TRIGONOMETRIC WIDTHS OF CLASSES OF PERIODIC FUNCTIONS OF MANY VARIABLES

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We obtain exact-order estimates for the trigonometric widths of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables in the space L_q for some relationships between the parameters p and q.

Introduction

In the present paper, we study the trigonometric widths of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables in the space L_q for $1 \le p, q, \theta \le \infty$.

To pose the problem, we introduce necessary notation and the definitions of the classes $B^{\Omega}_{p,\theta}$ and the investigated approximative characteristic.

Let $\mathbb{R}^d$, $d \ge 1$, be an d-dimensional Euclidean space with elements $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$, $(x, y) = x_1 y_1 + \ldots + x_d y_d$, and let $L_p(\pi_d)$ be a space of functions $f(x) = f(x_1, \ldots, x_d)$ 2π-periodic in each variable and summable to the power p, $1 \le p < \infty$ (resp., essentially bounded for $p = \infty$), in the cube

$$\pi_d = \prod_{j=1}^{d} [-\pi; \pi].$$

The norm in this space is defined as follows:

$$\|f\|_{L_p(\pi_d)} = \|f\|_p = \left((2\pi)^{-d} \int\limits_{\pi_d} |f(x)|^p \, dx \right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|f\|_{L_\infty(\pi_d)} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \pi_d} |f(x)|.$$

For $f \in L_p(\pi_d)$ and $h \in \mathbb{R}^d$, we set

$$\Delta_h f(x) = f(x+h) - f(x)$$

and define the multiple difference of order $l \in \mathbb{N}$ for a function $f(x)$ at the point $x = (x_1, \ldots, x_d)$ with steps h by the formula

$$\Delta_h^l f(x) = \Delta_h \Delta_h^{l-1} f(x), \qquad \Delta_h^0 f(x) = f(x).$$

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This difference can also be represented in the form

$$\Delta_h^l f(x) = \sum_{n=0}^{l} (-1)^{l+n} C_l^n f(x+nh).$$

Denote the modulus of continuity of the function $f \in L_p(\pi_d)$ of order $l \in \mathbb{N}$ by the formula

$$\Omega_l(f;t)_p = \sup_{|h|\le t} \|\Delta_h^l f(x)\|_p,$$

where $|h|$ is the Euclidean norm of h.

Let $\Omega(t)$ be a function of the type of modulus of continuity of order l defined on $\mathbb{R}_+ = \{t, t \ge 0\}$ and satisfying the following conditions:

(i) $\Omega(0) = 0$, $\Omega(t) > 0$ for $t > 0$;

(ii) $\Omega(t)$ is continuous;

(iii) $\Omega(t)$ increases;

(iv) for all $n \in \mathbb{Z}_+$, $\Omega(nt) \le Cn^l \Omega(t)$, where $l \in \mathbb{N}$ and $C \ge 0$ is a constant independent of n and t.

By Ψ_l we denote the set of these functions Ω. Note that if $f \in L_p(\pi_d)$, then $\Omega_l(f;t)_p \in \Psi_l$.

We write

(i) $\Omega \in S^\alpha$ if $\Omega(\tau)/\tau^\alpha$ almost increases for some $\alpha > 0$, i.e., there exists a constant $C_1 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\Omega(\tau_1)}{\tau_1^\alpha} \le C_1 \frac{\Omega(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \le \tau_2 \le 1;$$

(ii) $\Omega \in S_l$, $l > 0$ if $\Omega(\tau)/\tau^\gamma$ almost decreases for some $0 < \gamma < l$, i.e., there exists a constant $C_2 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\Omega(\tau_1)}{\tau_1^\gamma} \ge C_2 \frac{\Omega(\tau_2)}{\tau_2^\gamma}, \quad 0 < \tau_1 \le \tau_2 \le 1.$$

The conditions under which the function Ω belongs to the sets S^α and S_l are called the Bari–Stechkin conditions [1].

We also set $\Phi_{\alpha,l} = \Psi_l \cap S^\alpha \cap S_l$.

For the sake of clarity, we present an example of the function $\Omega \in \Phi_{\alpha,l}$:

$$\Omega(t) = \begin{cases} t^r \left(\log^+ \frac{1}{t}\right)^b, & t > 0, \\ 0, & t = 0, \end{cases}$$

where $\log^+ t = \max\{1, \log t\}$, $0 < r < l$, and b is a fixed real number.

We now directly pass to the definition of the spaces $B_{p,\theta}^\Omega$ [2, 3].

Let $1 \le p, \theta \le \infty$ and let $\Omega \in \Phi_{\alpha,l}$. Assume that $f \in B_{p,\theta}^\Omega$ if f satisfies the following conditions:

(i) $f \in L_p(\pi_d)$;

(ii) $\|f\|_{b^{\Omega}_{p,\theta}} < \infty$, where $\|f\|_{b^{\Omega}_{p,\theta}}$ is given by the formula

$$\|f\|_{b^{\Omega}_{p,\theta}} = \begin{cases} \left(\int\limits_0^{+\infty} \left(\dfrac{\Omega(f;t)_p}{\Omega(t)} \right)^{\theta} \dfrac{dt}{t} \right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_{t>0} \dfrac{\Omega_l(f,t)_p}{\Omega(t)}, & \theta = \infty. \end{cases}$$

The space $B^{\Omega}_{p,\theta}$ is a linear normed space with the norm

$$\|f\|_{B^{\Omega}_{p,\theta}} = \|f\|_p + \|f\|_{b^{\Omega}_{p,\theta}}.$$

If $\Omega(t) = t^r$, then the spaces $B^{\Omega}_{p,\theta}$ coincide with the Besov spaces $B^r_{p,\theta}$ [4]. In particular, for $\theta = \infty$, we get $B^r_{p,\infty} = H^r_p$, where H^r_p are the spaces introduced by Nikol'skii in [5]. If $\|f\|_{B^{\Omega}_{p,\theta}} \le 1$, then we say that the function f belongs to the class $B^{\Omega}_{p,\theta}$ and preserve for the classes the same notation as for the corresponding spaces $B^{\Omega}_{p,\theta}$.

In what follows, we use certain ordering relations. We now specify these relations. For two sequences $\mu_1(n)$ and $\mu_2(n)$, the relation $\mu_1 \asymp \mu_2$ means that there exist constants $C_3,\ C_4 > 0$ such that

$$C_3\mu_1(n) \le \mu_2(n) \le C_4\mu_1(n).$$

The relations $\mu_1 \ll \mu_2$ or $\mu_1 \gg \mu_2$ mean that $C\mu_1(n) \le \mu_2(n)$ and $\mu_2(n) \le C\mu_1(n)$, respectively. All constants C_i, $i = 1, 2, \dots$, encountered in the present paper may depend solely on the parameters contained in the definitions of the class, metric in which the analyzed approximation is realized, and dimension of the space $\mathbb{R}^d$.

By $V_m(t)$, $m \in \mathbb{N}$, $t \in \mathbb{R}$, we denote the de la Vallée-Poussin kernel of the form

$$V_m(t) = 1 + 2\sum_{k=1}^{m} \cos kt + 2\sum_{k=m+1}^{2m} \left(\frac{2m-k}{m} \right) \cos kt.$$

The multidimensional kernel $V_m(x)$, $m \in \mathbb{N}$, $x \in \mathbb{R}^{\mathrm{d}}$, is defined by the formula

$$V_m(x) = \prod_{j=1}^{d} V_m(x_j).$$

For a function $f \in L_p(\pi_d)$, we consider the operator $\mathbf{V}_m$ of convolution of this function with the kernel $V_m(x)$, i.e.,

$$\mathbf{V}_m f = f * V_m = \mathrm{V}_m(f, x).$$

Thus, $V_m(f, x)$ is the de la Vallée-Poussin multiple sum of the function $f(x)$. For $f \in L_p(\pi_d)$, we set

$$\sigma_0(f, x) = V_1(f, x), \qquad \sigma_s(f, x) = V_{2^s}(f, x) - V_{2^{s-1}}(f, x), \quad s \in \mathbb{N}.$$

In this notation, for $1 \le p \le \infty$, the classes $B^{\Omega}_{p,\theta}$ can be defined as follows (to within absolute constants; see [3]):

$$B^{\Omega}_{p,\theta} = \{f \in L_p(\pi_d) \colon \|f\|_{B^{\Omega}_{p,\theta}} \le 1\},$$

where

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \begin{cases} \left(\sum\limits_{s \in \mathbb{Z}_+} \left(\dfrac{\|\sigma_s(f, \cdot)\|_p}{\Omega(2^{-s})}\right)^{\theta}\right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_{s \in \mathbb{Z}_+} \dfrac{\|\sigma_s(f, \cdot)\|_p}{\Omega(2^{-s})}, & \theta = \infty. \end{cases} \tag{1}$$

Note that, for $1 < p < \infty$, we can write an equivalent relation for the norms of functions from the classes $B^{\Omega}_{p,\theta}$, $1 \le \theta \le \infty$, by using "blocks" of the Fourier series of the function $f(x)$ instead of $\sigma_s(f, x)$ in relation (1)

1. Definitions of the Approximative Characteristics and Auxiliary Statements

We define approximative characteristics of the classes $B^{\Omega}_{p,\theta}$ studied in the present paper.

Let $F \subset L_q(\pi_d)$ be a functional class. The trigonometric width of the class F in the space L_q is defined by the formula [6]

$$d^T_m(F, L_q) = \inf_{\Omega_m} \sup_{f \in F} \inf_{t(\Omega_m, x)} \|f(\cdot) - t(\Omega_m, \cdot)\|_q, \tag{2}$$

where

$$t(\Omega_m, x) = \sum_{j=1}^{m} c_j \, e^{i(k^j, x)}, \quad \Omega_m = \{k^1, \ldots, k^m\}$$

is a collection of vectors $k^j = (k^j_1, \ldots, k^j_d)$, $j = \overline{1, m}$, from the integer-valued lattice Z^d and c_j are arbitrary numbers.

For the first time, the notion of trigonometric width was introduced by Ismagilov [6]. For different functional classes, quantity (2) was studied in numerous works. For the detailed presentation and the corresponding references, we refer the reader, e.g., to [7–10].

In estimating the widths $d^T_m(B^{\Omega}_{p,\theta}, L_q)$, we use the well-known estimates for the best m-term trigonometric approximations of functions from the classes $B^{\Omega}_{p,\theta}$ and the approximations of these classes by trigonometric polynomials with spectra in cubic domains. To formulate the corresponding results, we introduce the required notation and definitions.

Let $f \in L_q(\pi_d)$ and let $e_m(f,\ L_q)$ be the best m-term trigonometric approximation of the function f in the space L_q defined as follows:

$$e_m(f,\ L_q) = \inf_{\{k^j\}_{j=1}^m} \inf_{\{c_j\}_{j=1}^m} \left\| f(\cdot) - \sum_{j=1}^m c_j\ e^{i(k^j,\cdot)} \right\|_q ,$$

where $\{k^j\}_{j=1}^m$ is a collection of vectors $k^j = \{k_1^j, \ldots, k_d^j\}$ with integer-valued coordinates, c_j are arbitrary numbers, and $(k^j, x) = k_1^j x_1 + \ldots + k_d^j x_d$.

If F is a functional class, then we set

$$e_m(F,\ L_q) = \sup_{f \in F} e_m(f,\ L_q). \tag{3}$$

The quantity $e_m(f,\ L_2)$ for a function of one variable was introduced by Stechkin in [11] in formulating a criterion of absolute convergence for orthogonal series. Later, the quantities $e_m(f, L_q)$ and $e_m(F, L_q)$, $1 \le q \le \infty$, were studied from the viewpoint of approximation of individual functions and classes of functions, respectively. The first estimates for the quantities $e_m(f, L_\infty)$ for some specific functions were established by Ismagilov [6]. The systematic investigation of quantity (3) for the Sobolev ($W^r_{p,\alpha}$) and Nikol'skii (H^r_p) classes of periodic functions of many variables was originated by Temlyakov [12]. The subsequent investigations of the quantities $e_m(F, L_q)$ in the classes of functions $W^r_{p,\alpha}$ and H^r_p were performed by Belinskii [8, 13]. We also mention the works [14–16] devoted to the study of quantity (3) for some important classes of functions.

Further, let $T_{\square_{2^n}} = \{t(x) \colon t(x) = \sum_{k \in \square_{2^n}} c_k e^{i(k,x)},\ c_k \in \mathbb{C},$ where

$$\square_{2^n} = \left\{k = (k_1, \ldots, k_d) \colon |k_j| < 2^n,\ 1 \le j \le d\right\}.$$

For $f \in L_q$, $1 \le q \le \infty$, we set

$$E_{\square_{2^n}}(f,\ L_q) = \inf_{t(\cdot) \in T_{\square_{2^n}}} \|f(\cdot) - t(\cdot)\|_q .$$

Moreover, for the functional class $F \subset L_q$, we, respectively, set

$$E_{\square_{2^n}}(F, L_q) = \sup_{f \in F} E_{\square_{2^n}}(f, L_q).$$

We now formulate several statements used to establish the required results.

Theorem A [5]. *Let $n_j \in \mathbb{N}$, $j = \overline{1, d}$, and let*

$$t(x) = \sum_{|k_j| \le n_j} c_k\ e^{i(k,x)}.$$

Then the following inequality holds for $1 \le q < p \le \infty$:

$$\|t\|_p \le 2^d \prod_{j=1}^d n_j^{1/q - 1/p} \|t\|_q . \tag{4}$$

Inequality (4) was proved by Nikol'skii. It is called the "inequality of different metrics." In the case $d = 1$ and $p = \infty$, the corresponding inequality was proved by Jackson [17].

Lemma A [18]. *Let $2 \le q < \infty$. Then, for any trigonometric polynomial*

$$P(\Theta_m, x) = \sum_{j=1}^{m} e^{i(k^j, x)}$$

and any $n \le m$, one can find a trigonometric polynomial $\widetilde{P}(\Theta_n, x)$ containing at most n harmonics and a constant $C_q > 0$ such that

$$\left\| P(\Theta_m, \cdot) - \widetilde{P}(\Theta_n, \cdot) \right\|_q \le C_q m n^{-1/2}.$$

Moreover, $\Theta_n \subset \Theta_m$, all coefficients $\widetilde{P}(\Theta_n, x)$ are equal, and their absolute values do not exceed mn^{-1}.

Now let $\mu(s)$, $s = 0, 1, 2, \ldots,$ be a subset of an integer-valued lattice of the form

$$\mu(s) = \left\{ k = (k_1, \ldots, k_d) \colon 2^{s-1} \le \max_{j=\overline{1,d}} |k_j| < 2^s \right\}.$$

For $f \in L_p(\pi_d)$, we introduce the notation

$$f_0(x) = \hat{f}(0) \qquad \text{and} \qquad f_s(x) = \sum_{k \in \mu(s)} \hat{f}(k) e^{i(k,x)}, \quad s = 1, 2, \ldots,$$

where

$$\hat{f}(k) = (2\pi)^{-d} \int_{\pi_d} f(t) e^{-i(k,t)} dt$$

is the Fourier coefficient of the function f.

Theorem B [19]. *Let $f \in L_p(\pi_d)$, $1 < p < \infty$. Then there exist constants $C_5(p)$ and $C_6(p)$ such that*

$$C_5(p) \| f \|_p \le \left\| \left(\sum_{s=0}^{\infty} |f_s|^2 \right)^{1/2} \right\|_p \le C_6(p) \| f \|_p. \tag{5}$$

Theorem C [20]. *Let $1 \le p, q, \theta \le \infty$ and let $\Omega \in \Phi_{\alpha,l}$ with some $\alpha > \alpha(p,q)$, where*

$$\alpha(p,q) = \begin{cases} d(1/p - 1/q)_+, & 1 \le p \le q \le 2 \ \textit{or}\ 1 \le q \le p \le \infty, \\ \max\{d/p; d/2\} & \textit{otherwise}. \end{cases}$$

Then the following estimate is true for any $m \in \mathbb{N}$:

$$e_m(B^{\Omega}_{p,\theta}, L_q) \asymp \Omega(m^{-1/d})m^{(1/p-\max\{1/q;1/2\})_+},$$

where $a_+ = \max\{a, 0\}$.

Theorem D [21]. *Let* $1 \le p, q, \theta \le \infty$ *and let a function* $\Omega \in \Phi_{\alpha,l}$ *with some* $\alpha > d(1/p - 1/q)_+$. *Then*

$$E_{\Box_{2^n}}(B^{\Omega}_{p,\theta}, L_q) \asymp \Omega(2^{-n})2^{nd(1/p-1/q)_+},$$

where $a_+ = \max\{a; 0\}$.

2. Estimation of the Trigonometric Widths of the Classes $B^{\Omega}_{p,\theta}$ in the Space L_q

The following statement is true:

Theorem 1. *Let* $1 \le p < 2 \le q < p/(p-1)$, *let* $1 \le \theta \le \infty$, *and let the function* Ω *belong to* $\Phi_{\alpha,l}$ *for some* $\alpha > d$. *Then*

$$d^T_m(B^{\Omega}_{p,\theta}, L_q) \asymp \Omega(m^{-1/d})m^{1/p-1/2}. \tag{6}$$

Proof. Note that the lower bound in (6) follows from Theorem C. Moreover, according to the definitions of the quantities $e_m(F, L_q)$ and $d^T_m(F, L_q)$, we have

$$e_m(F, L_q) \le d^T_m(F, L_q) \tag{7}$$

Therefore, we can write (even for $\alpha > d/p$)

$$d^T_m(B^{\Omega}_{p,\theta}, L_q) \ge e_m(B^{\Omega}_{p,\theta},\ L_q) \gg \Omega(m^{-1/d})m^{1/p-1/2}.$$

The lower bound is established.

We now establish the upper bound. Since the right-hand side of (6) is independent of θ and the classes $B^{\Omega}_{p,\theta}$ are extended as the parameter θ increases, i.e., for $1 \le \theta \le \theta' \le \infty$, we have the inclusions

$$B^{\Omega}_{p,1} \subset B^{\Omega}_{p,\theta} \subset B^{\Omega}_{p,\theta'} \subset B^{\Omega}_{p,\infty} \equiv H^{\Omega}_p,$$

it suffices to establish the upper bound for $d^T_m(B^{\Omega}_{p,\infty},\ L_q)$, i.e., $d^T_m(H^{\Omega}_p, L_q)$.

We take an arbitrary $m \in \mathbb{N}$ and choose $n \in \mathbb{N}$ such that

$$2^{(n-1)d} \le m \le 2^{nd},$$

i.e., $m \asymp 2^{nd}$.

For $s = 0, 1, 2, \ldots,$ we set

$$m_s = \begin{cases} 2^{sd}, & 0 \le s < n, \\ \left[\Omega^{-1}(2^{-n}) 2^{sd} \Omega(2^{-s})\right] + 1, & n \le s \le n_0, \\ 0, & s > n_0, \end{cases}$$

where $[a]$ is the integer part of the number a and

$$n_0 = \left[n \frac{\alpha/d - 1/p + 1/2}{\alpha/d - 1/p + 1/q} \right] + 1.$$

This enables us to estimate the sum $\sum_{s=0}^{n_0} m_s$. Thus, we get

$$\sum_{s=0}^{n_0} m_s \le \sum_{s=0}^{n-1} 2^{sd} + \sum_{s=n}^{n_0} \Omega^{-1}(2^{-n}) 2^{sd} \Omega(2^{-s}) + \sum_{s=n}^{n_0} 1$$

$$\ll 2^{nd} + \Omega^{-1}(2^{-n}) \sum_{s=n}^{n_0} \frac{\Omega(2^{-s})}{2^{-\alpha s}} 2^{-s(\alpha-d)} + (n_0 - n + 1) = \mathcal{J}_1.$$

Since $\Omega(t) \in S^\alpha$ with some $\alpha > d$, the following relation is true:

$$\frac{\Omega(2^{-s})}{2^{-\alpha s}} \le \frac{\Omega(2^{-n})}{2^{-\alpha n}}, \quad s \ge n.$$

We can continue the estimate for $\mathcal{J}_1$ as follows:

$$\mathcal{J}_1 \ll 2^{nd} + \Omega^{-1}(2^{-n}) \frac{\Omega(2^{-n})}{2^{-\alpha n}} \sum_{s=n}^{n_0} 2^{-s(\alpha-d)} + (n_0 - n + 1)$$

$$\ll 2^{nd} + 2^{\alpha n} 2^{-n(\alpha-d)} + (n_0 - n + 1)$$

$$= 2^{nd} + 2^{nd} + (n_0 - n + 1) \ll 2^{nd} \left(1 + \frac{n_0 - n + 1}{2^{nd}}\right) \asymp m.$$

Hence, we get

$$\sum_{s=0}^{n_0} m_s \ll m.$$

Consider a trigonometric polynomial

$$t_s(x) = \sum_{k \in \mu(s)} e^{i(k,x)}.$$

Note that, for any s, this polynomial has $|\mu(s)|$ terms, i.e., their number has the order $2^{(s+1)d}$. By $|A|$ we denote the number of elements of the finite set $A \subset \mathbb{Z}^d$.

Further, since the inequality $m_s \le 2^{(s+1)d}$ holds for any $s = 0, 1, 2, \ldots,$ by Lemma A, there exist a trigonometric polynomial $t(\Theta_{m_s}, x)$ containing at most m_s harmonics and a constant C_q such that

$$\left\| t_s(\cdot) - t(\Theta_{m_s}, \cdot) \right\|_q \le C_q 2^{(s+1)d} m_s^{-1/2} \ll 2^{sd} m_s^{-1/2}.$$

Moreover, $\Theta_{m_s} \subset \Theta_{2^{(s+1)d}}$, all coefficients $t(\Theta_{m_s}, x)$ are equal, and their absolute values do not exceed $2^{(s+1)d} m_s^{-1}$.

We now construct a subspace of trigonometric polynomials with the "numbers" of harmonics from the union of the sets

$$P = \bigcup\nolimits_{0 \le s < n} \mu(s) \qquad \text{and} \qquad Q = \bigcup\nolimits_{n \le s \le n_0} \Theta_{m_s}$$

and show that the approximation by polynomials from this space realizes the order of the trigonometric width $d_m^T(H_p^\Omega, L_q)$ for $1 \le p < 2 \le q < p/(p-1)$.

Let f be an arbitrary function from the class H_p^Ω. For this function, we consider an approximating polynomial with the "numbers" of harmonics from $P \bigcup Q$ of the form

$$t(x) = \sum_{s=0}^{n-1} f_s(x) + \sum_{s=n}^{n_0} (t(\Theta_{m_s}, x) * f_s(x)).$$

Then

$$\| f(\cdot) - t(\cdot) \|_q \le \left\| \sum_{s=n}^{n_0} f_s(\cdot) - (f_s(\cdot) * t(\Theta_{m_s}, \cdot)) \right\|_q + \left\| \sum_{s > n_0} f_s(\cdot) \right\|_q = \mathcal{J}_2 + \mathcal{J}_3. \tag{8}$$

First, we establish the upper bound of the term $\mathcal{J}_3$ for $p \ne 1$. Note that, for $f \in H_p^\Omega$, we have

$$\|\sigma_s(f, \cdot)\|_p \le \Omega(2^{-s}), \quad s = 0, 1, 2, \ldots.$$

Thus, according to the Minkowski inequality and the "inequality of different metrics," we get

$$\begin{aligned} \mathcal{J}_3 = \left\| \sum_{s > n_0} f_s(\cdot) \right\|_q \sum_{s > n_0} \| f_s(\cdot) \|_q \\ \ll \sum_{s > n_0} 2^{sd(1/p - 1/q)} \|\sigma_s(f, \cdot)\|_p \\ \le \sum_{s > n_0} 2^{sd(1/p - 1/q)} \Omega(2^{-s}) = \sum_{s > n_0} \frac{\Omega(2^{-s})}{2^{-\alpha s}} 2^{-sd(\alpha/d - 1/p + 1/q)}. \end{aligned}$$

Since $\Omega(t) \in S^\alpha$, the inequality $\alpha > d$ implies the following formula:

$$\frac{\Omega(2^{-s})}{2^{-\alpha s}} \ll \frac{\Omega(2^{-n})}{2^{-\alpha n}}, \quad s > n_0 > n.$$

Hence, for the term $\mathcal{J}_3$, we now get

$$\mathcal{J}_3 \ll \frac{\Omega(2^{-n})}{2^{-\alpha n}} \sum_{s>n_0} 2^{-sd(\alpha/d-1/p+1/q)}$$

$$\ll \frac{\Omega(2^{-n})}{2^{-\alpha n}} 2^{-n\frac{\alpha/d-1/p+1/2}{\alpha/d-1/p+1/q}d(\alpha/d-1/p+1/q)}$$

$$= \frac{\Omega(2^{-n})}{2^{-\alpha n}} 2^{-nd(\alpha/d-1/p+1/2)}$$

$$= \Omega(2^{-n})2^{nd(1/p-1/2)} \asymp \Omega(m^{-1/d})m^{1/p-1/2}. \tag{9}$$

We now find the upper bound of the quantity $\mathcal{J}_2$. To this end, for each $s \in [n, n_0]$, we consider a linear operator T_s acting upon the function $f(x) \in L_p$ as follows:

$$T_s f(x) = f(x) * (t_s(x) - t(\Theta_{m_s}; x)).$$

Then the following assertion is true:

Lemma B [22]**.** *Let* $1 < p < 2 < q < p/(p-1)$. *Then the norm of the operator* T_s *from* L_p *in* $L_q(\|T_s\|_{p\to q})$ *satisfies the relation*

$$\|T_s\|_{p\to q} = \sup_{\|f\|_p \le 1} \|T_s f\|_q \ll 2^{sd} m_s^{-(1/2+1/p')},$$

where $p' = p/(p-1)$.

First, let $p \in (1,2)$. We successively apply Theorem B, the Minkowski inequality, and Lemma B (for $n \le s \le n_0$). This yields

$$\mathcal{J}_2 \ll \left\| \left(\sum_{s=n}^{n_0} |f_s(\cdot) - (f_s(\cdot) * t(\Theta_{m_s}, \cdot))|^2 \right)^{1/2} \right\|_q$$

$$= \left\| \sum_{s=n}^{n_0} |f_s(\cdot) - (f_s(\cdot) * t(\Theta_{m_s}, \cdot))|^2 \right\|_{q/2}^{1/2}$$

$$\le \left(\sum_{s=n}^{n_0} \| |f_s(\cdot) - (f_s(\cdot) * t(\Theta_{m_s}, \cdot))|^2 \|_{q/2} \right)^{1/2}$$

$$= \left(\sum_{s=n}^{n_0} \| f_s(\cdot) - (f_s(\cdot) * t(\Theta_{m_s}, \cdot)) \|_q^2 \right)^{1/2}$$

$$= \left(\sum_{s=n}^{n_0} \| T_s f_s(\cdot) \|_q^2 \right)^{1/2} \le \left(\sum_{s=n}^{n_0} \| T_s \|_{p\to q}^2 \| f_s(\cdot) \|_p^2 \right)^{1/2}$$

$$\ll \left(\sum_{s=n}^{n_0} 2^{2sd} m_s^{-(1+2/p')} \| f_s(\cdot) \|_p^2 \right)^{1/2} . \tag{10}$$

Substituting the values of m_s in (10), after necessary transformations, we obtain

$$\mathcal{J}_2 \ll \left(\sum_{s=n}^{n_0} 2^{2sd} \Omega^{1+2/p'}(2^{-n}) 2^{-sd(1+2/p')} \Omega^{-(1+2/p')}(2^{-s}) \| \sigma_s(f,\cdot) \|_p^2 \right)^{1/2}$$

$$\le \Omega^{1/2+1/p'}(2^{-n}) \left(\sum_{s=n}^{n_0} \Omega^{-(1+2/p')}(2^{-s}) \Omega^2(2^{-s}) 2^{sd(1-2/p')} \right)^{1/2}$$

$$= \Omega^{3/2-1/p}(2^{-n}) \left(\sum_{s=n}^{n_0} \Omega^{2/p-1}(2^{-s}) 2^{sd(2/p-1)} \right)^{1/2}$$

$$= \Omega^{3/2-1/p}(2^{-n}) \left(\sum_{s=n}^{n_0} \left(\frac{\Omega(2^{-s})}{2^{-\alpha s}} \right)^{2/p-1} 2^{-s(\alpha-d)(2/p-1)} \right)^{1/2} .$$

In view of the fact that, according to the conditions of the theorem, the function $\Omega(t) \in S^\alpha$ with some $\alpha > d$ and the inequalities $2/p - 1 > 0$ and $\alpha - d > 0$ are true, we can continue the estimate of the quantity $\mathcal{J}_2$ as follows:

$$\mathcal{J}_2 \ll \Omega^{3/2-1/p}(2^{-n}) \left(\frac{\Omega(2^{-n})}{2^{-\alpha n}} \right)^{1/p-1/2} \left(\sum_{s=n}^{n_0} 2^{-s(\alpha-d)(2/p-1)} \right)^{1/2}$$

$$\ll \Omega(2^{-n}) 2^{\alpha n(1/p-1/2)} 2^{-n(\alpha-d)(1/p-1/2)}$$

$$= \Omega(2^{-n}) 2^{nd(1/p-1/2)} \asymp \Omega(m^{-1/d}) m^{1/p-1/2}. \tag{11}$$

Thus, substituting (9) and (11) in (8), we arrive at the estimate

$$\| f(\cdot) - t(\cdot) \|_q \ll \Omega(m^{-1/d}) m^{1/p-1/2}, \quad 1 < p < 2 \le q < \frac{p}{p-1}.$$

This yields the necessary upper bound for the width $d_m^T(H_p^\Omega, L_q)$ and, hence, for the width

$$d_m^T(B_{p,\theta}^\Omega, L_q), \qquad 1 < p < 2 \le q < p/(p-1), \qquad 1 \le \theta < \infty.$$

We now establish the upper bound for the trigonometric width $d_m^T(H_1^\Omega, L_q)$, $2 \le q < \infty$.

Let p_1 be a number satisfying the condition $1 < p_1 < 2$. In what follows, we determine its value more precisely. We estimate the term $\mathcal{J}_3$ as in the previous case. For the quantity $\mathcal{J}_2$, we first repeat the reasoning used above and obtain

$$\mathcal{J}_2 \ll \left(\sum_{s=n}^{n_0} \|T_s f_s(\cdot)\|_q^2\right)^{1/2} \leq \left(\sum_{s=n}^{n_0} \|T_s\|_{p_1\to q}^2 \|f_s(\cdot)\|_{p_1}^2\right)^{1/2}$$

$$\asymp \left(\sum_{s=n}^{n_0} \|T_s\|_{p_1\to q}^2 \|\sigma_s(f,\cdot)\|_{p_1}^2\right)^{1/2} \ll \left(\sum_{s=n}^{n_0} 2^{2sd} m_s^{-(1+2/p_1')} \|\sigma_s(f,\cdot)\|_{p_1}^2\right)^{1/2}. \tag{12}$$

Applying the inequality of different metrics to $\|\sigma_s(f,\cdot)\|_{p_1}$ and substituting the values of m_s in (12) , we find

$$\mathcal{J}_2 \ll \left(\sum_{s=n}^{n_0} 2^{2sd} m_s^{-(1+2/p_1')} 2^{2sd(1-1/p_1)} \|\sigma_s(f,\cdot)\|_1^2\right)^{1/2}$$

$$\leq \Omega^{1/2+1/p_1'}(2^{-n}) \left(\sum_{s=n}^{n_0} \Omega^{1-2/p_1'}(2^{-s}) 2^{sd}\right)^{1/2}$$

$$= \Omega^{1/2+1/p_1'}(2^{-n}) \left(\sum_{s=n}^{n_0} \left(\frac{\Omega(2^{-s})}{2^{-\alpha s}}\right)^{1-2/p_1'} 2^{-s(\alpha-2\alpha/p_1'-d)}\right)^{1/2}$$

$$\leq \Omega^{1/2+1/p_1'}(2^{-n}) \left(\frac{\Omega(2^{-n})}{2^{-\alpha n}}\right)^{1/2-1/p_1'} \left(\sum_{s=n}^{n_0} 2^{-s(\alpha-2\alpha/p_1'-d)}\right)^{1/2}$$

$$= \Omega(2^{-n}) 2^{\alpha n(1/2-1/p_1')} \left(\sum_{s=n}^{n_0} 2^{-s(\alpha-2\alpha/p_1'-d)}\right)^{1/2}.$$

We now choose a number p_1 for which the inequality $\alpha - 2\alpha/p_1' - d > 0$, where $1/p_1 + 1/p_1' = 1$, is satisfied. This is possible because, according to the conditions of the theorem, $\alpha > d$.

Thus, for the quantity $\mathcal{J}_2$, we get

$$\mathcal{J}_2 \ll \Omega(2^{-n}) 2^{\alpha n(1/2-1/p_1')} 2^{-n(\alpha/2-\alpha/p_1'-d/2)} \ll \Omega(2^{-n}) 2^{nd/2} \asymp \Omega(m^{-1/d}) m^{1/2}.$$

In view of the estimate for the quantity $\mathcal{J}_3$, this yields the required estimate for the width $d_m^T(H_1^\Omega, L_q)$ and, hence, the estimate for $d_m^T(B_{1,\theta}^\Omega, L_q)$, $1 \leq \theta < \infty$.

The theorem is proved.

In conclusion, we present an assertion for the orders of the trigonometric widths $d_m^T(B_{p,\theta}^\Omega, L_q)$ for some other relations between the parameters p and q. This assertion is a corollary of the well-known results.

Theorem 2. *Let* $1 \le q \le p \le \infty$ *or* $1 \le p \le q \le 2$ *and* $1 \le \theta \le \infty$ *and let the function* Ω *belong to* $\Phi_{\alpha,l}$ *for some* $\alpha > d(1/p - 1/q)_+$. *Then the following order estimate is true:*

$$d_m^T(B_{p,\theta}^{\Omega}, L_q) \asymp \Omega(m^{-1/d}) m^{(1/p-1/q)_+}. \tag{13}$$

The upper bound in (13) is obtained from Theorem D by using the inequality

$$d_m^T(B_{p,\theta}^{\Omega}, L_q) \le E_{\square_{2^n}}(B_{p,\theta}^{\Omega}, L_q), \quad m \asymp 2^{nd},$$

The lower bound is a corollary of Theorem C.

Remark 1. If

$$\Omega(t) = t^r, \qquad r > d, \quad 1 \le p < 2 \le q < p/(p-1) \quad 1 \le \theta \le \infty,$$

then

$$d_m^T(B_{p,\theta}^r)_q \asymp m^{-r/d+1/p-1/2}. \tag{14}$$

Estimate (14) was established in [22].

Remark 2. For the relationships between the parameters p and q satisfying the conditions of Theorems 1 and 2, according to Theorem C, we can write

$$d_m^T(B_{p,\theta}^{\Omega}, L_q) \asymp e_m(B_{p,\theta}^{\Omega}, L_q).$$

Remark 3. The problem of orders of the widths $d_m^T(B_{p,\theta}^{\Omega}, L_q)$ in the cases $2 \le p < q \le \infty$ and $1 < p < 2$, $p/(p-1) < q \le \infty$ remains open.

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