, $\mathcal{P}_l$ and $\mathcal{P}_r$ are the spectral projectors onto the left and right deflating subspaces of $\lambda\mathbf{E}-\mathbf{A}$ corresponding to the finite eigenvalues with negative real part.

If $\lambda\mathcal{E}-\mathcal{A}$ is periodic stable, the matrices $\mathbf{E}$ and $\mathbf{A}$ are both nonsingular. Using the relation $\mathcal{P}_l\,\mathbf{E}=\mathbf{E}\,\mathcal{P}_r$, we can transform the PCALE (10) to the projected standard Lyapunov equation

$$(\mathbf{E}^{-1}\mathbf{A})\mathcal{G}^{cr}+\mathcal{G}^{cr}(\mathbf{E}^{-1}\mathbf{A})^T=-2\mathcal{P}_r\mathbf{E}^{-1}\mathcal{B}(\mathbf{E}^{-1}\mathcal{B})^T\mathcal{P}_r^T,\quad \mathcal{G}^{cr}=\mathcal{P}_r\mathcal{G}^{cr}\mathcal{P}_r^T. \tag{11}$$

In this case, an approximate solution of (10) can be computed by the ADI method applied to (11). The ADI iteration for (11) is given by

$$\begin{aligned}\mathcal{G}_i^{cr}&=(\mathbf{E}^{-1}\mathbf{A}+\tau_i I)^{-1}(\mathbf{E}^{-1}\mathbf{A}-\bar\tau_i I)\mathcal{G}_{i-1}^{cr}(\mathbf{E}^{-1}\mathbf{A}-\tau_i I)^T(\mathbf{E}^{-1}\mathbf{A}+\bar\tau_i I)^{-T}\\&\quad -4Re(\tau_i)(\mathbf{E}^{-1}\mathbf{A}+\tau_i I)^{-1}\mathcal{P}_r\mathbf{E}^{-1}\mathcal{B}(\mathbf{E}^{-1}\mathcal{B})^T\mathcal{P}_r^T(\mathbf{E}^{-1}\mathbf{A}+\bar\tau_i I)^{-T},\end{aligned} \tag{12}$$

with an initial matrix $\mathcal{G}_0^{cr}=0$ and the shift parameters $\tau_1,\tau_2,\ldots,\tau_i\in\mathbb{C}^-$. It follows from

$$\begin{aligned}\mathcal{P}_r(\mathbf{E}^{-1}\mathbf{A}-\bar\tau_i I)&=(\mathbf{E}^{-1}\mathbf{A}-\bar\tau_i I)\mathcal{P}_r,\\ \mathcal{P}_r(\mathbf{E}^{-1}\mathbf{A}+\tau_i I)^{-1}&=(\mathbf{E}^{-1}\mathbf{A}+\tau_i I)^{-1}\mathcal{P}_r,\end{aligned}$$

that $\mathcal{G}_i^{cr}=\mathcal{P}_r\mathcal{G}_i^{cr}\mathcal{P}_r^T$, i.e., the second equation in (10) is satisfied. Iteration (12) can be written as

$$\begin{aligned}\mathcal{G}_i^{cr}&=(\mathbf{A}+\tau_i\mathbf{E})^{-1}(\mathbf{A}-\bar\tau_i\mathbf{E})\mathcal{G}_{i-1}^{cr}(\mathbf{A}-\tau_i\mathbf{E})^T(\mathbf{A}+\bar\tau_i\mathbf{E})^{-T}\\&\quad -4Re(\tau_i)(\mathbf{A}+\tau_i\mathbf{E})^{-1}\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T(\mathbf{A}+\bar\tau_i\mathbf{E})^{-T}.\end{aligned} \tag{13}$$

Similarly to [27], we can establish the convergence of the ADI iteration (13).

Then the *causal* and *noncausal Hankel singular values* of the periodic descriptor system (1) are defined as

$$\begin{aligned}\sigma_{k,j}&=\sqrt{\lambda_j(G_k^{cr}E_{k-1}^TG_k^{co}E_{k-1})}=\zeta_j(L_k^TE_{k-1}R_k),\\ \theta_{k,j}&=\sqrt{\lambda_j(G_k^{ncr}A_k^TG_{k+1}^{nco}A_k)}=\zeta_j(\tilde Y_{k+1}A_k\tilde X_k),\end{aligned}$$

respectively, where $\lambda_j(.)$ and $\zeta_j(.)$ denote the eigenvalues and singular values of the corresponding product matrices.

For a balanced system, truncation of states related to the small causal Hankel singular values does not change system properties essentially. Unfortunately, we can not do the same for the noncausal Hankel singular values. If we truncate the states that correspond to the small non-zero noncausal Hankel singular values, then the pencil for the reduced-order system may get finite eigenvalues outside the unit circle that will lead to additional errors in the approximate system.

Let

$$\begin{aligned}L_k^TE_{k-1}R_k&=[U_{k,1},U_{k,2}]\begin{bmatrix}\Sigma_{k,1}&\\&\Sigma_{k,2}\end{bmatrix}[V_{k,1},V_{k,2}]^T,\\ \tilde Y_{k+1}^TA_k\tilde X_k&=U_{k,3}\Theta_kV_{k,3}^T,\end{aligned} \tag{27}$$

be singular value decompositions, where $[U_{k,1},U_{k,2}]$, $[V_{k,1},V_{k,2}]$, $U_{k,3}$ and $V_{k,3}$ are orthogonal,

$$\Sigma_{k,1}=\mathrm{diag}(\sigma_{k,1},\ldots,\sigma_{k,r_k^f}),\quad \Sigma_{k,2}=\mathrm{diag}(\sigma_{k,r_{k+1}^f},\ldots,\sigma_{n_k^f}),$$

with $\sigma_{k,1}\geq\cdots\geq\sigma_{k,r_k^f}>\sigma_{k,r_{k+1}^f}\geq\ldots\geq\sigma_{k,n_k^f}>0$, and $\Theta_k=\mathrm{diag}(\theta_{k,1},\ldots,\theta_{k,r_k^\infty})$ is nonsingular for $k=0,1,\ldots,K-1$. Note that similar to continuous-time descriptor systems [25], the number of non-zero noncausal Hankel singular values of (1) can be estimated as

$$r_k^\infty=\mathrm{rank}(\tilde Y_{k+1}^TA_k\tilde X_k)\leq\min(\nu_km_k,\,\nu_kp_k,\,n_k^\infty),$$

where ν_k are the indices of the regular set of periodic matrix pairs $\{(E_k,A_k)\}_{k=0}^{K-1}$. This estimate shows that if the index ν_k times the number m_k of inputs or the number p_k of the outputs in much smaller than the dimension n_k^∞ of its k-th periodic deflated subspace corresponding to the infinite eigenvalues, then the order of system (1) can be reduced significantly.

We compute the reduced-order system (26) as

$$\tilde E_k=S_{k,r}^TE_kT_{k+1,r},\quad \tilde A_k=S_{k,r}^TA_kT_{k,r},\quad \tilde B_k=S_{k,r}^TB_k,\quad \tilde C_k=C_kT_{k,r}, \tag{28}$$

where the projection matrices have the form

$$\begin{aligned}S_{k,r}&=[\,L_{k+1}U_{k+1,1}\Sigma_{k+1,1}^{-1/2},\;\tilde Y_{k+1}U_{k,3}\Theta_k^{-1/2}\,]\in\mathbb{R}^{\mu_{k+1},r_{k+1}},\\ T_{k,r}&=[\,R_kV_{k,1}\Sigma_{k,1}^{-1/2},\quad\tilde X_kV_{k,3}\Theta_k^{-1/2}\,]\in\mathbb{R}^{n_k,r_k},\end{aligned}$$

iteration as in the causal case.

Remark 3. The causal and noncausal observabiliy Gramians of the periodic descriptor system (1) can also be determined from the corresponding PLDALEs that are dual to the PLDALE (7) and (8), see [3] for details. Applying Algorithm 2 and Algorithm 3 to such equations, we find, respectively, the low-rank Cholesky factors L_k of the causal observability Gramians $G_k^{co} \approx L_k L_k^T$ and the Cholesky factor $\tilde{Y}_k$ of the noncausal observability Gramians $G_k^{nco} = \tilde{Y}_k \tilde{Y}_k^T$.

5 Application to model order reduction

Model order reduction (MOR) is an approach, where the dynamical system is approximated by a reduced-order model that is in some measure close to the original system. For the periodic descriptor system (1), a reduced-order model of dimension r has the form

$$\tilde{E}_k \tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k, \quad \tilde{y}_k = \tilde{C}_k \tilde{x}_k, \qquad k \in \mathbb{Z}, \tag{26}$$

where $\tilde{E}_k \in \mathbb{R}^{\gamma_{k+1} \times r_{k+1}}$, $\tilde{A}_k \in \mathbb{R}^{\gamma_{k+1} \times r_k}$, $\tilde{B}_k \in \mathbb{R}^{\gamma_{k+1} \times m_k}$, $\tilde{C}_k \in \mathbb{R}^{p_k \times r_k}$ are K-periodic matrices, $\sum_{k=0}^{K-1} \gamma_k = \sum_{k=0}^{K-1} r_k = r$, and $r \ll n$. Apart from having a much smaller state-space dimension, it is also important that the reduced-order model preserves physical properties of the original system such as regularity and stability, and that the approximation error is small.

In this section, we present a generalization of a balanced truncation model reduction method to periodic descriptor systems. Balanced truncation for periodic standard discrete-time system has been considered in [10, 31]. An extension of such important concepts as balanced realization and Hankel singular values to periodic descriptor systems has been presented in [9].

Definition 5.1. *A realization (E_k, A_k, B_k, C_k) of a periodic descriptor system (1) is called* balanced *if*

$$G_k^{cr} = G_k^{co} = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix}, \quad G_k^{ncr} = G_{k+1}^{nco} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta_k \end{bmatrix},$$

where $\Sigma_k = \operatorname{diag}(\sigma_{k,1}, \ldots, \sigma_{k,n_k^f})$ and $\Theta_k = \operatorname{diag}(\theta_{k,1}, \ldots, \theta_{k,n_k^\infty})$, $k = 0, 1, \ldots, K-1$.

Assume that the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable. Consider the Cholesky factorizations of the causal and noncausal Gramians

$$G_k^{cr} = R_k R_k^T, \qquad G_k^{co} = L_k L_k^T,$$
$$G_k^{ncr} = \tilde{X}_k \tilde{X}_k^T, \qquad G_k^{nco} = \tilde{Y}_k \tilde{Y}_k^T.$$

Proposition 1. *Consider a periodic descriptor system (1), where the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable. If $\tau_1, \tau_2, \ldots, \tau_i \in \mathbb{C}^-$, then the ADI iteration (13) converges to the solution $\mathcal{G}^{cr}$ of the projected PCALE (10).*

Proof. Let $\mathcal{G}^{cr}$ be the solution of the projected PCALE (10). After the i-th iteration, the error $\mathcal{G}_i^{cr} - \mathcal{G}^{cr}$ can be computed from (13) recursively as

$$\mathcal{G}_i^{cr} - \mathcal{G}^{cr} = \mathbf{R}_i \mathcal{G}^{cr} \mathbf{R}_i^*, \tag{14}$$

with

$$\mathbf{R}_i = \mathcal{P}_r (\mathbf{A} + \tau_i \mathbf{E})^{-1} (\mathbf{A} - \bar{\tau}_i \mathbf{E}) \cdots (\mathbf{A} + \tau_1 \mathbf{E})^{-1} (\mathbf{A} - \bar{\tau}_1 \mathbf{E}). \tag{15}$$

Since the periodic descriptor system (1) is asymptotically stable, the Cayley transformed pencil $\lambda \mathbf{E} - \mathbf{A}$ has only the eigenvalues with negative real part and also the eigenvalue $\lambda = 1$. Then it can be transformed into the Weierstrass canonical form

$$\mathbf{E} = \mathbf{W} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \mathbf{Z}, \quad \mathbf{A} = \mathbf{W} \begin{bmatrix} J & 0 \\ 0 & J_1 \end{bmatrix} \mathbf{Z}, \tag{16}$$

where the matrices $\mathbf{W}$ and $\mathbf{Z}$ are nonsingular, J has eigenvalues with negative real part and J_1 has the eigenvalue $\lambda = 1$ only. In this case, the projectors $\mathcal{P}_l$ and $\mathcal{P}_r$ can be represented as

$$\mathcal{P}_l = \mathbf{W} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathbf{W}^{-1}, \quad \mathcal{P}_r = \mathbf{Z}^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Z}. \tag{17}$$

Substituting (16) and (17) in (15), we obtain that

$$\mathbf{R}_i = \mathbf{Z}^{-1} \begin{bmatrix} \mathbf{J}_i & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Z}, \tag{18}$$

with

$$\mathbf{J}_i = (J + \tau_i I)^{-1} (J - \bar{\tau}_i I) \cdot \ldots \cdot (J + \tau_1 I)^{-1} (J - \bar{\tau}_1 I).$$

Since all eigenvalues of J have negative real part, then all eigenvalues of $(J + \tau_l I)^{-1}(J - \bar{\tau}_l I)$, $l = 1, \ldots, i$, lie inside the unit circle. Therefore,

$$\lim_{i \to \infty} \mathbf{R}_i = 0,$$

and, hence, the right-hand side of equation (14) tends to zero. In other words, $\mathcal{G}_i^{cr}$ converges toward the solution $\mathcal{G}^{cr}$. □

Proposition 2. *Consider the PCALE (10). Assume that the pencil $\lambda \mathbf{E} - \mathbf{A}$ is in Weierstrass canonical form (16), where J is diagonal. Then the i-th iterate $\mathcal{G}_i^{cr}$ of the ADI method (13) satisfies the estimate*

$$\|\mathcal{G}_i^{cr} - \mathcal{G}^{cr}\|_2 \le \rho^2(\mathbf{R}_i) \|\mathbf{Z}\|_2^2 \|\mathbf{Z}^{-1}\|_2^2 \|\mathcal{G}^{cr}\|_2, \tag{19}$$

where $\rho(\mathbf{R}_i)$ is the spectral radius of the matrix $\mathbf{R}_i$ in (15).

Proof. We have from (14) and (18) that

$$\|\mathcal{G}_i^{cr} - \mathcal{G}^{cr}\|_2 = \|\mathbf{R}_i \mathcal{G}^{cr} \mathbf{R}_i^*\|_2 \leq \rho^2(\mathbf{R}_i)\ \|\mathbf{Z}\|_2^2\ \|\mathbf{Z}^{-1}\|_2^2\ \|\mathcal{G}^{cr}\|_2.$$

Thus, estimate (19) is established. □

Note that the solutions of the PLDALE (7) and the PCALE (10) are identical and have the block diagonal structure. Hence, an approximation to the solution of (10) is also an approximate solution of the PLDALE (7).

3.1 Computing shift parameters

The convergence rate of the ADI iteration (13) is determined by the spectral radius of the matrix $\mathbf{R}_i$ and depends strongly on the choice of ADI parameters. The minimization of this spectral radius with respect to the shift parameters $\tau_1, \tau_2, \ldots, \tau_i \in \mathbb{C}^-$ leads to the generalized minimax problem

$$\{\tau_1, \tau_2, \ldots, \tau_i\} = \underset{\tau_1,\tau_2,\ldots,\tau_i \in \mathbb{C}^-}{\arg\min} \quad \max_{t \in \mathrm{Sp}_s(\mathbf{E},\mathbf{A})} \quad \prod_{j=1}^{i} \frac{|(t-\bar{\tau}_j)|}{|(t+\tau_j)|}, \tag{20}$$

where $\mathrm{Sp}_s(\mathbf{E}, \mathbf{A})$ denotes the set of stable eigenvalues of the pencil $\lambda\mathbf{E} - \mathbf{A}$, i.e., the finite eigenvalues with negative real part. The optimal shift parameters are, in general, difficult to compute, especially, for large-scale problems, since the spectrum of the pencil $\lambda\mathbf{E} - \mathbf{A}$ is unknown. An alternative approach for computing the suboptimal parameters has been proposed in [20] for standard problem with $\mathbf{E} = I$. It is based on replacing the eigenspectrum by a set of largest and smallest in modulus Ritz values that approximate the eigenvalues of $\mathbf{A}$. The Ritz values can be computed by an Arnoldi process applied to the matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$. This approach can also be extended to the generalized problem (20).

Since the pencil $\lambda\mathbf{E} - \mathbf{A}$ has finite eigenvalues with negative real part (stable eigenvalues) and also an eigenvalue $\lambda = 1$, an equivalent expression for (20) can be written as

$$\{\tau_1, \tau_2, \ldots, \tau_i\} = \underset{\tau_1,\tau_2,\ldots,\tau_i \in \mathbb{C}^-}{\arg\min} \quad \max_{t \in \mathrm{Sp}(\mathbf{E}^{-1}\mathbf{A})\setminus\{1\}} \quad \prod_{j=1}^{i} \frac{|(t-\bar{\tau}_j)|}{|(t+\tau_j)|}, \tag{21}$$

where $\mathrm{Sp}(\mathbf{E}^{-1}\mathbf{A})$ denotes the spectrum of the matrix $\mathbf{E}^{-1}\mathbf{A}$. Thus, the suboptimal ADI shift parameters $\tau_1, \tau_2, \ldots, \tau_i$ can be computed by the heuristic procedure [20] from the set of largest approximate stable eigenvalues of $\mathbf{E}^{-1}\mathbf{A}$ and $\mathbf{A}^{-1}\mathbf{E}$.

Therefore, the Cholesky factor X of the solution $\mathcal{G}^{ncr} = XX^T$ of (24) and also of the PLDALE (8) takes the form

$$X = [\,\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B},\ \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B},\ \ldots,\ (\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\,]. \tag{25}$$

The computation of this factor is presented in Algorithm 3.

Algorithm 3: Generalized Smith method for the noncausal PLDALE.

INPUT : $\mathcal{A}, \mathcal{E}, \mathcal{B}$, spectral projector $\mathcal{Q}_r$.

OUTPUT: A low-rank Cholesky factor X_ν such that $\mathcal{G}^{ncr} = X_\nu X_\nu^T$.

```
W_1 = Q_r A^{-1} B
X_1 = W_1
for i = 2, 3, ..., ν do
    W_i = A^{-1} E W_{i-1}
    X_i = [X_{i-1}, W_i]
end
```

We note that if the index ν is unknown, then Algorithm 3 can be stopped as soon as $\|W_i\|_F \leq tol$ or $\|W_i\|_F / \|X_i\|_F \leq tol$ a user-defined tolerance tol. In practice, most of the systems we handle are index-1 or index-2 problem. For an index-1 problem, the algorithm only needs to compute the first block column of X in (25), and for an index-2 problem, it computes only the first two block columns of X. In that sense, the solution can be obtained with few computations.

Remark 2. In order to guarantee that the second equation in (24) and also in (8) is satisfied in finite precision arithmetic, we have to project W_i onto the image of $\mathcal{Q}_r$ by pre-multiplication with $\mathcal{Q}_r$.

Note that the generalized Smith iteration does not preserve the block diagonal structure at every iteration step in Algorithm 3, but after ν iteration we obtain $\mathcal{G}_\nu^{ncr} = \mathcal{G}^{ncr}$ and $\mathcal{G}^{ncr} = X_\nu X_\nu^T$ has block diagonal structure analogous to the solution of the PLDALE (8) given by $\mathcal{G}^{ncr} = \mathrm{diag}(G_1^{ncr}, \ldots, G_{K-1}^{ncr}, G_0^{ncr})$, where G_k^{ncr} are the periodic solutions of the PPDALE (6) for $k = 0, 1, \ldots, K-1$. Since $\mathcal{G}^{ncr} = X_\nu X_\nu^T$ has block diagonal structure, we can partition the Cholesky factor X_ν as

$$X_\nu = [\,\tilde{X}_1^T,\ \ldots,\ \tilde{X}_{K-1}^T,\ \tilde{X}_0^T\,]^T$$

with $\tilde{X}_k \in \mathbb{R}^{n_k, \nu m}$. Then the noncausal reachability Gramians of system (1) can be computed in factored form $G_k^{ncr} = \tilde{X}_k \tilde{X}_k^T$.

In the case of singular $\mathcal{A}$, we can again use the generalized Cayley transformation (9) and compute the Cholesky factor of the PLDALE (8) via the ADI

precision, is sufficient to achieve an error of the machine precision magnitude for the solution $\mathcal{G}^{cr}$ of the PLDALE (7). We summarize the resulting ADI iteration in Algorithm 2.

Algorithm 2: Low-rank ADI iteration for the causal PLDALE.

INPUT : $\mathcal{A}, \mathcal{E}, \mathcal{B}, \mathcal{P}_l$ and shift parameters $\tau_1, \tau_2, \ldots, \tau_i$.

OUTPUT: A low-rank Cholesky factor Z_i such that $\mathcal{G}^{cr} \approx Z_i Z_i^T$.

$V_1 = 2\sqrt{-Re(\tau_1)}\big((1+\tau_1)\mathcal{A} + (1-\tau_1)\mathcal{E}\big)^{-1}\mathcal{P}_l\mathcal{B}, \quad Z_1 = V_1$

for $i = 2, 3, \ldots$ **do**

  $V_i = \sqrt{\dfrac{Re(\tau_i)}{Re(\tau_{i-1})}}\Big(V_{i-1} - (\tau_i + \bar{\tau}_{i-1})\big((1+\tau_i)\mathcal{A} + (1-\tau_i)\mathcal{E}\big)^{-1}(\mathcal{A}-\mathcal{E})V_{i-1}\Big)$

  $Z_i = [\, Z_{i-1},\ V_i\,]$

  $[\mathcal{R}_i, \Pi_i, r_n] = \mathrm{RRQR}(Z_i^T, \sqrt{\epsilon})$

  Update $Z_i = \Pi_i \mathcal{R}_i^T [\, I_{r_n},\ 0\,]^T$

end

4 Smith method for noncausal lifted Lyapunov equations

Consider now the PLDALE (8). For nonsingular $\mathcal{A}$, this equation is equivalent to the PLDALE

$$\mathcal{G}^{ncr} - (\mathcal{A}^{-1}\mathcal{E})\mathcal{G}^{ncr}(\mathcal{A}^{-1}\mathcal{E})^T = \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T, \quad \mathcal{G}^{ncr} = \mathcal{Q}_r\mathcal{G}^{ncr}\mathcal{Q}_r^T. \tag{24}$$

Such an equation can be solved by the Smith method [23] given by

$$\begin{aligned} \mathcal{G}_1^{ncr} &= \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T, \\ \mathcal{G}_i^{ncr} &= \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T + (\mathcal{A}^{-1}\mathcal{E})\mathcal{G}_{i-1}^{ncr}(\mathcal{A}^{-1}\mathcal{E})^T. \end{aligned}$$

Note that $\mathcal{Q}_r$ is the spectral projector onto the invariant subspace of the matrix $\mathcal{A}^{-1}\mathcal{E}$ corresponding to the zero eigenvalues. Then $\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{E} = \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r$ is nilpotent with the nilpotency index ν, where ν is the index of the periodic descriptor system (1). In this case, after ν iteration we obtain

$$\mathcal{G}_\nu^{ncr} = \sum_{k=0}^{\nu-1}(\mathcal{A}^{-1}\mathcal{E})^k\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T((\mathcal{A}^{-1}\mathcal{E})^T)^k = \mathcal{G}^{ncr}.$$

3.2 Low-rank ADI method

Recently, an efficient modification of the ADI method has been proposed to compute an low-rank approximation to the solution of standard Lyapunov equations with large-scale matrix coefficients [14, 20]. It is known as the *low-rank alternating direction implicit* (LR-ADI) method. This method was extended to projected Lyapunov equations in [27].

As in the standard case [1, 21], it can be observed that the eigenvalues of the symmetric solutions of projected Lyapunov equations with low-rank right-hand sides often decay very rapidly. Such solutions can be well approximated by low-rank matrices. That means that we can find a matrix Z with a small number of columns such that ZZ^T is an approximation to the solution $\mathcal{G}^{cr}$ of PCALE (10). This matrix Z is referred to as a *low-rank Cholesky factor* of the solution of (10).

Since the matrices $\mathcal{G}_i^{cr}$ in the ADI iteration (13) are Hermitian and positive semidefinite, the Cholesky factor Z_i of $\mathcal{G}_i^{cr} = Z_i Z_i^*$ can be represented as

$$Z_i = \big[2\sqrt{-Re(\tau_i)}(\mathbf{A}+\tau_i\mathbf{E})^{-1}\mathcal{P}_l\mathcal{B}, \quad (\mathbf{A}+\tau_i\mathbf{E})^{-1}(\mathbf{A}-\bar{\tau}_i\mathbf{E})Z_{i-1}\big], \tag{22}$$

and $Z_0 = 0$. Introducing the matrices $\Phi_j = (\mathbf{A} - \bar{\tau}_j\mathbf{E})$ and $\Psi_j = (\mathbf{A}+\tau_j\mathbf{E})^{-1}$, we can express the i-th iteration as

$$Z_i = 2[\sqrt{-Re(\tau_i)}\Psi_i\mathcal{P}_l\mathcal{B}, \sqrt{-Re(\tau_{i-1})}\Psi_i(\Phi_i\Psi_{i-1})\mathcal{P}_l\mathcal{B}, \ldots, \sqrt{-Re(\tau_1)}\Psi_i\Phi_i\cdots\Psi_2(\Phi_2\Psi_1)\mathcal{P}_l\mathcal{B}].$$

Since Ψ_j and Φ_j commute, we can reorder these matrices and rewrite Z_i in the form

$$Z_i = [\, G,\ F_{i-1}G,\ F_{i-2}F_{i-1}G,\ \ldots,\ F_1F_2\cdots F_{i-1}G\,],$$

where

$$G = 2\sqrt{-Re(\tau_i)}(\mathbf{A}+\tau_i\mathbf{E})^{-1}\mathcal{P}_l\mathcal{B}$$

and

$$F_j = \sqrt{\frac{Re(\tau_j)}{Re(\tau_{j+1})}}\Psi_j\Phi_{j+1} = \sqrt{\frac{Re(\tau_j)}{Re(\tau_{j+1})}}\Big(I - (\tau_j + \overline{\tau}_{j+1})(\mathbf{A}+\tau_j\mathbf{E})^{-1}\mathbf{E}\Big).$$

Reenumerating the shift parameters in reverse order, we obtain Algorithm 1 for computing the low-rank Cholesky factor of the solution of (10), which is also the solution of the PLDALE (7).

Note that if the complex shift parameters appear in complex conjugate pairs $\{\tau_i, \tau_{i+1} = \overline{\tau}_i\}$, then performing a double step as described in [27] we can keep the low-rank Cholesky factor Z_i to be real.

Algorithm 1: Low-rank ADI iteration for the causal PLDALE.

INPUT : $\mathbf{A}, \mathbf{E}, \mathcal{B}, \mathcal{P}_l$ and shift parameters $\tau_1, \tau_2, \ldots, \tau_i$.

OUTPUT: a low-rank Cholesky factor Z_i such that $\mathcal{G}^{cr} \approx Z_i Z_i^*$.

1 $V_1 = 2\sqrt{-Re(\tau_1)}(\mathbf{A} + \tau_1\mathbf{E})^{-1}\mathcal{P}_l\mathcal{B}$
2 $Z_1 = V_1$
3 **for** $i = 2, 3, \ldots$ **do**
4 $\quad V_i = \sqrt{\frac{Re(\tau_i)}{Re(\tau_{i-1})}}(I - (\tau_i + \bar{\tau}_{i-1})(\mathbf{A} + \tau_i\mathbf{E})^{-1}\mathbf{E})V_{i-1}$
5 $\quad Z_i = [Z_{i-1},\ V_i]$
6 **end**

The ADI iteration can be stopped as soon as the *normalized residual norm* given by

$$\eta(Z_i) = \frac{\|\mathbf{A}Z_iZ_i^T\mathbf{E}^T + \mathbf{E}Z_iZ_i^T\mathbf{A}^T + 2\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F}{\|2\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T\|_F} \tag{23}$$

satisfies the condition $\eta(Z_i) < tol$ with a user-defined tolerance tol or a stagnation of normalized residual norms is observed. If the number of shift parameters is smaller than the number of iterations required to attain a prescribed tolerance, then we reuse these parameters in a cyclic manner.

If $\mathcal{G}_i^{cr} = Z_iZ_i^T$ converges to the solution of (10), then

$$\lim_{i\to\infty} Z_iZ_i^T = \lim_{i\to\infty}(\mathcal{G}_i^{cr} - \mathcal{G}_{i-1}^{cr}) = 0.$$

Therefore, the stopping criterion in Algorithm 1 can also be defined by the condition $\|V_i\|_F \le tol$ or $\|V_i\|_F/\|Z_i\|_F \le tol$ with some tolerance tol.

Note that we do not need to compute the matrices $\mathbf{E}$ and $\mathbf{A}$ explicitly in Algorithm 1. Instead, we can rewrite the iteration for the original matrices $\mathcal{E}$ and $\mathcal{A}$. Using $\mathbf{E} = \mathcal{A} - \mathcal{E}$ and $\mathbf{A} = \mathcal{A} + \mathcal{E}$, we have

$$(\mathbf{A} + \tau_i\mathbf{E})^{-1}\mathbf{E} = \big((1+\tau_i)\mathcal{A} + (1-\tau_i)\mathcal{E}\big)^{-1}(\mathcal{A} - \mathcal{E}).$$

Then Steps 1 and 4 in Algorithm 1 can be reformulated as

$$\begin{aligned} V_1 &= 2\sqrt{-Re(\tau_1)}\big((1+\tau_1)\mathcal{A} + (1-\tau_1)\mathcal{E}\big)^{-1}\mathcal{P}_l\,\mathcal{B}, \\ V_i &= \sqrt{\frac{Re(\tau_i)}{Re(\tau_{i-1})}}\Big(V_{i-1} - (\tau_i + \bar{\tau}_{i-1})\big((1+\tau_i)\mathcal{A} + (1-\tau_i)\mathcal{E}\big)^{-1}(\mathcal{A} - \mathcal{E})V_{i-1}\Big). \end{aligned}$$

The minimax problem (21) for the ADI parameters can be reformulated accordingly.

Remark 1. In exact arithmetic, the matrices Z_i satisfy $Z_i = \mathcal{P}_rZ_i$ and, hence, the second equation in (10) is fulfilled for the low-rank approximation $Z_iZ_i^T$. However, in finite precision arithmetic, a drift-off effect may occur. In this case, we need to project V_i onto the image of $\mathcal{P}_r$ by pre-multiplication with $\mathcal{P}_r$. In order to limit the additional computation cost, we can do this, for example, at every second or third iteration step.

We observe that the ADI iteration does not preserve the block diagonal structure at every iteration step in Algorithm 1. This is due to the specific structure of the matrices $\mathbf{E}$ and $\mathbf{A}$. But we can show that after the successful i-th iteration step, the approximate Gramian $\mathcal{G}_i^{cr} = Z_iZ_i^T$ has almost block diagonal structure analogous to the solution of the PLDALE (7) given by $\mathcal{G}_i^{cr} = \mathrm{diag}(G_1^{cr}, \ldots, G_{K-1}^{cr}, G_0^{cr})$, where G_k^{cr} are the periodic solutions of the PPDALE (5) for $k = 0, 1, \ldots, K-1$. Since $\mathcal{G}_i^{cr}$ is almost block diagonal after the i-th iteration and $\mathcal{G}_i^{cr} = Z_iZ_i^T$, we let

$$Z_i = [R_1^T,\ \ldots,\ R_{K-1}^T,\ R_0^T\,]^T$$

with $R_k \in \mathbb{R}^{n_k\times r_c}$. Then $R_kR_k^T$ is an approximation to the causal reachability Gramian G_k^{cr}, $k = 0, 1, \ldots, K-1$.

3.3 Column compression for the LR-ADI method

For a fast convergence of the ADI iteration, it is very important to choose a set of good shift parameters. Since we are working with suboptimal parameters, the desired convergence in the LR-ADI iteration may not be achieved in fewer iteration steps and, as a result, the number of columns of the approximate Cholesky factor may grow. In order to keep the low-rank structure in Z_i, we truncate those columns that do not carry any additional information in the subsequent iteration steps. This truncation approach saves memory space and lowers the computational cost, because residual computations required in the stopping criteria will also incorporate these redundant columns.

Assume that $Z_i \in \mathbb{R}^{n\times r_c}$ has the numerical rank $\mathrm{rank}(Z_i, \tau) = r_n < r_c$ with a prescribed tolerance τ. Then we compute the rank-revealing QR decomposition (RRQR)

$$Z_i^T = Q_i\mathcal{R}_i\Pi_i^T, \qquad \mathcal{R}_i = \begin{bmatrix} \mathcal{R}_{i,11} & \mathcal{R}_{i,12} \\ 0 & \mathcal{R}_{i,22} \end{bmatrix},$$

where Q_i is orthogonal, Π_i is a permutation matrix, $\mathcal{R}_{i,11} \in \mathbb{R}^{r_n\times r_n}$ is upper triangular and $\|\mathcal{R}_{i,22}\|_F \le \tau$. Setting $\mathcal{R}_{i,22} \approx 0$ and $Z_i^T = [\mathcal{R}_{i,11}\ \mathcal{R}_{i,12}]\Pi_i^T$, we find that $Z_iZ_i^T \approx \mathcal{G}^{cr}$. Note that we do not need to compute Q_i, since this matrix cancels out in the product $Z_iZ_i^T$ due to its orthogonal property. In practice, the rank determination has to be performed on the basis of the truncation tolerance τ in the RRQR. In [5], it is shown that a tolerance $\tau = \sqrt{\epsilon}$, where ϵ is the machine