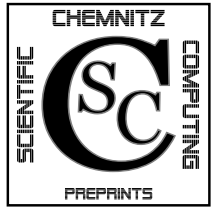
TECHNISCHE UNIVERSITÄT CHEMNITZ

Marcus Meyer Julia Müller

Identification of mechanical strains by measurements of a deformed electrical potential field

CSC/08-06



Chemnitz Scientific Computing Preprints

Chemnitz Scientific Computing Preprints – ISSN 1864-0087

Impressum:

Chemnitz Scientific Computing Preprints — ISSN 1864-0087
(1995–2005: Preprintreihe des Chemnitzer SFB393)

Herausgeber:
Professuren für
Numerische und Angewandte Mathematik
an der Fakultät für Mathematik
der Technischen Universität Chemnitz

Postanschrift:
TU Chemnitz, Fakultät für Mathematik
09107 Chemnitz
Sitz:
Reichenhainer Str. 41, 09126 Chemnitz

`http://www.tu-chemnitz.de/mathematik/csc/`

References

[1] Bertsekas, D. P.: *Constrained Optimization and Lagrange Multiplier Methods.* Athena Scientific (1996).

[2] Braess, D.: *Finite Elemente.* Springer, Berlin (2003).

[3] Engl, H. W.; Hanke, M.; Neubauer, A. : *Regularization of Inverse Problems.* Dordrecht, Kluwer Academic Publishers (2000).

[4] Fenner, G. E.: *Strain Sensitive Apparatus and Elements therefore.* United States Patent Office : 3.251.222 (1966).

[5] Hackbusch, W. : *Theorie und Numerik elliptischer Differentialgleichungen.* Teubner, Stuttgart (1996).

[6] Hein, T.; Meyer, M.: *Simultaneous identification of independent parameters in elliptic equations - numerical studies.* J. Inv. Ill-Posed Problems 16, 417-433 (2008).

[7] Hoffmann, K.: *Eine Einführung in die Technik des Messens mit Dehnungsmessstreifen.* Hottinger Baldwin Messtechnik GmbH (1987).

[8] Hofmann, B.: *Regularization for Applied Inverse and Ill-Posed Problems.* Teubner Verlag, Leipzig (1986).

[9] Kreissig, R.; Benedix, U.: *Höhere technische Mechanik.* Springer, Wien New York (2002).

[10] MathWorks, Inc.: *Partial Differential Equation Toolbox Users's Guide.* Software-Documentation, www.mathworks.com (2006).

[11] Müller, J.: *Identifikation des mechanischen Beanspruchungs- und Deformationszustandes durch experimentelle Analyse der Verzerrung eines eingeprägten elektrischen Potentialfeldes.* TU Chemnitz, Diploma Thesis (2008).

[12] Nocedal, J.; Wright, S.J.: *Numerical Optimization.* Springer, New York (1999).

[13] Schwab, A. J.: *Begriffswelt der Feldtheorie.* Springer, Berlin (2002).

[14] Stockmann, M.: *Mikromechanische Analyse der Wirkungsmechanismen elektrischer Dehnungsmessstreifen.* TU Chemnitz, Habilitation Thesis (2000).

[15] Stockmann, M.; Naumann, J.; Schumann, J.; Mönch, I.: *Differential Strain Gauge: A Sensor Device for Macro- and Microsystems.* In: Strain (in press) (2008).

TECHNISCHE UNIVERSITÄT CHEMNITZ

Chemnitz Scientific Computing Preprints

Marcus Meyer **Julia Müller**

Identification of mechanical strains by measurements of a deformed electrical potential field

CSC/08-06

Abstract

In this paper we discuss the inverse problem of the identification of mechanical stresses by measuring the deformation of an electric potential field in a so called differential strain gauge (D-DMS). We derive a mathematical model, where the forward operator is given in terms of an elliptic boundary value problem. Derivatives of the forward operator are considered and the solution of the inverse problem via a least-squares minimization is introduced. Here, the discretized problem is solved with the Gauss-Newton method. Numerical studies of practical interest are presented.

CSC/08-06 **ISSN 1864-0087** **November 2008**

Contents

1 Introduction **1**

2 Mechanical background **2**
2.1 Geometry of the deformed strain gauge 2
2.2 Basic equations for the electrical potential field 3

3 The forward operator **4**
3.1 The boundary value problem and weak formulation 4
3.2 Derivatives of the forward operator 8

4 Solution of the inverse problem **10**
4.1 Finite elements discretization 10
4.2 The discrete inverse problem 12
4.3 Least-squares minimization 13

5 Numerical studies **14**
5.1 Numerical implementation 15
5.2 Results 16

Author's addresses:

Marcus Meyer
Julia Müller
TU Chemnitz
Fakultät für Mathematik
Reichenhainer Str. 41
D-09107 Chemnitz

`http://www.tu-chemnitz.de/mathematik/inverse_probleme/`
`marcus.meyer@mathematik.tu-chemnitz.de`

No.	Exact strains	$p^\dagger$	δ_{rel}	f_{rel}	K_0	Time in *sec*
1	$\varepsilon_x = 0.01$ $\varepsilon_y = -0.003$ $\gamma = 0.005$	$\begin{pmatrix} 1.013039 \\ 0.005 \end{pmatrix}$	0	0.00734	14	41.92
			0.001	0.03965	20	129.11
			0.01	0.3309	20	142.06
2	$\varepsilon_x = 0.005$ $\varepsilon_y = 0.003$ $\gamma = 0.001$	$\begin{pmatrix} 1.001994 \\ 0.001 \end{pmatrix}$	0	0.0457	14	41.80
			0.0001	0.0654	14	42.17
			0.001	0.2465	20	103.84
3	$\varepsilon_x = 0.05$ $\varepsilon_y = 0.03$ $\gamma = 0.01$	$\begin{pmatrix} 1.01942 \\ 0.01 \end{pmatrix}$	0	0.0047	13	39.44
			0.001	0.0251	20	108.38
			0.01	0.2098	20	113.61
4	$\varepsilon_x = 0.01$ $\varepsilon_y = 0.01$ $\gamma = 0.01$	$\begin{pmatrix} 1 \\ 0.01 \end{pmatrix}$	0	0.0102	18	52.14
			0.001	0.0547	18	52.27
			0.01	0.4571	20	103.75
5	$\varepsilon_x = 0.01$ $\varepsilon_y = -0.003$ $\gamma = 0$	$\begin{pmatrix} 1.013039 \\ 0 \end{pmatrix}$	0	0.0078	14	42.39
			0.001	0.0468	20	129.06
			0.01	0.3558	20	141.78

Table 1: Results for varying deformation states and noise levels

level) of the measurements has to be scaled depending on the dimension of the identified parameters.

Vice versa it is natural, that a given measuring tolerance of the technical equipment limits the attainable accuracy of the identified parameters. Let us consider a realistic setting with deformations $\varepsilon_x = 0$ and $\varepsilon_y = \gamma = 0.001$ and a supplied voltage of $U_B = 0.1V$. Then the variation of the potential field is of order $10^{-5}V$. We find out by numerical tests that in this case the identification error is smaller than 10%, if the accuracy of the measurements is more precisely than $10^{-6}V$ (what here corresponds to a relative error $\delta_{rel} \leq 10^{-5}$, which means in this case, that the absolute measuring error is smaller then 10% of the potential variation). Note, that effective instruments reach an absolute accuracy of $10^{-8}V$ what would be sufficient for the above example. We refer to [11, Section 4] for details on practical questions.

which seems to be a proper choice due to the fact, that the deformations are assumed to be small. Hence, $\gamma \approx 0$ and $\kappa_0 = \frac{1+\varepsilon_{x0}}{1+\varepsilon_{y0}} \approx 1$ is plausible. The stopping criterion for the iteration process is defined as follows. Either the iteration stops at $k = K_0$ if the residual norm is small enough

$$||\underline{F}_m(p_{K_0}) - \underline{y}^{\delta}_{data}||_2 \leq 10^{-6}$$

or if the norm of the search step falls below some tolerance

$$||d_{K_0}||_2 \leq 10^{-6} \, .$$

As a quality criterion for the accuracy of the reached results we define the relative error

$$f_{rel} := \frac{\left|\left|\begin{pmatrix} \kappa^{\delta} - 1 \\ \gamma^{\delta} \end{pmatrix} - \begin{pmatrix} \kappa^{\dagger} - 1 \\ \gamma^{\dagger} \end{pmatrix}\right|\right|_2}{\left|\left|\begin{pmatrix} \kappa^{\dagger} - 1 \\ \gamma^{\dagger} \end{pmatrix}\right|\right|_2} \tag{29}$$

with $\kappa^{\dagger}$, $\gamma^{\dagger}$ denoting the (known) exact parameters and κ^{δ}, γ^{δ} being the identified parameters. Note, that due to the definition of κ this interpretation of the relative error is natural, since f_{rel} measures the error in ε_x and not in $1 + \varepsilon_x$.

5.2 Results

Concluding this paper we present some numerical results listed in table 1. Here, the inverse problem was solved for some variations of κ and γ and the influence of noise in the measured data was studied.

The results show, that the identification of the strain parameters is stable, i.e. the results are quite exact even for noisy data as long as the noise level is not too large. In this context we mention, that due to the small number of unknown parameters the discretization of the problem has a regularizing character. Thus, no additional regularization has to be introduced.

As one can see, the order of magnitude of the noise level δ_{rel} has a varying influence on the results quality depending on the order of magnitude of the identified deformations. See e.g. cases no. 1 and no. 2 in table 1. Here, the identified parameters are of order 10^{-2} and 10^{-3}, and thus the relative measuring error has to be smaller by the factor of 10 in case no. 2, while the solution errors are nearby in the same region. The cases 3, 4, and 5 yield similar results. This behavior can be explained by the fact, that small deformations of the D-DMS cause only a small change in the potential field. Then, with a fixed supplied source voltage, a fixed noise level much more falsifies the results for small deformations than for larger deformations. As a consequence, the precision (i.e. the maximum noise

1 Introduction

For a lot of problems in experimental mechanics it is of interest to measure the deformation of a body, i.e. to identify components of the strain tensor ε. This can be done with strain gauges (DMS), which is a quite common and well known technique. In recent times engineers try to develop improvements for such measuring methods, driven by the quest for a high-level accuracy. In the focus of such considerations there is a measuring principle called D-DMS (differential strain gauge). Here, a sensor consisting of a small piece of foil is fixed at a body and (in undeformed state) an electrical potential field is applied at this sensor. If the underlying body is deformed, the potential field also changes, and from the measured difference of the electric potential the deformation can be recovered. See e.g. [4], [7] and [14] for details on DMS and D-DMS strain gauges.

From the mathematical point of view the indirect measurement of strains via measured potentials of an electric field is modelled as an inverse parameter identification problem. In this context the components of the strain tensor denote the parameters to be determined. In general, such identification problems may be ill-posed, e.g. the unknown parameters must not depend continuously on the measured data, which results in instability of the problem. Then regularization methods have to be applied. See [3] or [8] for a survey on regularization and ill-posed problems. We note, that for the identification problem we consider here, no additional regularization has to be introduced, due to the discrete structure of the problem with only a small number of unknown parameters.

In this paper we focus on the mathematical formulation and numerical solution of the inverse problem. For a more detailed discussion of the application of D-DMS strain gauges and more extensive numerical studies we refer to [11]. While solving the above mentioned identification problem we follow the approaches introduced in [6].

The mechanical background of the inverse problem is considered in section 2. We describe the relation of strains and deformations of the D-DMS sensor. Basic equations for the electric potential field are introduced. Section 3 is devoted to the forward operator, which is defined via a boundary value problem, whose solution is the electric potential $q = q(\varepsilon)$ depending on the strain parameters. Derivatives of the forward operator are introduced. Due to the fact, that the unknown strain parameters are contained in the shape of the (deformed) domain of the boundary value problem, the domain is transformed to the unity square. This results in a modified boundary value problem and standard methods as explained in [6] are applicable. In section 4 the discretization of the problem is considered and a formulation of the inverse problem as a least-squares minimization is introduced. For the numerical solution the Gauss-Newton method is applied. Additionally, in the closing section 5 some results of numerical studies are presented.

2 Mechanical background

2.1 Geometry of the deformed strain gauge

Without loss of generality we describe the undeformed quadratic D-DMS sensor as the unity square in the x-y-plane. Bearing in mind, that the sensor foil is thin, deformations in z-direction can be neglected. Therefore the considered problem is 2-dimensional. In the deformed state we assume, that the shape of the D-DMS looks like a parallelogram as displayed in figure 1. Note, that translational or rotational displacements of the underlying body do not influence the strain gauge. Thus, the deformation can be described by the strain tensor ε, which is introduced in the following sense (see e.g. [9]).

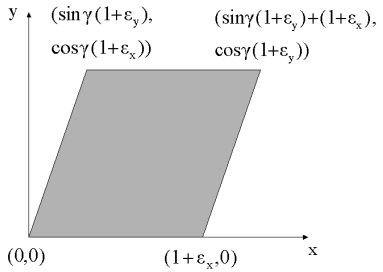


Figure 1: Geometry of the deformed D-DMS strain gauge referring to the strain components $\varepsilon_x, \varepsilon_y$ and γ

Let (x_0, y_0) denote the position vector of a material point in the undeformed state. The position vector of the same point in the deformed state is named (x, y). Then, with the displacement vector $u(x, y) = (u_x(x, y), u_y(x, y))$ the relation

$$(x, y) = (x_0, y_0) + u(x_0, y_0)$$

holds. Assuming small deformations and small deformation gradients

$$\frac{\partial u_i}{\partial j} \ll 1 \; \forall \, i, j = x, y$$

the 2-dimensional symmetric strain tensor is defined as

$$\varepsilon(u) = \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{pmatrix}$$

part of the Dirichlet boundary the potential is $q = 1$ and at the second part the potential is $q = -1$. The difference of two potentials then results in a voltage. In practice, the voltage is supplied with two voltage source points (SP-) and (SP+) (see figure 4) representing small circles. The border of these circles denotes the two parts of Dirichlet boundary mentioned above.

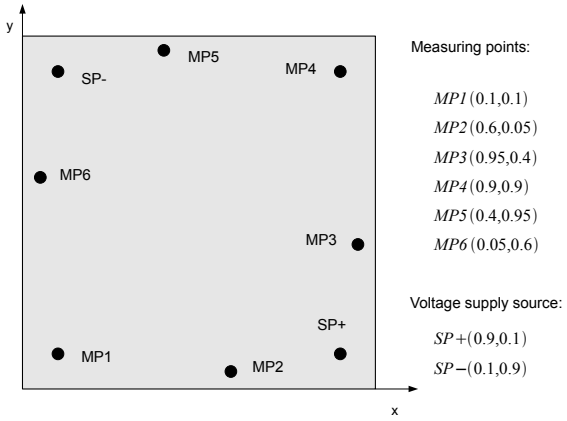


Figure 4: Arrangement of measuring points (MP) and voltage supplying points (SP) at the D-DMS (coordinates relating to the unity square)

For the sake of completeness in figure 4 the arrangement of the data measuring points (MP) is shown. We here use 6 measuring points, thus $\underline{y}_{data} \in \mathbb{R}^6$.

5.1 Numerical implementation

We solve the inverse problem with simulated data, i.e. at first $\underline{y}_{data}$ is computed as the solution of a forward problem with a discretization of $n = 151295$ nodes. Then vice versa the identification problem is solved on a coarser mesh with $n = 37951$ nodes and perturbed data. Therefore, for a given relative noise level $\delta_{rel} \geq 0$ we define the noisy data vector

$$\underline{y}^{\delta}_{data} := \underline{y}_{data} + \delta_{rel} \frac{||\underline{y}_{data}||_2}{||\underline{e}||_2} \underline{e}$$

with a Gaussian random vector $\underline{e} \in \mathbb{R}^6$. The initial value of the Gauss-Newton iteration is set as

$$p_0 = (\kappa_0, \gamma_0)^T = (1, 0)^T \, ,$$

We refer to [12, Chapter 10] for details concerning the Gauss-Newton iteration.

5 Numerical studies

In the last section we present some numerical results. Therefore the discretized identification problem was implemented and solved in MATLAB R2008 including the Partial-Differential-Equation-Toolbox [10]. For a more detailed view on the numerical implementation and an extensive survey on the results we additionally refer to [11].

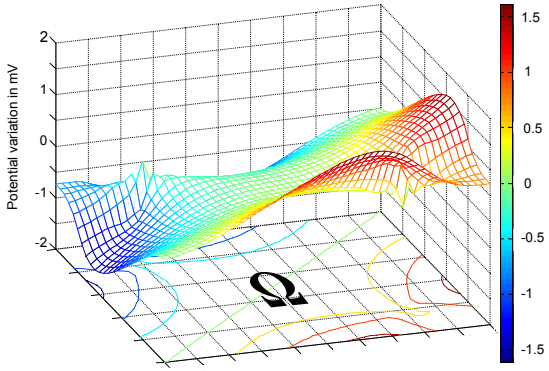


Figure 3: Variation of the potential field in mV for $\varepsilon_x = 0$, $\varepsilon_y = 0$, $\gamma = 0.005$ compared with the potential for undeformed state $\varepsilon_x = \varepsilon_y = \gamma = 0$ (supply voltage of $U_B = 2V$)

A simulation of the potential field at a deformed D-DMS strain gauge compared with the potential at an undeformed D-DMS makes the effect of a potential field deformation visible, which will later be used for the strain identification. In figure 3 the difference of the potential fields for a deformed ($\varepsilon_x = 0$, $\varepsilon_y = 0$, $\gamma = 0.005$) and undeformed ($\varepsilon_x = \varepsilon_y = \gamma = 0$) state is visualized. In this case we find out, that for a supplied voltage of $2V$ the variation of potentials has an order of magnitude of $\pm 1.5mV = \pm 1.5 \cdot 10^{-3}V$.

Note, that the supplied source voltage is identical in both cases. In particular the numerical results involve a source voltage of $2V$, i.e. for the boundary value problem (4) we have two different Dirichlet boundary conditions, where at one

with

$$\varepsilon_x := \frac{\partial u_x}{\partial x}, \quad \varepsilon_y := \frac{\partial u_y}{\partial y}, \quad \text{and} \quad \varepsilon_{xy} = \varepsilon_{yx} := \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right).$$

We set $\gamma := 2\varepsilon_{xy}$, i.e. γ describes the change of angles caused by the deformation. The numbers ε_x and ε_y represent changes of length in x-direction and y-direction.

The above mentioned strain tensor ε is well known from linear theory, holding for small deformations. For our purpose of describing a (small) D-DMS sensor, we are able to suppose, that ε is constant all over the sensor domain. Thus, the strains ε_x, ε_y, and γ are real numbers, which simplifies the problem considerably.

2.2 Basic equations for the electrical potential field

As explained in the introduction, the D-DMS strain gauge is the domain of an electrical potential field. For a detailed view on the mathematical description of electric fields and related Maxwell equations see e.g. [13].

Let $q = q(x, y)$ denote the electric potential. Then for the considered situation of a source free, irrotational, and static field the Laplace equation

$$\operatorname{div} \operatorname{grad} q(x, y) = \Delta q(x, y) = 0 \tag{1}$$

holds. Note, that the difference U_{12} of potentials $q(x_1, y_1)$ and $q(x_2, y_2)$ with $(x_1, y_1) \neq (x_2, y_2)$ denotes a voltage.

The static electric field is induced by a connected voltage, which has to be formulated as a Dirichlet boundary condition. Namely, at the boundary of an electrode we set the potential

$$q = g \quad \text{on} \quad \Gamma_D\,. \tag{2}$$

Then at the boundary of each electrode we have an equipotential line with a known potential $g \in \mathbb{R}$. The potential difference between two electrodes results in the connected voltage.

At the free boundary of the D-DMS sensor the equipotential lines are perpendicular to the boundary, such that the Neumann boundary condition

$$\frac{\partial q}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_N \tag{3}$$

holds. Here, ν defines the outer normal vector of the boundary Γ_N.

3 The forward operator

3.1 The boundary value problem and weak formulation

Let $\hat{\Omega} \subset \mathbb{R}^2$ be a domain with boundary $\partial\hat{\Omega} = \hat{\Gamma}_D \cup \hat{\Gamma}_N$, representing the deformed D-DMS as displayed in figure 1. Analogously, the undeformed unity square is denoted as domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. See figure 2 for an illustration of $\hat{\Omega}$ and Ω.

Resulting from formulae (1), (2), and (3), the potential field $q(x, y)$ in the deformed D-DMS strain gauge is derived as the solution of the elliptic boundary value problem

$$\begin{cases} \Delta q &= 0 \quad \text{in } \hat{\Omega}\,, \\ q &= g \quad \text{on } \hat{\Gamma}_D\,, \\ \frac{\partial q}{\partial \nu} &= 0 \quad \text{on } \hat{\Gamma}_N\,. \end{cases} \tag{4}$$

Remark 3.1 *The solution of (4) corresponds to the solution of the direct problem. Here, the parameters ε_x, ε_y, and γ are given and therewith the shape of the deformed domain $\hat{\Omega}$ is known. Thus, the potential q has to be computed. Vice versa for the inverse problem the potential q is given (on some measuring points) and the parameter $p = (\varepsilon_x, \varepsilon_y, \gamma)^T$ has to be estimated. Since all strains are assumed to be constant, $p \in \mathbb{R}^3$ holds.*

We solve the BVP (4) by introducing a weak formulation. Let

$$\hat{\mathbb{V}}_D := \{q \in H^1(\hat{\Omega}) : u|_{\hat{\Gamma}_D} = g\}$$

be the space of Ansatz functions and

$$\hat{\mathbb{V}}_0 := \{v \in H^1(\hat{\Omega}) : v|_{\hat{\Gamma}_D} = 0\}$$

denote the space of test functions. Here, H^1 is the Sobolov space $H^1(\Omega)$ with standard inner product. See e.g. [5] for details on Sobolev spaces and weak formulation of the Laplace equation.

Multiplying the Laplace equation with a test function $v \in \hat{\mathbb{V}}_0$ and subsequently integrating gives

$$\begin{aligned} \Delta q(x, y) = 0 \quad &\Rightarrow \quad \Delta q(x, y) \cdot v(x, y) = 0 \\ &\Rightarrow \quad \int_{\hat{\Omega}} \Delta q(x, y) \cdot v(x, y) d\hat{\Omega} = 0\,. \end{aligned}$$

With Green's first identity we derive

$$\int_{\hat{\Omega}} \Delta q\, v\, d\hat{\Omega} = -\int_{\hat{\Omega}} \nabla q \cdot \nabla v\, d\hat{\Omega} + \int_{\partial\hat{\Omega}} \frac{\partial q}{\partial \nu}\, v\, ds$$

these measuring points lying in the domain of the strain gauge. For simplicity we assume that the measuring points coincide with nodes of the finite elements mesh.

The discrete operator $\underline{F}_m : \mathcal{D}(\underline{F}_m) \subset \mathbb{R}^2 \to \mathbb{R}^m$ now describes a map of the parameters κ and γ onto the projection of the solution of (23) to the m measuring points. With an additional projection matrix $\underline{Q} \in \mathbb{R}^{m \times n}$, this reads as

$$\underline{F}_m(p) = \underline{Q}\, \underline{q}\,.$$

Eventually, we end up in the following definition.

Definition 4.1 (Discrete identification problem) *For given measuring points $\zeta_1, ..., \zeta_m \in \Omega$ and corresponding measuring data $\underline{y}_{data} := (y_1, ..., y_m)^T \in \mathbb{R}^m$ with $y_i := y(\zeta_i), i = 1, ..., m$, the parameter $p = (\kappa, \gamma)^T \in \mathcal{D}(\underline{F}_m) \subset \mathbb{R}^2$ has to be estimated, such that*

$$\underline{F}_m(p) = \underline{y}_{data} \tag{25}$$

holds.

4.3 Least-squares minimization

The inverse problem or the equation (25), resp., is in the following solved with a least-squares minimization. In particular, we consider the functional

$$J(p) := \frac{1}{2} ||\underline{F}_m(p) - \underline{y}_{data}||^2 \to \min_{p \in \mathcal{D}(\underline{F}_m)}\,. \tag{26}$$

By minimizing $J(p)$, a parameter $p = (\kappa, \gamma)^T$ is found, which fits the given measuring data best.

The minimization (26) denotes a nonlinear optimization problem, which we solve iteratively with a sequence of parameter updates

$$p_{k+1} := p_k + \beta_k\, d_k, \qquad k = 0, 1, \ldots, \tag{27}$$

for a given initial value p_0. The search direction d_k and the step size parameter β_k are chosen, such that the iteration process follows a descent direction, i.e. $J(p_{k+1}) < J(p_k)$.

Various algorithms for the appropriate calculation of d_k and β_k exist. For simplicity we use $\beta_k \equiv 1$ or, alternatively, β_k is decreased with a simple bisection algorithm if necessary.

For the calculation of d_k the Gauss-Newton algorithm is used. Here, for given iterate p_k the search direction is found as a solution of the corresponding normal equation

$$\underline{F}'_m(p_k)^T \underline{F}'_m(p_k) d_k = \underline{F}'_m(p_k)^T (\underline{y}_{data} - \underline{F}_m(p_k))\,. \tag{28}$$

$$\underline{F}(p) := \underline{q}, \qquad p \in \mathcal{D}(\underline{F})$$

with $\underline{q} \in \mathbb{R}^n$ being the solution of (23) for given parameter p. For details on the solution of such constrained minimization problems (23) with multiplier methods we refer to [1]. See [12] for a general survey on numerical optimization.

Derivatives of the discrete operator $\underline{F}$ are derived as solutions of appropriately chosen linear systems. Therefor we introduce an alternative formulation of the stiffness matrix $\underline{K}$. Namely, with the matrices $\underline{K}^{(k)} = (k_{ij}^{(k)}) \in \mathbb{R}^{n\times n}$, $1 \le k \le 3$, which are defined as

$$\begin{aligned} k_{ij}^{(1)} : &= \int_\Omega \frac{\partial\psi_i}{\partial\xi}\frac{\partial\psi_j}{\partial\xi}\, d\Omega\,, \\ k_{ij}^{(2)} : &= \int_\Omega \frac{\partial\psi_i}{\partial\eta}\frac{\partial\psi_j}{\partial\eta}\, d\Omega\,, \\ k_{ij}^{(3)} : &= \int_\Omega \left(\frac{\partial\psi_i}{\partial\xi}\frac{\partial\psi_j}{\partial\eta} + \frac{\partial\psi_i}{\partial\eta}\frac{\partial\psi_j}{\partial\xi}\right) d\Omega, \qquad 1 \le i,j \le n\,, \end{aligned}$$

obviously, the relation

$$\underline{K}(p) = \underline{K}^{(1)} + \kappa^2\underline{K}^{(2)} - \kappa\sin\gamma\underline{K}^{(3)}$$

holds. Thus, for given p_0 the derivatives are calculated corresponding to formula (22) as solutions of

$$\underline{K}(p_0)\underline{z} = \underline{f}_{deriv} \tag{24}$$

with

$$\underline{f}_{deriv} := h_\kappa\left(-2\kappa_0\underline{K}^{(2)}\underline{q}_0 + \sin\gamma_0\underline{K}^{(3)}\underline{q}_0\right) + h_\gamma\kappa_0\cos\gamma_0\,\underline{K}^{(3)}\underline{q}_0\,.$$

Here, the solution $\underline{z} := \underline{F}'(p_0)\underline{h}$ of (24) denotes the discrete directional derivative for a given direction $\underline{h} = (h_\kappa, h_\gamma)^T \in \mathbb{R}^2$.

4.2 The discrete inverse problem

For the solution of the inverse problem we have to respect, that only the information of a finite number of measuring points is available. Let $\zeta_1, ..., \zeta_m \in \Omega$ denote

where the integral on the boundary $\partial\hat\Omega$ vanishes due to

$$\int_{\partial\hat\Omega}\frac{\partial q}{\partial n}\,v\,ds = \underbrace{\int_{\hat\Gamma_D}\frac{\partial q}{\partial\nu}\,v\,ds}_{=0\ (v\in\hat V_0)} + \underbrace{\int_{\hat\Gamma_N}\frac{\partial q}{\partial\nu}\,v\,ds}_{=0\ (\frac{\partial q}{\partial n}=0\ \text{on}\ \hat\Gamma_N)}\,.$$

Thus we have

$$\int_{\hat\Omega}\Delta q\,v\,d\hat\Omega = -\int_{\hat\Omega}\nabla q\cdot\nabla v\,d\hat\Omega = 0\,. \tag{5}$$

In formula (5) the unknown parameter $p = (\varepsilon_x, \varepsilon_y, \gamma)^T$ is implicitly contained in the domain $\hat\Omega$. For the identification problem we prefer a formulation, where p is explicitly visible. This can be enforced by a slight modification as follows.

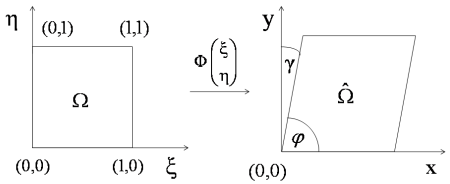


Figure 2: Transformation of unity square Ω to deformed domain $\hat\Omega$ by the use of mapping Φ

The idea is, to transform the deformed domain $\hat\Omega$ to the undeformed unity square Ω (see figure 2). Therefor a transformation mapping $\Phi : \Omega \to \hat\Omega$ is defined via

$$\begin{aligned} \Phi\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} &= \begin{pmatrix}\sin\gamma\cdot\eta(1+\varepsilon_y) + \xi(1+\varepsilon_x)\\ \cos\gamma\cdot\eta(1+\varepsilon_y)\end{pmatrix} \\ &= \begin{pmatrix}1+\varepsilon_x & \sin\gamma(1+\varepsilon_y)\\ 0 & \cos\gamma(1+\varepsilon_y)\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}. \end{aligned} \tag{6}$$

Note, that the x-y-coordinate system is used in the deformed domain $\hat\Omega$ and that the ξ-η-coordinates are applied to the unity square Ω. In the following we need the inverse transformation mapping $\Phi^{-1} : \hat\Omega \to \Omega$, defined as

$$\Phi^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\varepsilon_x} & -\frac{\sin\gamma}{\cos\gamma(1+\varepsilon_x)} \\ 0 & \frac{1}{\cos\gamma(1+\varepsilon_y)} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} . \tag{7}$$

Additionally we calculate

$$\det\,(\Phi^{-1})' = \begin{vmatrix} \frac{1}{1+\varepsilon_x} & -\frac{\sin\gamma}{\cos\gamma(1+\varepsilon_x)} \\ 0 & \frac{1}{\cos\gamma(1+\varepsilon_y)} \end{vmatrix} = \frac{1}{(1+\varepsilon_x)(1+\varepsilon_y)\cos\gamma} ,$$

keeping in mind, that due to the smallness assumption on the deformations and strains

$$\left|\frac{1}{(1+\varepsilon_x)(1+\varepsilon_y)\cos\gamma}\right| = \frac{1}{(1+\varepsilon_x)(1+\varepsilon_y)\cos\gamma} \tag{8}$$

holds. While using Nabla operators we have to respect the transformation mapping, i.e. we must differentiate, wether the Nabla operator refers to the x-y- or the ξ-η-coordinates. According to the notation for Ω and $\hat{\Omega}$ we define

$$\hat{\nabla} q(x,y) = \begin{pmatrix} \frac{\partial q}{\partial x} \\ \frac{\partial q}{\partial y} \end{pmatrix} \quad \text{and} \quad \nabla q(\xi,\eta) = \begin{pmatrix} \frac{\partial q}{\partial \xi} \\ \frac{\partial q}{\partial \eta} \end{pmatrix} .$$

Applying the chain rule we derive

$$\begin{aligned} \begin{pmatrix} \frac{\partial q(x,y)}{\partial x} \\ \frac{\partial q(x,y)}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{\partial q(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial q(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial q(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial q(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \end{pmatrix} \\ \hat{\nabla} q(x,y) &= \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix}\begin{pmatrix} \frac{\partial q(\xi,\eta)}{\partial \xi} \\ \frac{\partial q(\xi,\eta)}{\partial \eta} \end{pmatrix} \\ \Rightarrow \hat{\nabla} q &= (\Phi')^{-T}\nabla q . \end{aligned}$$

Then the scalar product of gradients $\hat{\nabla} q \cdot \hat{\nabla} v$ reads as

$$\begin{aligned} \hat{\nabla} q \cdot \hat{\nabla} v = \quad & (\hat{\nabla} q)^T(\hat{\nabla} v) \\ = \quad & ((\Phi')^{-T}\nabla q)^T((\Phi')^{-T}\nabla v) \\ = \quad & (\nabla q)^T((\Phi')^{-1}(\Phi')^{-T})\nabla v \end{aligned}$$

with the matrix

$$(\Phi')^{-1}(\Phi')^{-T} = \begin{pmatrix} \frac{1}{1+\varepsilon_x} & -\frac{\sin\gamma}{\cos\gamma(1+\varepsilon_x)} \\ 0 & \frac{1}{\cos\gamma(1+\varepsilon_y)} \end{pmatrix}\begin{pmatrix} \frac{1}{1+\varepsilon_x} & 0 \\ -\frac{\sin\gamma}{\cos\gamma(1+\varepsilon_x)} & \frac{1}{\cos\gamma(1+\varepsilon_y)} \end{pmatrix}$$

Let $\mathcal{T}$ define a triangulation of the domain Ω with k elements and n nodes P_i, $1 \le i \le n$. According to this assumption, we choose ansatz functions $\psi_1, \ldots, \psi_n$ with

$$\psi_i(P_j) = \delta_{ij}, \quad 1 \le i, j \le n .$$

Here, δ_{ij} denotes the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \ne j , \\ 1 & \text{for } i = j . \end{cases}$$

Then we search an approximate solution of the differential equation (13) in the FE-space

$$\mathbb{V}^{(n)} := \text{span}\,\{\psi_1, \ldots, \psi_n\} .$$

Analogously to (15), subspaces of $\mathbb{V}^{(n)}$ are defined, whose elements fulfill homogenous ($\mathbb{V}_0^{(n)}$) and inhomogenous ($\mathbb{V}_D^{(n)}$) boundary conditions, resp. Furthermore, the mapping $\Psi : \mathbb{R}^n \to \mathbb{V}_D^{(n)}$ is introduced, such that

$$q^{(n)} := \Psi\underline{q} = \sum_{i=1}^{n} q_i\,\psi_i \approx q, \qquad \underline{q} \in \mathbb{R}^n$$

holds. Our aim is, to find an approximate solution $q^{(n)}$ of the differential equation (13) in the finite dimensional subspace $\mathbb{V}_D^{(n)}$. Thus, $q^{(n)} \in \mathbb{V}_D^{(n)}$ denotes the solution of the equation

$$a(q^{(n)}, v^{(n)}; p) = 0 \quad \forall v^{(n)} \in \mathbb{V}_0^{(n)} .$$

As usual, the vector $\underline{q} := (q_1, ..., q_n)^T \in \mathbb{R}^n$ is interpreted as the discrete solution of the forward problem (14). Finding the discrete solution corresponds to the solution of the minimization problem

$$\underline{q}^T\underline{K}(p)\underline{q} \to \min \qquad \text{for} \quad \underline{q} \in \mathbb{R}^n \;\Leftrightarrow\; q^{(n)} \in \mathbb{V}_D^{(n)} \tag{23}$$

with stiffness matrix $\underline{K}(p) := (k_{ij}) \in \mathbb{R}^{n\times n}$ defined as

$$\begin{aligned} k_{ij} : \quad &= \quad a(\psi_j, \psi_i; \kappa, \gamma) \\ &= \quad \int_\Omega \begin{pmatrix} 1 & -\kappa\sin\gamma \\ -\kappa\sin\gamma & \kappa^2 \end{pmatrix} \nabla\psi_j \cdot \nabla\psi_i \, d\Omega \qquad 1 \le i, j \le n . \end{aligned}$$

Hence, the discrete forward operator $\underline{F} : \mathcal{D}(\underline{F}) \subset \mathbb{R}^2 \to \mathbb{R}^n$ is defined as

Thus, we have to consider the limit

$$\lim_{\tau\to 0} \frac{\kappa_0 \sin\gamma_0 - \kappa_0 \sin(\gamma_0+\tau)}{\tau} ,$$

which is of type '$\frac{0}{0}$'. Applying l'Hospitals rule yields

$$\lim_{\tau\to 0} \frac{(\kappa_0 \sin\gamma_0 - \kappa_0 \sin(\gamma_0+\tau))'}{(\tau)'} = \lim_{\tau\to 0} -\kappa_0 \cos(\gamma_0+\tau)$$

and finally

$$\lim_{\tau\to 0} \frac{\kappa_0 \sin\gamma_0 - \kappa_0 \sin(\gamma_0+\tau)}{\tau} = -\kappa_0 \cos\gamma_0 \tag{18}$$

holds. Equation (18) implies for $\tau \to 0$, that the directional derivative $\omega^{(\gamma)} \in \mathbb{V}_0$ denotes the solution of the variational problem

$$a(\omega^{(\gamma)}, v; p_0) = \kappa_0 \cos\gamma_0 \int_\Omega \left(\frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega \qquad \forall\, v \in \mathbb{V}_0 . \tag{19}$$

Consequently, the gradient of the forward operator is defined as

$$\nabla F(p_0) = \begin{pmatrix} \omega^{(\kappa)} \\ \omega^{(\gamma)} \end{pmatrix} , \tag{20}$$

and arbitrary directional derivatives can be computed with

$$z := F'(p_0) \begin{pmatrix} h_\kappa \\ h_\gamma \end{pmatrix} = \nabla F(p_0) \cdot \begin{pmatrix} h_\kappa \\ h_\gamma \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} h_\kappa \\ h_\gamma \end{pmatrix} \in \mathbb{R}^2 . \tag{21}$$

In other words, z fulfills

$$\begin{aligned} a(z, v; p_0) &= h_\kappa \left(-2\kappa_0 \int_\Omega \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \eta}\, d\Omega + \sin\gamma_0 \int_\Omega \frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \xi}\, d\Omega \right) \\ &\quad + h_\gamma \cdot \kappa_0 \cos\gamma_0 \int_\Omega \left(\frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega \qquad \forall\, v \in \mathbb{V}_0. \end{aligned} \tag{22}$$

4 Solution of the inverse problem

4.1 Finite elements discretization

In the following we introduce a common finite elements discretization of the problem. See e.g. [2] for the implementation of such methods.

$$= \frac{1}{\cos^2\gamma} \begin{pmatrix} \frac{1}{(1+\varepsilon_x)^2} & -\frac{\sin\gamma}{(1+\varepsilon_x)(1+\varepsilon_y)} \\ -\frac{\sin\gamma}{(1+\varepsilon_x)(1+\varepsilon_y)} & \frac{1}{(1+\varepsilon_y)^2} \end{pmatrix} .$$

Exploiting the above results, we now transform the weak formulation (5) from the x-y- to the ξ-η-coordinates. Thus, in the following the modified formula

$$\begin{aligned} \int_{\hat\Omega} \hat\nabla q \cdot \hat\nabla v \, d\hat\Omega &= \int_\Omega (\nabla q)^T ((\Phi')^{-1}(\Phi')^{-T}) \nabla v |\det(\Phi^{-1})'| \, d\Omega \\ &= \frac{1}{(1+\varepsilon_x)(1+\varepsilon_y)\cos\gamma} \frac{1}{\cos^2\gamma} \int_\Omega \left[\frac{1}{(1+\varepsilon_x)^2}\frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \xi} + \right. \\ &\quad \left. + \frac{1}{(1+\varepsilon_y)^2}\frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \eta} - \frac{\sin\gamma}{(1+\varepsilon_x)(1+\varepsilon_y)} \left(\frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \xi} \right) \right] d\Omega \\ &= 0 \end{aligned}$$

is considered. Supposing $p \in \mathbb{R}^3$ being a constant, we simplify the results and get

$$\begin{aligned} 0 &= \int_0^1 \int_0^1 \left[\frac{1}{(1+\varepsilon_x)^2}\frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \xi} + \frac{1}{(1+\varepsilon_y)^2}\frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \eta} - \right. \\ &\quad \left. - \frac{\sin\gamma}{(1+\varepsilon_x)(1+\varepsilon_y)} \left(\frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \xi} \right) \right] d\xi \, d\eta . \end{aligned} \tag{9}$$

By multiplying equation (9) with $(1+\varepsilon_x)^2$ we derive

$$\int_0^1 \int_0^1 \frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \xi} + \frac{(1+\varepsilon_x)^2}{(1+\varepsilon_y)^2}\frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \eta} - \sin\gamma \frac{(1+\varepsilon_x)}{(1+\varepsilon_y)} \left(\frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\xi \, d\eta = 0. \tag{10}$$

From the last equation we realize, that the independent identification of both parameters ε_x and ε_y is not possible. By the considered approach merely the ratio of strains is identifiable, which is defined as the parameter

$$\kappa = \frac{1+\varepsilon_x}{1+\varepsilon_y} . \tag{11}$$

Note, that this observation coincides with experimental results presented in [15].

Applying equations (9) and (11), a bilinear form referring to (4) is introduced as

$$a(q,v;p) := \int_\Omega \frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \xi} + \kappa^2 \frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \eta} - \kappa \sin\gamma \left(\frac{\partial q}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega \qquad (12)$$

with $q \in \mathbb{V}_D$, $v \in \mathbb{V}_0$. Thus the bilinear form $a(.,.;p) : \mathbb{V}_D \times \mathbb{V}_0 \to \mathbb{R}$ depends on the parameter $p = (\kappa, \gamma)^T$ instead of the initially defined $p = (\varepsilon_x, \varepsilon_y, \gamma)^T$. We point out, that the bilinear form (12) refers to the modified boundary value problem

$$\begin{cases} \nabla \left(\begin{pmatrix} 1 & -\kappa \sin\gamma \\ -\kappa \sin\gamma & \kappa^2 \end{pmatrix} \nabla q \right) & = 0 \quad \text{in } \Omega, \\ q & = g \quad \text{on } \Gamma_D, \\ \dfrac{\partial q}{\partial \nu} & = 0 \quad \text{on } \Gamma_N \end{cases} , \qquad (13)$$

where the modified differential equation corresponds to the initial equation in (4), iff $\gamma = 0$ and $\varepsilon_x = \varepsilon_y$. Hence, in the undeformed case as well as for uniform expansion of the D-DMS, no change of the electric potential will come into effect.

Now, a weak solution $q \in H^1(\Omega)$ of the boundary value problem (4) is found as a solution of the variational problem

$$\begin{cases} a(q,v;p) = 0 \qquad \forall\, v \in \mathbb{V}_0 \\ q \in \mathbb{V}_D \end{cases} . \qquad (14)$$

Here, we use the redefined spaces

$$\begin{aligned} \mathbb{V}_D &:= \{q \in H^1(\Omega) : q|_{\Gamma_D} = g\} \\ \mathbb{V}_0 &:= \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\} . \end{aligned} \qquad (15)$$

Eventually, the forward operator $F : \mathcal{D}(F) \subset \mathbb{R}^2 \to \mathbb{V}_D$ of the considered identification problem is defined as

$$F(p) = q$$

with q solving (14).

3.2 Derivatives of the forward operator

In the following section we consider derivatives of the nonlinear forward operator F. Let $q_0 := F(p_0)$ with $p_0 = (\kappa_0, \gamma_0)^T \in \mathcal{D}(F)$ be given. We set $\Delta p := (\tau, 0)^T$ and define $p_\tau := p_0 + \Delta p$ as well as $q_\tau := F(p_\tau)$.

Then the formal definition of a directional derivative is

$$\omega^{(\kappa)} := \lim_{\tau \to 0} \frac{1}{\tau}(F(p_\tau) - F(p_0)).$$

Clearly, $\omega^{(\kappa)} \in \mathbb{V}_0$ and the equation

$$a(q_\tau, v; p_\tau) - a(q_0, v; p_0) = 0$$

holds. The last fact means in detail

$$\begin{aligned} 0 \;=\; & \frac{1}{\tau}\int_\Omega \frac{\partial q_\tau}{\partial \xi}\frac{\partial v}{\partial \xi} + (\kappa_0+\tau)^2 \frac{\partial q_\tau}{\partial \eta}\frac{\partial v}{\partial \eta} - (\kappa_0+\tau)\sin\gamma_0 \left(\frac{\partial q_\tau}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial u_\tau}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega \\ & -\frac{1}{\tau}\int_\Omega \frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \xi} + {\kappa_0}^2 \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \eta} - \kappa_0 \sin\gamma_0 \left(\frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega \\ =\; & \int_\Omega \frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \xi}\frac{\partial v}{\partial \xi} + (\kappa_0+\tau)^2 \frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \eta}\frac{\partial v}{\partial \eta} \\ & -(\kappa_0+\tau)\sin\gamma_0 \left(\frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega \\ & + \int_\Omega (2\kappa_0+\tau)\frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \eta} - \sin\gamma_0 \left(\frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega . \end{aligned} \qquad (16)$$

Obviously, the limit $\tau \to 0$ implies

$$a(\omega^{(\kappa)}, v; p_0) = -2\kappa_0 \int_\Omega \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \eta}\, d\Omega + \sin\gamma_0 \int_\Omega \frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \xi}\, d\Omega \quad \forall\, v \in \mathbb{V}_0 . \qquad (17)$$

Analogously we derive for $p_\tau := p_0 + \Delta p$, $\Delta p := (0, \tau)^T$ and

$$\omega^{(\gamma)} := \lim_{\tau \to 0} \frac{1}{\tau}(F(p_\tau) - F(p_0))$$

the equation

$$\begin{aligned} 0 \;=\; & \int_\Omega \frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \xi}\frac{\partial v}{\partial \xi} + {\kappa_0}^2 \frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \eta}\frac{\partial v}{\partial \eta} \\ & -\kappa_0 \sin(\gamma_0+\tau) \left(\frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial \left(\frac{q_\tau - q_0}{\tau}\right)}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega \\ & + \int_\Omega \frac{\kappa_0 \sin\gamma_0 - \kappa_0 \sin(\gamma_0+\tau)}{\tau} \left(\frac{\partial q_0}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial q_0}{\partial \eta}\frac{\partial v}{\partial \xi} \right) d\Omega . \end{aligned}$$