Local Mean Characterization of Besov-Triebel-Lizorkin Type Spaces with Dominating Mixed Smoothness on Rectangular Domains

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December 8, 2008

Abstract

We study the restrictions $S^{\bar{r}}_{p,q}B(\Omega)$ and $S^{\bar{r}}_{p,q}F(\Omega)$ of the Besov- and Triebel-Lizorkin type spaces with dominating mixed smoothness, where Ω denotes a rectangular domain. For a general range of parameters ($\bar{r} \in \mathbb{R}^d, p, q > 0$) we give explicit intrinsic characterizations of $S^{\bar{r}}_{p,q}B(\Omega)$ and $S^{\bar{r}}_{p,q}F(\Omega)$ in terms of Peetre type maximal functions and local means. The approach is based on a paper by Rychkov [9], where the isotropic problem for Lipschitz domains is treated, as well as on characterizations given by Vybíral [22] and Bazarkhanov [2].

Key Words. Besov type spaces of dominating mixed smoothness, Triebel-Lizorkin type spaces of dominating mixed smoothness, restriction extension problem, rectangular domains, Peetre type maximal functions, local means characterization
AMS subject classifications. 42B25, 42B35, 46E35, 46F05.

1 Introduction

Sobolev and Besov spaces with dominating mixed smoothness properties have attracted recently much attention in approximation theory as well as numerical analysis. Using such scales, one is able to overcome the curse of dimensionality to some extent by using so called sparse grid and hyperbolic cross constructions. Let us mainly refer to [3, 12, 15, 21, 20] and the references given there. However, [3] deals with functions having second bounded mixed derivative in $L_2([0,1]^d)$. This is just one reason out of

many more that indicates the necessity for the study of spaces on domains.
The main aim of this paper is to take the first step for giving intrinsic characterizations for the whole scale of Besov as well as Lizorkin-Triebel spaces $S^{\bar{r}}_{p,q}B(\Omega)$ and $S^{\bar{r}}_{p,q}F(\Omega)$, respectively (subsequently we will use the symbol $S^{\bar{r}}_{p,q}A(\Omega)$ for both scales). These spaces are defined via restriction of the corresponding function spaces on $\mathbb{R}^d$ to the domain Ω and equipped with the quotient norm. We are interested in characterizing the spaces by the use of information taken exclusively from the interior of Ω. This problem is closely related to the construction of a bounded extension operator $\mathcal{E}$: $S^{\bar{r}}_{p,q}A(\Omega) \to S^{\bar{r}}_{p,q}A(\mathbb{R}^d)$. It turned out that in case

$$\Omega = \mathcal{I}^1 \times ... \times \mathcal{I}^d \,, \tag{1}$$

where $\mathcal{I}^k \subset \mathbb{R}$ are infinite intervals, such an extension operator exists. Our method is the "tensorized" analogue of Rychkov's approach to the same problem within the scales $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ on a special Lipschitz domain Ω. We obtain intrinsic characterizations of $S^{\bar{r}}_{p,q}A(\Omega)$ in terms of Peetre type maximal functions and local means. This represents a powerful tool for further work in this direction. Actually, one is interested in characterizations in terms of differences and derivatives. In a forthcoming paper we will carry over the results from [19] to spaces on domains by applying the present ones.
The main results of the paper are stated in Sections 3 and 4. The theorems given there are the direct counterpart of the recent results by Vybíral [22] and Bazarkhanov [2] for the corresponding spaces on $\mathbb{R}^d$. These spaces have an almost 50 year old history. First spaces were introduced by Nikol'skij in [6] in the early sixties of the last century. He introduced in particular Sobolev-type spaces

$$\begin{aligned} S^{(r_1,r_2)}_p W(\mathbb{R}^2) = \Big\{ f \in L_p(\mathbb{R}^2) : \|f|S^{(r_1,r_2)}_p W(\mathbb{R}^2)\| = \|f|L_p(\mathbb{R}^2)\| \\ + \left\| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \big| L_p(\mathbb{R}^2) \right\| + \left\| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \big| L_p(\mathbb{R}^2) \right\| + \left\| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} \big| L_p(\mathbb{R}^2) \right\| < \infty \Big\} \,. \end{aligned}$$

The mixed derivative plays the dominant role here and led to the name of these classes of spaces. The detailed study of this matter is due to many researchers from the former Soviet Union, for instance Amanov, Besov, Lizorkin, Nikol'skij and Potapov to mention just a few. Amanov gave a first systematic treatment in [1], where he defined scales of spaces in terms of differences and derivatives. As far as the Fourier analytic treatment of Besov and Lizorkin-Triebel type spaces $S^{\bar{r}}_{p,q}B$ and $S^{\bar{r}}_{p,q}F$ is considered, we mainly refer to the monograph by Schmeisser and Triebel [13]. Several types of equivalent quasi-norms, embedding and trace theorems as well as characterizations by differences

are proved there. For a complete survey through existing results and further references in connection with this topic we refer to Schmeisser [11].
The paper is organized as follows. In Section 2 we recall and collect some basic tools like distributions and convolutions, vector valued Lebesgue spaces, maximal functions, maximal inequalities and the definition of function spaces on $\mathbb{R}^d$ and Ω. Section 3 is devoted to the characterization of $S^{\bar{r}}_{p,q}A(\Omega)$ via Peetre type maximal functions, where Ω satisfies (1). In Section 4 we characterize the same spaces in terms of local means.

2 Some Preparation

2.1 Notation

Let us first introduce some basic notation. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ denote the real numbers, complex numbers, natural numbers, natural numbers including 0 and the integers. The dimension of the underlying Euclidean space for function spaces is denoted by d, its elements will be denoted by $x, y, z, ...$ and $|x|$ is used for the Euclidean norm. Further vector-valued quantities, like indices, will be written as $\bar{r} = (r_1, ..., r_d)$,$\bar{k} = (k_1, ..., k_d), \bar{\ell} = (\ell_1, ..., \ell_d)$,.... We shall use $|\bar{k}|_1$ for the ℓ_1^d-norm of a vector $\bar{k}$. As usual $\lambda \cdot \bar{k} = (\lambda \cdot k_1, ..., \lambda \cdot k_d)$, $\lambda \in \mathbb{R}$, and $\bar{k} + \bar{\ell} = (k_1 + \ell_1, ..., k_d + \ell_d)$. To simplify notation let us also use $\lambda + \bar{k}$ in the sense of $(\lambda + k_1, ..., \lambda + k_d)$. Similarly, we abbreviate the relations $r_i > s_i$ ($r_i \geq s_i$) for $i = 1, ..., d$ by $\bar{r} > \bar{s}$ ($\bar{r} \geq \bar{s}$) and by $\bar{r} > s$ ($\bar{r} \geq s$) if additionally $\bar{s} = (s, ..., s)$ with $s \in \mathbb{R}$.
For a multi-index $\bar{\alpha}$ we define the differential operator $D^{\bar{\alpha}}$ by

$$D^{\bar{\alpha}} = \frac{\partial^{|\bar{\alpha}|_1}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} .$$

Given a function $g : \mathbb{R} \to \mathbb{C}$, we denote by $L_g \in \mathbb{N}_0$ the maximal number such that g has vanishing moments, i.e.

$$\int_{\mathbb{R}} x^\alpha g(x)\, dx = 0 \quad , \quad \alpha = 0, ..., L_g .$$

The dyadic dilates of g are defined by $g_j(x) = 2^j g(2^j x)$. Note that $g_j(x)$ for $j = 0$ is not covered by the last formula, but is just a value of a function g_0. For any function $h : \Omega \subset \mathbb{R}^d \to \mathbb{C}$ we define $h_\Omega : \mathbb{R}^d \to \mathbb{C}$ by

$$g_\Omega(x) = \begin{cases} g(x) & : \ x \in \Omega \\ 0 & : \ x \notin \Omega \end{cases} .$$

The letter c denotes a constant, which may vary from line to line but is always independent of f, unless the opposite is not explicitly stated. We also use the notation

$a \asymp b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}\, a \le b \le c\, a\,.$$

2.2 Distributions

Let Ω be a domain (i.e. an open connected set) and $C_0^\infty(\Omega)$ the space of all infinitely differentiable and compactly supported functions $f : \Omega \to \mathbb{C}$. The topology on $C_0^\infty(\Omega)$ is given via the convergence of sequences, see [14] or [18, I.4]. A sequence $\{\varphi_j\}_j \subset C_0^\infty(\Omega)$ converges to a $\varphi \in C_0^\infty(\Omega)$ if $\operatorname{supp}\varphi_j \subset K$, $j = 1, 2, ...$, where $K \subset \mathbb{R}^d$ is a compact subset, and $\{D^{\bar{\alpha}}\varphi_j\}_j$ converges uniformly to φ for every multi-index $\bar{\alpha} \in \mathbb{N}_0^d$. The resulting topological vector space is denoted by $D(\Omega)$. Clearly, we have $D(\Omega_1) \hookrightarrow D(\Omega_2)$ if $\Omega_1 \subset \Omega_2$ with an obvious interpretation (extension by zero). We may consider an element $\varphi \in D(\Omega_1)$ in $D(\Omega_2)$ using the same symbol. The space $D'(\Omega)$ contains all continuous linear forms T. In terms of the above defined topology this means the following. A linear mapping $T : D(\Omega) \to \mathbb{C}$ belongs to $D'(\Omega)$ if and only if $\varphi_j \underset{D}{\to} \varphi$ implies $T(\varphi_j) \underset{\mathbb{C}}{\to} T(\varphi)$. We equip the space $D'(\Omega)$ with the weak topology meaning the following. A sequence $\{f_j\}_j \subset D'(\Omega)$ converges to $f \in D'(\Omega)$ if and only if $f_j(\varphi)$ converges to $f(\varphi)$ in $\mathbb{C}$ for every fixed $\varphi \in D(\Omega)$. A locally integrable function $f : \Omega \to \mathbb{C}$ is interpreted as a distribution by

$$f(\varphi) = \int\limits_{\mathbb{R}^d} f(x)\varphi(x)\, dx\,. \tag{2}$$

As usual $S(\mathbb{R}^d)$ is used for the locally convex space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$ where the topology is generated by the family of norms

$$\|\varphi\|_{k,\ell} = \sup_{x\in\mathbb{R}^d, |\bar{\alpha}|_1 \le \ell} |D^{\bar{\alpha}}\varphi(x)|(1 + |x|)^k \quad , \quad \varphi \in S(\mathbb{R}^d)\,.$$

In other words, a sequence $\{\varphi_j\}_j \subset S(\mathbb{R}^d)$ converges to $\varphi \in S(\mathbb{R}^d)$ in $S(\mathbb{R}^d)$ if and only if

$$\|\varphi - \varphi_j\|_{k,\ell} \xrightarrow[j\to\infty]{} 0$$

holds for all pairs $(k, \ell) \in \mathbb{N}_0^2$. The set-theoretical as well as topological embedding $D(\mathbb{R}^d) \hookrightarrow S(\mathbb{R}^d)$ holds true. The space $S'(\mathbb{R}^d)$ is called the set of all tempered distributions on $\mathbb{R}^d$ and defined as the topological dual of $S(\mathbb{R}^d)$. Indeed, a linear mapping $f : S(\mathbb{R}^d) \to \mathbb{C}$ belongs to $S'(\mathbb{R}^d)$ if and only if there exist numbers $k, \ell \in \mathbb{N}_0$ and a constant $c = c_f$ such that

$$|f(\varphi)| \leq c_f \sup_{x\in\mathbb{R}^d, |\bar{\alpha}|_1 \leq \ell} |D^{\bar{\alpha}}\varphi(x)|(1+|x|)^k \tag{3}$$

for all $\varphi \in S(\mathbb{R}^d)$. We interpret a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ with at most polynomial growth as a tempered distribution using (2).
$S'(\mathbb{R}^d)$ is equipped with the weak topology as well.

Moreover, we shall need the subspace $S'(\Omega) \subset D'(\Omega)$ consisting of all distributions having finite order and at most polynomial growth at infinity. Indeed, f belongs to $S'(\Omega)$ if and only if the estimate

$$|f(\varphi)| \leq c \sup_{x\in\Omega, |\bar{\alpha}|_1 \leq \ell} |D^{\bar{\alpha}}\varphi(x)|(1+|x|)^k \tag{4}$$

for all $\varphi \in S(\mathbb{R}^d)$ is true for sum numbers k, ℓ, c depending on f.

The restriction of a tempered distribution to Ω is defined as follows. If $f \in S'(\mathbb{R}^d)$ is a tempered distribution then $f|_\Omega$ denotes the restricted mapping f to $D(\Omega)$. Comparing (3) and (4) it becomes obvious that a sufficient condition for g to be in $S'(\Omega)$ is the existence of $f \in S'(\mathbb{R}^d)$ such that $g = f|_\Omega$.

2.3 Convolutions

In the classical sense the convolution $\varphi * \psi$ of two integrable functions φ, ψ is defined via the integral

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^d} \varphi(x-y)\psi(y)\,dy = \int_{\mathbb{R}^d} \psi(x-y)\varphi(y)\,dy. \tag{5}$$

If $\varphi, \psi \in S(\mathbb{R}^d)$ then (5) still belongs to $S(\mathbb{R}^d)$. This concept can be generalized to $S'(\mathbb{R}^d)$. Given an $f \in S'(\mathbb{R}^d)$ and $\varphi \in S(\mathbb{R}^d)$ we define the convolution $(\varphi * f)$ to be the $\mathbb{R}^d$-function

$$(\varphi * f)(x) = f(\varphi(x-\cdot))\,, \tag{6}$$

which makes sense pointwise and is again an element of $S'(\mathbb{R}^d)$. Indeed, we have

$$f(\varphi(x-\cdot))(\psi) = f(\varphi(-\cdot) * \psi) \tag{7}$$

for $\psi \in S(\mathbb{R}^d)$. With an additional assumption this concept makes also sense in $D'(\Omega)$. Having domains K and Ω such that $x + K \subset \Omega$ for all $x \in \Omega$ and $\varphi \in D(K)$, $\psi \in D(\Omega)$ then (5) defines an an element of $D(\Omega)$. For $\varphi \in D(-K)$ and $f \in D'(\Omega)$, (6) makes sense pointwise on Ω and is as well an element of $D'(\Omega)$, where (7) holds for $\psi \in D(\Omega)$. However, one has to be careful while interpreting the function $(\varphi * f)_\Omega(x)$ as a tempered distribution by (2). It is not clear that the function has at most polynomial growth at infinity.

2.4 Lebesgue spaces

Let Ω be a domain, where the whole $\mathbb{R}^d$ is admitted as well. The spaces $L_p(\Omega)$ denote the classical Lebesgue space with $0 < p \leq \infty$ and

$$\|f|L_p(\Omega)\| = \Big(\int\limits_\Omega |f(x)|^p \, dx \Big)^{1/p} . \tag{8}$$

If $p = \infty$ we modify (8) by

$$\|f|L_\infty(\Omega)\| = \operatorname*{ess-sup}_{x\in\Omega} |f(x)| \, .$$

Having a sequence of complex-valued functions $\{f_{\bar{k}}\}_{\bar{k}\in\mathbb{N}_0^d}$ on Ω, we put

$$\|f_{\bar{k}}|\ell_q(L_p(\Omega))\| = \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} \|f_{\bar{k}}|L_p(\Omega)\|^q \Big)^{1/q}$$

and

$$\|f_{\bar{k}}|L_p(\ell_q,\Omega)\| = \Big\| \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} |f_{\bar{k}}(x)|^q \Big)^{1/q} \Big| L_p(\Omega) \Big\| ,$$

where we modify appropriately in the case $q = \infty$. If $\Omega = \mathbb{R}^d$ we simply write $L_p(\ell_q)$ and $\ell_q(L_p)$.

2.5 Maximal functions

The Hardy-Littlewood maximal function

For a locally integrable function f we denote by $Mf(x)$ the Hardy-Littlewood maximal function defined by

$$(Mf)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int\limits_Q |f(y)| \, dy \quad , \quad x \in \mathbb{R}^d , \tag{9}$$

where the supremum is taken over all cubes centered at x with sides parallel to the coordinate axes. The following theorem is due to Fefferman and Stein [5].

Theorem 2.1 For $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $c > 0$, such that

$$\|Mf_{\bar{k}}|L_p(\ell_q)\| \leq c\|f_{\bar{k}}|L_p(\ell_q)\|$$

holds for all sequences $\{f_{\bar{k}}\}_{\bar{k}\in\mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

Now we define a one-dimensional version of (9)

$$(M_i f)(x) = \sup_{s>0} \frac{1}{2s} \int\limits_{x_i-s}^{x_i+s} |f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d)|\, dt \quad , \quad x \in \mathbb{R}^d\,.$$

Precisely, one has to prove, that this operator maps two equivalent representatives of f to the same equivalence class. Having this in mind, one can state the following version of Theorem 2.1.

Theorem 2.2 For $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $c > 0$ such that

$$\|M_i f_{\bar{k}} | L_p(\ell_q)\| \leq c \|f_{\bar{k}} | L_p(\ell_q)\| \quad , \quad i = 1, ..., d\,,$$

holds for all sequences $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

In the sequel we will use the operator $\bar{M} = M_1 \circ \cdots \circ M_d$ as well. Of course, we have a version of the maximal inequality in Theorem 2.2 also for this operator.

The Peetre maximal function

Let us introduce next a version of the Peetre maximal operator introduced in [7]. This construction is adapted to the case of function spaces with dominating mixed smoothness. Given a system $\{\psi_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d} \subset S(\mathbb{R}^d)$, a tempered distribution $f \in S'(\mathbb{R}^d)$ and a vector $\bar{a} > 0$ we define the system of maximal functions

$$(\psi^*_{\bar{k}} f)_{\bar{a}}(x) = \sup_{y \in \mathbb{R}^d} \frac{|(\psi_{\bar{k}} * f)(y)|}{\prod\limits_{i=1}^{d} (1 + 2^{k_i}|x_i - y_i|)^{a_i}} \quad , \quad x \in \mathbb{R}^d\,. \tag{10}$$

Since $(\psi_{\bar{k}} * f)(y)$ makes sense pointwise everything is well-defined. However, the value "∞" is also possible here.

2.6 Helpful lemmas

Let us give some technical lemmas in order to split the proof of our main theorem into several pieces. The following Lemma corresponds to Lemma 1.17 in [22], where a more general case is treated. We omit the proof and refer to [22].

Lemma 2.3 *Let $g, h \in S'(\mathbb{R})$ such that $L_g \geq M$. Then for every $N \in \mathbb{N}_0$ there exists a constant C_N such that*

$$\sup_{z \in \mathbb{R}} |(g_b * h)(z)|(1 + |z|)^N \leq c_N\, b^{M+1} \quad , \quad 0 < b < 1\,,$$

where $g_b(\cdot) = b^{-1} g(b^{-1} \cdot)$.

Furthermore, we shall need the following convolution inequality which corresponds to [22, Lem. 1.18]. Here, $L_p(\mathbb{R}^d)$ is replaced by $L_p(\Omega)$. The modifications in the proof are obvious.

Lemma 2.4 *Let* $0 < p, q \le \infty$ *and* $\delta > 0$. *Let* $\{g_{\bar{k}}\}_{\bar{k}\in\mathbb{N}_0^d}$ *be a sequence of nonnegative measurable functions on* $\mathbb{R}^d$ *and put*

$$G_{\bar{\ell}}(x) = \sum_{\bar{k}\in\mathbb{N}_0^d} 2^{-|\bar{k}-\bar{\ell}|_1\delta} g_{\bar{k}}(x) \quad , \quad x \in \mathbb{R}^d .$$

Then there is some constant $C = C(p, q, \delta)$, *such that*

$$\|G_{\bar{\ell}}|\ell_q(L_p(\Omega))\| \le C\|g_{\bar{k}}|\ell_q(L_p(\Omega))\|$$

and

$$\|G_{\bar{\ell}}|L_p(\ell_q)\| \le C\|g_{\bar{k}}|L_p(\ell_q, \Omega)\|$$

hold true.

2.7 Admissible domains

We need to explain, what we shall understand by a "rectangular" domain. Let $b \in \{-1, 1\}$ and $\mathcal{I}_b = b \cdot (0, \infty)$. According to a given vector $\bar{b} \in \{-1, 1\}^d$ we put

$$\mathcal{D}_{\bar{b}} = \{M_1 \times \cdots \times M_d \ : \ M_i = \mathbb{R} \vee M_i = \mathcal{I}_{b_i}, i = 1, ..., d\} .$$

By definition, $\mathcal{Q}_{\bar{b}} \in \mathcal{D}_{\bar{b}}$, where $\mathcal{Q}_{\bar{b}} = \mathcal{I}_{b_1} \times \cdots \times \mathcal{I}_{b_d}$ and for an arbitrary $\Omega \in \mathcal{D}_{\bar{b}}$ we have

$$x + \mathcal{Q}_{\bar{b}} \subset \Omega \tag{11}$$

for all $x \in \Omega$. This construction provides the following. If $\varphi \in D(-\mathcal{Q}_{\bar{b}})$ then the dilated function $\varphi_{\bar{a}}(a_1\cdot, ..., a_d\cdot)$, where $\bar{a} = (a_1, ..., a_d)$ denotes a vector with strictly positive components, is an element of $D(-\mathcal{Q}_{\bar{b}})$ as well. Now (11) ensures that $(\varphi_{\bar{a}} * f)(x)$ makes sense for $f \in D'(\Omega)$ (see Paragraph 2.3). We define a set Ω to be admissible if it belongs to the class

$$\mathcal{D} = \bigcup_{\substack{\bar{b}\in\{-1,1\}^d \\ y\in\mathbb{R}^d}} y + \mathcal{D}_{\bar{b}} ,$$

where $y + \mathcal{D}_{\bar{b}} = \{y + M \ : \ M \in \mathcal{D}_{\bar{b}}\}$. In other words, $\mathcal{D}$ is the collection of all sets Ω satisfying (1) where the $\mathcal{I}^k$, $k = 1, ..., d$, denote infinite (connected) intervals in $\mathbb{R}$.

2.8 Function spaces on $\mathbb{R}^d$

For the definition of the B- and F-spaces with dominating mixed smoothness on $\mathbb{R}^d$ we will rather use an equivalent characterization given in [22, Thm. 1.23] than the classical approach in [13, Chapt. 2]. We start with a family of functions $\varphi_0^i \in S(\mathbb{R})$, $i = 1, ..., d$, satisfying

$$\begin{cases} \int\limits_{\mathbb{R}} \varphi_0^i(x)\,dx \neq 0 \quad \text{and} \\ \qquad\quad L_{\varphi^i} \geq R_i \quad \text{for} \quad \varphi(x) = \varphi_0(x) - 1/2\varphi(x/2). \end{cases} \tag{12}$$

Note that such a family of functions always exist for a given $\bar{R} \in \mathbb{R}^d$. Choose for instance $\varphi^i \in D(-\infty, 0)$ like it is done in [4, Lem. 3.16].
The conditions in (12) are equivalent to the ones in [22, Thm. 1.23]. This can be seen by well-known properties of the Fourier transform.
Here and subsequently we will use the following notation. For $\bar{k} \in \mathbb{N}_0^d$ we put

$$\varphi_{\bar{k}}(x_1, ..., x_d) = \varphi_{k_1}^1(x_1) \cdot ... \cdot \varphi_{k_d}^d(x_d)\,.$$

See Paragraph 2.1 for the meaning of $\varphi_{k_i}^i(x)$.

Definition 2.5 *Let $\bar{r} \in \mathbb{R}^d$, $\bar{R} + 1 > \bar{r}$ and φ_0^i, $i = 1, ..., d$, be given by (12).*
(i) *Let $0 < p, q \leq \infty$ and $\bar{a} > 1/p$. Then $S_{p,q}^{\bar{r}}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that*

$$\begin{aligned} \|f|S_{p,q}^{\bar{r}}B(\mathbb{R}^d)\| &= \|2^{\bar{k}\bar{r}}(\varphi_{\bar{k}} * f)|\ell_q(L_p)\| \\ &\asymp \|2^{\bar{k}\bar{r}}(\varphi_{\bar{k}}^* f)_{\bar{a}}|\ell_q(L_p)\| < \infty\,. \end{aligned} \tag{13}$$

(ii) *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\bar{a} > 1/\min(p,q)$. Then $S_{p,q}^{\bar{r}}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that*

$$\begin{aligned} \|f|S_{p,q}^{\bar{r}}F(\mathbb{R}^d)\| &= \|2^{\bar{k}\bar{r}}(\varphi_{\bar{k}} * f)|L_p(\ell_q)\| \\ &\asymp \|2^{\bar{k}\bar{r}}(\varphi_{\bar{k}}^* f)_{\bar{a}}|L_p(\ell_q)\| < \infty\,. \end{aligned} \tag{14}$$

2.9 Function spaces on domains

During this paragraph Ω denotes an arbitrary domain. Later we will restrict to admissible domains (see Paragraph 2.7). We start by defining the spaces $S_{p,q}^{\bar{r}}B(\Omega)$ and $S_{p,q}^{\bar{r}}F(\Omega)$ by restriction of the elements of the corresponding spaces on $\mathbb{R}^d$.

Definition 2.6

(i) If $0 < p, q \leq \infty$ *and* $\bar{r} \in \mathbb{R}^d$ *then* $S^{\bar{r}}_{p,q}B(\Omega)$ *is the space of all* $f \in D'(\Omega)$ *such that there exists a* $g \in S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$ *satisfying* $f = g|_\Omega$*. It is endowed with the quotient norm*

$$\|f|S^{\bar{r}}_{p,q}B(\Omega)\| = \inf\{\|g|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\| \ : \ g|_\Omega = f\}\,.$$

(ii) For $0 < p < \infty$, $0 < q \leq \infty$ *and* $\bar{r} \in \mathbb{R}^d$ *the space* $S^{\bar{r}}_{p,q}F(\Omega)$ *is defined analogously with norm*

$$\|f|S^{\bar{r}}_{p,q}F(\Omega)\| = \inf\{\|g|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\| \ : \ g|_\Omega = f\}\,.$$

3 Peetre type characterizations on domains

Let us first introduce the analogue of the Peetre maximal function (10) for a domain Ω. We will use the more or less obvious modification. Let $K, \Omega \subset \mathbb{R}^d$ domains such that $K + \Omega \subset \Omega$. Let further $\{\psi_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d} \subset D(-K)$ and $f \in D'(\Omega)$ a distribution. Analogously to (10) we define with $\bar{a} > 0$ the system of maximal functions on Ω

$$(\psi^{\Omega}_{\bar{k}} f)_{\bar{a}}(x) = \sup_{y \in \Omega} \frac{|(\psi_{\bar{k}} * f)(y)|}{\prod\limits_{i=1}^{d}(1 + 2^{k_i}|x_i - y_i|)^{a_i}} \quad , \quad x \in \Omega\,. \tag{15}$$

Next, we state a decomposition result proved by Rychkov in [9], which represents somehow the core of the matter. We just need a special case. Here $\delta \in D'(\mathbb{R})$ denotes the well-known δ-function given by

$$\delta(\varphi) = \varphi(0) \quad , \quad \varphi \in D(\mathbb{R})\,.$$

Lemma 3.1 *Let* $\varphi_0 \in D(0, \infty)$ *having nonzero integral and let* $\varphi(x) = \varphi_0(x) - 1/2\varphi(x/2)$ *(compare with (12)). Then for any given* $L \in \mathbb{N}_0$ *there exist functions* $\psi_0, \psi \in D(0, \infty)$ *such that* $L_\psi \geq L$ *and*

(i)

$$\delta = \sum_{j=0}^{\infty} \psi_j * \varphi_j \quad in \quad D'(\mathbb{R})\,,$$

(ii)

$$\sum_{j=0}^{k} \psi_j * \varphi_j = (\psi_0 * \varphi_0)_k\,,$$

(iii)

$$\int_{\mathbb{R}} \psi_0 * \varphi_0 \, dx = 1\,.$$

Proof. We refer mainly to Rychkov [9, Prop. 2.1]. In the proof given there formula (2.5) corresponds to (i). To see (ii) let us first put $\gamma_0 = \psi_0 * \varphi_0$ and $\gamma = \psi * \varphi$. An elementary calculation yields $\gamma_j = \psi_j * \varphi_j$ for $j = 0, 1, ...$ We further put $f_2(\varphi) = f(\varphi(2\cdot))$ for $f \in D'(\mathbb{R})$ which gives in particular $\delta_2 = \delta$. Hence, one can rewrite (i) in the following way

$$\begin{aligned}\delta = \delta_2 &= 1/2\gamma_0(1/2\cdot) + \frac{1}{2}\sum_{j=1}^{\infty}\gamma_j(1/2\cdot)\\ &= 1/2\gamma_0(1/2\cdot) + \sum_{j=1}^{\infty} 2^{j-1}\gamma(2^{j-1}\cdot)\\ &= 1/2\gamma_0(1/2\cdot) + \gamma(\cdot) + \sum_{j=1}^{\infty} 2^j\gamma(2^j\cdot)\,.\end{aligned}$$

Using (i) again we get

$$\delta = 1/2\gamma_0(1/2\cdot) + \gamma(\cdot) - \gamma_0(\cdot) + \delta$$

and end up with

$$\gamma(\cdot) = \gamma_0(\cdot) - 1/2\gamma_0(1/2\cdot)\,.$$

Therefore,

$$\sum_{j=0}^{k}\gamma_j = (\gamma_0)_k \quad , \quad k \in \mathbb{N}\,.$$

To see (iii) we choose $\eta \in D'(\mathbb{R})$ such that $\eta(x) = 1$ if $|x| \le \varepsilon$ and obtain for k big enough

$$\int_{\mathbb{R}} \psi_0 * \varphi_0(x)\,dx = \int_{\mathbb{R}} (\psi_0 * \varphi_0)_k(x)\,dx = (\psi_0 * \varphi_0)_k(\eta) \xrightarrow[k\to\infty]{} \delta(\eta) = 1.$$

■

Remark 3.2 *Lemma 3.1 also holds true if $\varphi_0 \in D(-\infty, 0)$. Then we find $\psi_0, \psi \in D(-\infty, 0)$ such that (i), (ii) and (iii) are valid.*

Proposition 3.3 *Let $\bar{b} \in \{-1, 1\}^d$, $\Omega \in \mathcal{D}_{\bar{b}}$ and $\varphi_0^i \in D(-\mathcal{I}_{b_i})$ having nonzero integral for $i = 1, ..., d$. Let further $\varphi^i(x) = \varphi_0^i(x) - 1/2\varphi(x/2)$. Then for any given $\bar{L} \in \mathbb{N}_0^d$ there exist functions $\psi_0^i, \psi^i \in D(-\mathcal{I}_{b_i})$ such that $L_{\psi^i} \ge L_i$ and*

$$f = \sum_{\bar{\ell}\in\mathbb{N}_0^d} \psi_{\bar{\ell}} * \varphi_{\bar{\ell}} * f \quad (\textit{convergence in} \quad D'(\Omega))$$

for all $f \in D'(\Omega)$.

Proof. Using the previous lemma, we find functions ψ_0^i, ψ^i such that in particular

$$\sum_{\ell_1=0}^{k_1} \cdots \sum_{\ell_d=0}^{k_d} \psi_{\bar{\ell}} * \varphi_{\bar{\ell}}(x) = (\psi_0^1 * \varphi_0^1)_{k_1}(x_1) \cdot ... \cdot (\psi_0^d * \varphi_0^d)_{k_d}(x_d) \quad , \quad x \in -\mathcal{Q}_{\bar{b}}\,, \bar{k} \in \mathbb{N}_0^d\,.$$

Recall Paragraph 2.3/(7). We want to prove that

$$f\Big(\gamma * \sum_{\ell_1=0}^{k_1} \cdots \sum_{\ell_d=0}^{k_d} \psi_{\bar{\ell}} * \varphi_{\bar{\ell}}(-\cdot)\Big) \xrightarrow[\bar{k}\to\infty]{} f(\gamma) \quad , \quad \gamma \in D(\Omega)\,.$$

Since f is a continuous linear functional on $D(\Omega)$ it suffices to prove that

$$\gamma_{\bar{k}} = \gamma * \sum_{\ell_1=0}^{k_1} \cdots \sum_{\ell_d=0}^{k_d} \psi_{\bar{\ell}} * \varphi_{\bar{\ell}}(-\cdot) \xrightarrow[\bar{k}\to\infty]{} \gamma \quad \text{in} \quad D(\Omega)\,.$$

Obviously, $\operatorname{supp} \gamma_k \subset M \subset \Omega$ for all $\bar{k} \in \mathbb{N}_0^d$, where M is closed and bounded. Using Lemma 3.1/(iii) we obtain for $\bar{\alpha} \in \mathbb{N}_0^d$

$$|D^{\bar{\alpha}}\gamma(x) - D^{\bar{\alpha}}\gamma_{\bar{k}}(x)| \leq \int\limits_{\mathbb{R}^d} |D^{\bar{\alpha}}\gamma(x) - D^{\bar{\alpha}}\gamma(x_1 - 2^{-k_1}y_1, ..., x_d - 2^{-k_d}y_d)| \times$$
$$\times\, |(\psi_0^1 * \varphi_0^1)(y_1) \cdot ... \cdot (\psi_0^d * \varphi_0^d)(y_d)|\, dy\,,$$

where the right-hand side converges uniformly to 0 if $\bar{k}$ goes to infinity. ■

Let us state the main result of this section. We are going to characterize the spaces $S^{\bar{r}}_{p,q}B(\Omega)$ and $S^{\bar{r}}_{p,q}F(\Omega)$ in terms of Peetre type maximal functions (15) for the full range of parameters.

Theorem 3.4 *Let $\bar{b} \in \{-1,1\}^d$ and $\Omega \in \mathcal{D}_{\bar{b}}$. Let further $0 < p, q \leq \infty$ ($p < \infty$ in the F-case), $\bar{r} \in \mathbb{R}^d$, $\bar{R} + 1 > \bar{r}$ and $\varphi_0^i \in D(-\mathcal{I}_{b_i})$ be given by (12) for $i = 1, ..., d$. Then*

(i) if $\bar{a} > 1/p$ we have

$$\|f|S^{\bar{r}}_{p,q}B(\Omega)\| \asymp \Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} \|(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}|L_p(\Omega)\|^q\Big)^{1/q} \tag{16}$$

(ii) and if $\bar{a} > 1/\min(p,q)$ we get

$$\|f|S^{\bar{r}}_{p,q}F(\Omega)\| \asymp \Big\|\Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} |(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}(x)|^q\Big)^{1/q}\Big|L_p(\Omega)\Big\| \tag{17}$$

for all $f \in D'(\Omega)$.

Proof. As always we restrict us to the proof of (ii). The proof of (i) follows similar but with much simpler arguments.

Step 1. We start with an $f \in D'(\Omega)$ and $g \in S'(\mathbb{R}^d)$ satisfying $g|_\Omega = f$. Clearly, we have

$$(\varphi_{\bar{\ell}} * f)(y) = (\varphi_{\bar{\ell}} * g)(y) \quad , \quad y \in \Omega \,.$$

Hence,

$$(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}(x) \leq (\varphi_{\bar{\ell}}^{*} g)_{\bar{a}}(x) \quad , \quad x \in \Omega \,.$$

With Definition 2.5/(ii) we conclude

$$\|g|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\| \geq \Big\| \Big(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} |(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}(x)|^q \Big)^{1/q} \Big| L_p(\Omega) \Big\|$$

for all $g \in S'(\mathbb{R}^d)$ with $g|_\Omega = f$, which implies "$\geq$" in (ii) (see also Definition 2.6/(ii)).

Step 2. The main goal of this step is to construct a linear bounded extension operator

$$\mathcal{E} : S^{\bar{r}}_{p,q}F(\Omega) \to S^{\bar{r}}_{p,q}F(\mathbb{R}^d) \,.$$

To gain this we combine the work of Vybiral [22] and the ideas of Rychkov [9] for the isotropic case. Proposition 3.3 tells us that for a given vector $\bar{L} \in \mathbb{N}_0^d$ we can find functions $\psi_0^i, \psi^i \in D(-\mathcal{I}_{b_i})$ with $L_{\psi^i} \geq L$ such that

$$f = \sum_{\bar{\ell} \in \mathbb{N}_0^d} \psi_{\bar{\ell}} * \varphi_{\bar{\ell}} * f \quad \text{in} \quad D'(\Omega). \tag{18}$$

Let us assume the finiteness of rhs(17). Otherwise there is nothing to prove. We claim that the mapping $\mathcal{E}$ given by

$$\mathcal{E} f = \sum_{\bar{k} \in \mathbb{N}_0^d} \psi_{\bar{k}} * (\varphi_{\bar{k}} * f)_\Omega$$

is well-defined in $S'(\mathbb{R}^d)$ under this condition. Moreover, we will prove $\mathcal{E} f|_\Omega = f$ and

$$\|\mathcal{E} f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\| \leq c \Big\| \Big(\sum_{\bar{k} \in \mathbb{N}_0^d} 2^{\bar{r}\bar{k}q} |(\varphi_{\bar{k}}^{\Omega} f)_{\bar{a}}(x)|^q \Big)^{1/q} \Big| L_p(\Omega) \Big\| \,, \tag{19}$$

where c is independent of f. This implies immediately "$\leq$" in (ii).

We put $g_{\bar{k}}(x) = (\varphi_{\bar{k}} * f)_\Omega(x)$ and

$$G_{\bar{k},\bar{a}}(x) = \sup_{y \in \mathbb{R}^d} \frac{|g_{\bar{k}}(y)|}{\prod_{i=1}^{d} (1 + 2^{k_i}|x_i - y_i|)^{a_i}} \quad , \quad x \in \mathbb{R}^d \,. \tag{20}$$

Now we estimate as follows

$$
\begin{aligned}
|(\varphi_{\bar{\ell}} * \psi_{\bar{k}} * g_{\bar{k}})(x)| &\le \int\limits_{\mathbb{R}^d} |(\varphi_{\bar{\ell}} * \psi_{\bar{k}})(z)| \cdot |g_{\bar{k}}(x-z)|\, dz \\
&\le G_{\bar{k},\bar{a}}(x) \int\limits_{\mathbb{R}^d} |(\varphi_{\bar{\ell}} * \psi_{\bar{k}})(z)| \cdot \prod_{i=1}^{d} (1 + 2^{k_i}|z_i|)^{a_i}\, dz \\
&\le G_{\bar{k},\bar{a}}(x) \prod_{i=1}^{d} J^i_{\ell_i,k_i}\,,
\end{aligned}
$$

where

$$
J^i_{\ell_i,k_i} = \int\limits_{\mathbb{R}} |(\varphi^i_{\ell_i} * \psi^i_{k_i})(t)|(1+2^{k_i}|t|)^{a_i}\, dt \quad , \quad i = 1, ..., d\,. \tag{21}
$$

We first observe that for $t \in \mathbb{R}$

$$
(\varphi^i_{\ell_i} * \psi^i_{k_i})(t) = 2^{k_i}[2^{\ell_i - k_i}\varphi(2^{\ell_i - k_i}\cdot) * \psi](2^{k_i}t) = 2^{\ell_i}[\varphi * 2^{k_i-\ell_i}\psi(2^{k_i-\ell_i}\cdot)](2^{\ell_i}t) \tag{22}
$$

with a proper modification if $k_i = 0$ or $\ell_i = 0$. If $\ell_i \ge k_i$ we put the second term from (22) into the corresponding term in (21). The change of variable $2^{k_i}t \to t$ and Lemma 2.3 yield

$$
\begin{aligned}
J^i_{\ell_i,k_i} &= \int\limits_{\mathbb{R}} [\varphi_{\ell_i - k_i} * \psi](t)(1+|t|)^{a_i}\, dt \\
&\le c \sup_{t\in\mathbb{R}} [\varphi_{\ell_i-k_i} * \psi](t)(1+|t|)^{a_i+2} \\
&\le c2^{(k_i-\ell_i)(R_i+1)}\,.
\end{aligned}
$$

If $k_i > \ell_i$ we use the third term in (22) to put it into (21) and obtain similar, but this time with $2^{\ell_i}t \to t$, the estimate

$$
\begin{aligned}
J^i_{\ell_i,k_i} &= \int\limits_{\mathbb{R}} [\varphi * \psi_{k_i-\ell_i}](t)(1+|2^{k_i-\ell_i}t|)^{a_i}\, dt \\
&\le c2^{(k_i-\ell_i)a_i} \sup_{t\in\mathbb{R}} [\varphi * \psi_{k_i-\ell_i}](t)(1+|t|)^{a_i+2} \\
&\le c2^{(\ell_i-k_i)(L_i+1-a_i)}\,,
\end{aligned}
$$

where L_i, $i = 1, ..., d$, corresponds to L_{ψ^i} and can be chosen sufficiently large according to Proposition 3.3. Therefore we obtain

$$
\begin{aligned}
2^{\bar{r}\bar{\ell}}|(\varphi_{\bar{\ell}} * \psi_{\bar{k}} * g_{\bar{k}})(x)| &\le G_{\bar{k},\bar{a}}(x) 2^{\bar{k}\bar{r}} \prod_{i=1}^{d} J^i_{\ell_i,k_i} 2^{(\ell_i-k_i)r_i} \\
&\le G_{\bar{k},\bar{a}}(x) 2^{\bar{k}\bar{r}} \prod_{i=1}^{d} \left\{ \begin{array}{lcl} 2^{(\ell_i-k_i)(L_i+1-a_i+r_i)} & : & k_i > \ell_i \\ 2^{(k_i-\ell_i)(R_i+1-r_i)} & : & \ell_i \ge k_i \end{array} \right. .
\end{aligned}
$$

We choose $L_i \geq a_i + |r_i|$ and obtain with

$$\delta = \min\{1, R_i + 1 - r_i \ : \ i = 1, ..., d\}$$

the estimate

$$2^{\bar{r}\bar{\ell}}|(\varphi_{\bar{\ell}} * \psi_{\bar{k}} * g_{\bar{k}})(x)| \leq c\, G_{\bar{k},\bar{a}}(x) 2^{\bar{k}\bar{r}} 2^{-|\bar{\ell}-\bar{k}|_1 \delta} \,. \tag{23}$$

Next we show the convergence of the sum $\sum_{\bar{k}\in\mathbb{N}_0^d} \psi_{\bar{k}} * g_{\bar{k}}$ in $S'(\mathbb{R}^d)$ whenever $\{2^{\bar{r}\bar{k}} G_{\bar{k},\bar{a}}\} \in L_p(\ell_q)$. Let us refer here to (25) below to motivate this condition. First of all, this condition ensures that (20) belongs to $L_p(\mathbb{R}^d)$ for all $\bar{k} \in \mathbb{N}_0^d$. Hence $g_{\bar{k}}$ has at most polynomial growth and is therefore an element of $S'(\mathbb{R}^d)$. Hence, all $\psi_{\bar{k}} * g_{\bar{k}}$ represent elements of $S'(\mathbb{R}^d)$ as well. Moreover, we conclude

$$\sum_{\bar{k}\in\mathbb{N}_0^d} \|\psi_{\bar{k}} * g_{\bar{k}} | S^{\bar{r}-2\delta}_{p,q} F(\mathbb{R}^d)\| < \infty$$

by (23). Namely,

$$\begin{aligned}
\|\psi_{\bar{k}} * g_{\bar{k}} | S^{\bar{r}-2\delta}_{p,q} F(\mathbb{R}^d)\| &= \|2^{-2\delta|\bar{\ell}|_1} 2^{\bar{r}\bar{\ell}} \varphi_{\bar{\ell}} * \psi_{\bar{k}} * g_{\bar{k}} | L_p(\ell_q)\| \\
&\leq c\|2^{-2\delta|\bar{\ell}|_1} 2^{-|\bar{\ell}-\bar{k}|_1\delta} 2^{\bar{k}\bar{r}} G_{\bar{k},\bar{a}} | L_p(\ell_q)\| \\
&= c 2^{\bar{k}\bar{r}} \|G_{\bar{k}} | L_p(\mathbb{R}^d)\| \Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{-\delta q(2|\bar{\ell}|_1 + |\bar{\ell}-\bar{k}|_1)} \Big)^{1/q} \\
&\leq c\|2^{\bar{k}\bar{r}} G_{\bar{k},\bar{a}} | L_p(\ell_q)\| 2^{-\delta|\bar{k}|_1} \Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{-\delta q|\bar{\ell}|_1} \Big)^{1/q} \\
&\leq c 2^{-\delta|\bar{k}|_1} \|2^{\bar{k}\bar{r}} G_{\bar{k},\bar{a}} | L_p(\ell_q)\| \,.
\end{aligned}$$

Hence, the sum converges in $S^{\bar{r}-2\delta}_{p,q} F(\mathbb{R}^d)$. Because of the topological embedding in $S'(\mathbb{R}^d)$ we have the convergence of $g = \mathcal{E} f = \sum_{\bar{k}\in\mathbb{N}_0^d} \psi_{\bar{k}} * g_{\bar{k}}$ in $S'(\mathbb{R}^d)$. By definition ($S'(\mathbb{R}^d)$ is equipped with the weak topology, cf. also (6)) we have for every $\bar{\ell} \in \mathbb{N}_0^d$ the pointwise identity

$$(\varphi_{\bar{\ell}} * g)(x) = \sum_{\bar{k}\in\mathbb{N}_0^d} (\varphi_{\bar{\ell}} * \psi_{\bar{k}} * g_{\bar{k}})(x) \quad , \quad x \in \mathbb{R}^d \,.$$

This together with (23) yields

$$2^{\bar{r}\bar{\ell}} |(\varphi_{\bar{\ell}} * g)(x)| \leq c \sum_{\bar{k}\in\mathbb{N}_0^d} G_{\bar{k},\bar{a}}(x) 2^{\bar{k}\bar{r}} 2^{-|\bar{\ell}-\bar{k}|_1\delta} \,. \tag{24}$$

And by Definition 2.5/(ii) we derive

$$\|g | S^{\bar{r}}_{p,q} F(\mathbb{R}^d)\| \leq c\|H_{\bar{\ell}} | L_p(\ell_q)\| \,,$$

where $\{H_{\bar\ell}\}_{\bar\ell\in\mathbb{N}_0^d}$ is given by the right-hand side of (24). Applying Lemma 2.4 we see immediately

$$\|g|S^{\bar r}_{p,q}F(\mathbb{R}^d)\| \le c\|2^{\bar k\bar r}G_{\bar k,\bar a}|L_p(\ell_q)\|\,.$$

It remains to prove that

$$\|2^{\bar k\bar r}G_{\bar k,\bar a}|L_p(\ell_q)\| \le \Big\|\Big(\sum_{\bar k\in\mathbb{N}_0^d}2^{\bar r\bar k q}|(\varphi^\Omega_{\bar k}f)_{\bar a}(x)|^q\Big)^{1/q}\Big|L_p(\Omega)\Big\|\,. \tag{25}$$

Now $g_{\bar k}=(\varphi_{\bar k}*f)_\Omega$ and (20) imply that for $x\in\mathbb{R}^d$

$$G_{\bar k,\bar a}(x)=\sup_{y\in\Omega}\frac{(\varphi_k*f)(y)}{\prod_{i=1}^d(1+2^{k_i}|x_i-y_i|)^{a_i}} \le \begin{cases}(\varphi^\Omega_{\bar k}f)_{\bar a}(x) & : \ x\in\Omega\\ (\varphi^\Omega_{\bar k}f)_{\bar a}(A(x)) & : \ x\notin\bar\Omega\end{cases}\,. \tag{26}$$

Here $A(x)=(m^1(x_1),...,m^d(x_d))\in\bar\Omega$ is defined as follows. For $i=1,...,d$ the sets $\Omega_i=\{x_i : x\in\Omega\}$ represent the projections of Ω to every axis and for $t\in\mathbb{R}$ we define $m^i(t)=t$ if $t\in\Omega_i$ and $m^i(t)=-t$ otherwise. Clearly, Ω_i is either $(0,\infty)$, $(-\infty,0)$ or $\mathbb{R}$ (cf. Paragraph 2.7) and $\Omega=\Omega_1\times...\times\Omega_d$. Decomposing lhs(25) and using (26) yields

$$\begin{aligned}&\|2^{\bar k\bar r}G_{\bar k,\bar a}|L_p(\ell_q)\|\\ &\le \Big\|\Big(\sum_{\bar k\in\mathbb{N}_0^d}2^{\bar r\bar k q}|(\varphi^\Omega_{\bar k}f)_{\bar a}(x)|^q\Big)^{1/q}\Big|L_p(\Omega)\Big\| + \sum_{\bar e:\mathcal{I}_{\bar e}\not\subseteq\Omega}\Big\|\Big(\sum_{\bar k\in\mathbb{N}_0^d}2^{\bar r\bar k q}|(\varphi^\Omega_{\bar k}f)_{\bar a}(A(x))|^q\Big)^{1/q}\Big|L_p(\mathcal{I}_{\bar e})\Big\|.\end{aligned}$$

Observe that $A|_{\mathcal{I}_{\bar e}}\to\Omega$ is affine and injective for every $\bar e$. Hence, by change of variable, every summand is less or equal than the first one and we are done with proving (19). Let us finally point out that for $\gamma\in D(\Omega)$ we obtain by definition

$$\begin{aligned}(\mathcal{E}f)(\gamma) &= \sum_{\bar k\in\mathbb{N}_0^d}[\psi_{\bar k}*(\varphi_{\bar k}*f)_\Omega](\gamma)\\ &= \sum_{\bar k\in\mathbb{N}_0^d}(\varphi_{\bar k}*f)_\Omega(\psi_{\bar k}(-\cdot)*\gamma)\,.\end{aligned}$$

It is $\psi_{\bar k}(-\cdot)\in D(\mathcal{Q}_{\bar b})$ and $\psi_{\bar k}(-\cdot)*\gamma\in D(\Omega)$ (see Paragraph 2.3). Then of course

$$(\varphi_{\bar k}*f)_\Omega(\psi_{\bar k}(-\cdot)*\gamma)=(\varphi_{\bar k}*f)(\psi_{\bar k}(-\cdot)*\gamma)=[\psi_{\bar k}*\varphi_{\bar k}*f](\gamma)$$

and by (18)

$$(\mathcal{E}f)(\gamma)=\sum_{\bar k\in\mathbb{N}_0^d}[\psi_{\bar k}*\varphi_{\bar k}*f](\gamma)=f(\gamma).$$

Consequently, we have $\mathcal{E}f|_\Omega=f$. ■

Remark 3.5 *For simplicity we stated the theorem only for* $\Omega \in \mathcal{D}_{\bar{b}}$. *Of course, the theorem also holds true for all* $\Gamma \in \mathcal{D}_{\bar{b}} + y$, $y \in \mathbb{R}^d$. *This is the consequence of a simple translation argument. Due to Definitions 2.5, 2.6 and*

$$[\varphi * f(\cdot - y)](\cdot) = (\varphi * f)(\cdot - y)$$

the distribution f *belongs to* $S^{\bar{r}}_{p,q}A(\Omega)$ *if and only if* $f(\cdot - y)$ *belongs to* $S^{\bar{r}}_{p,q}A(y + \Omega)$.

4 Characterization via local means

This section is devoted to the characterization of the spaces $S^{\bar{r}}_{p,q}B(\Omega)$ and $S^{\bar{r}}_{p,q}F(\Omega)$ in terms of local averages or "pure" convolutions. As in the previous section, Ω denotes an admissible domain. The approach given here is a modified mixture of the approaches of Rychkov in [9, Thm. 3.2] and Vybiral in [22, Thm. 1.22]. For the intended characterization via local means we have to restrict to distributions from $S'(\Omega)$ (see Paragraph 2.2/(4)). This is a technical assumption and we do not know if it can be relaxed. Let us start with the following proposition to specify all distributions from $S'(\Omega)$ as restrictions of tempered distributions. The following result is a proper modification of the corresponding one proved by Rychkov in [9] for the isotropic case and Lipschitz domains

Proposition 4.1 *Let* $\Omega \in \mathcal{D}$. *For every* $f \in S'(\Omega)$ *there is a* g *such that* $g|_\Omega = f$.

Proof. Assume (4) and consider the distribution

$$F(x) = (1 + |x|)^{-L} f \in D'(\Omega)\,,$$

where L is supposed to be large. We are going to show that $F \in S^{\bar{r}}_{\infty,\infty}B(\Omega)$, where $\bar{r}$ is determined later. Now we have for arbitrary $\psi \in D(\Omega)$ using (4)

$$\begin{aligned} |F(\psi)| &= f((1 + |x|^2)^{-L}\psi(x)) \\ &\le \sup_{x\in\Omega, |\bar{\alpha}|_1 \le M} |D^{\bar{\alpha}}[(1 + |\cdot|^2)^{-L}\psi(\cdot)](x)|(1 + |x|)^M \\ &\le \sup_{x\in\Omega, |\bar{\alpha}|_1 \le M} (1 + |x|)^M \sum_{\bar{\beta}\le\bar{\alpha}} |D^{\bar{\alpha}-\bar{\beta}}[(1 + |\cdot|^2)^{-L}](x)| \cdot |D^{\bar{\beta}}\psi(x)|\,. \end{aligned}$$

Clearly, we have

$$\sum_{\bar{\beta}\le\bar{\alpha}} |D^{\bar{\alpha}-\bar{\beta}}[(1 + |\cdot|^2)^{-L}](x)| \le c(1 + |x|^2)^{-L}\,.$$

Hence, if $L > M/2$ we obtain

$$|F(\psi)| \le c \sup_{x\in\Omega, |\bar{\alpha}|_1 \le M} |D^{\bar{\alpha}}\psi(x)|$$

Let now φ_0^i, $i = 1, ..., d$, be given according to Theorem 3.4. Then

$$\begin{aligned} |(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}(x)| &\leq \sup_{y\in\Omega} |F(\varphi_{\bar{\ell}}(y-\cdot))| \\ &\leq c \sup_{x\in\mathbb{R}^d, |\bar{\alpha}|_1\leq M} |D^{\bar{\alpha}}\varphi_{\bar{\ell}}(x)| \\ &\leq c 2^{|\bar{\ell}|_1(M+1)} . \end{aligned}$$

Consequently, if $r \leq -(M+1)$ we obtain by Theorem 3.4 that $F \in S^{\bar{r}}_{\infty,\infty}B(\Omega)$. Hence, by definition there exists $G \in S'(\mathbb{R}^d)$ such that $F = G|_\Omega$. Finally the distribution $g = (1+|x|)^L G$ satisfies $g|_\Omega = f$. ■

This proposition shows the identity

$$S'(\Omega) = S'(\mathbb{R}^d)|_\Omega = \{g|_\Omega \ : \ g \in S'(\mathbb{R}^d)\} .$$

The following theorem gives a characterization of $S^{\bar{r}}_{p,q}B(\Omega)$ and $S^{\bar{r}}_{p,q}F(\Omega)$ in terms of pure convolutions. Of course, according to their definition it is quite natural to ask just distributions from $S'(\Omega) = S'(\mathbb{R}^d)|_\Omega$. However, if we start with a distribution f from $D'(\Omega)$ it is not clear how to decide if such a distribution has an extension in $S'(\mathbb{R}^d)$. In general this is a kind of disadvantage in comparison to Theorem 3.4. Anyhow, if $\bar{r}$ is large enough we deal with spaces of functions from $L_p(\Omega) \cap L_1(\Omega)$ which represent tempered distributions after extending by 0 outside Ω. For all practical concerns the following main result represents a powerful tool.

Theorem 4.2 *Let $\bar{b} \in \{-1,1\}^d$ and $\Omega \in \mathcal{D}_{\bar{b}}$. Let further $0 < p,q \leq \infty$ ($p < \infty$ in the F-case), $\bar{r} \in \mathbb{R}^d$, $\bar{R}+1 > \bar{r}$ and $\varphi_0^i \in D(-\mathcal{I}_{b_i})$ be given by (12) for $i = 1, ..., d$. Then*

(i) $\|f|S^{\bar{r}}_{p,q}B(\Omega)\| \asymp \Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q}\|\varphi_{\bar{\ell}} * f|L_p(\Omega)\|^q\Big)^{1/q}$ *and*

(ii) $\|f|S^{\bar{r}}_{p,q}F(\Omega)\| \asymp \Big\|\Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q}|(\varphi_{\bar{\ell}} * f)(x)|^q\Big)^{1/q}\Big|L_p(\Omega)\Big\|$

for all $f \in S'(\Omega)$.

Proof. We divide the proof of the "$\leq$" inequality in (ii) into three steps.

Step 1. Using Proposition 3.3 and the decomposition of a distribution f in a sum of convolutions we want to obtain a somehow index shifted representation for $(\varphi_{\bar{\ell}} * f)(x)$. Let us start with some necessary definitions. For $\gamma \in D(\Omega)$ we define for $\bar{u} \in \mathbb{Z}^d$ the function $\gamma^{\bar{u}}(x) = 2^{|\bar{u}|_1}\gamma(2^{u_1}x_1, ..., 2^{u_d}x_d)$ which is again an element of $D(\Omega)$. Similarly, starting with $g \in D'(\Omega)$ the distribution $g^{\bar{u}} \in D'(\Omega)$ is defined by $g^{\bar{u}}(\gamma) = g(\gamma^{-\bar{u}})$ for $\gamma \in D(\Omega)$. Let $\bar{N} \in \mathbb{N}_0^d$ be an arbitrary vector of nonnegative integers. By Proposition 3.3 we find functions $\psi^i, \psi_0^i \in D(-\mathcal{I}_{b_i})$ such that $L_{\psi^i} \geq N_i$, $i = 1, ..., d$, and

$$f^{-\bar{\ell}} = \sum_{\bar{m} \in \mathbb{N}_0^d} \psi_{\bar{m}} * \varphi_{\bar{m}} * f^{-\bar{\ell}} \tag{27}$$

holds in $D'(\Omega)$. Dilating (27) and using $(\eta * f^{-\bar{\ell}})^{\bar{\ell}} = (\eta^{\bar{\ell}} * f)$ we obtain

$$\begin{aligned} f &= \sum_{\bar{m} \in \mathbb{N}_0^d} \psi_{\bar{m}}^{\bar{\ell}} * (\varphi_{\bar{m}} * f^{-\bar{\ell}})^{\bar{\ell}} \\ &= \sum_{\bar{m} \in \mathbb{N}_0^d} \psi_{\bar{m}}^{\bar{\ell}} * \varphi_{\bar{m}}^{\bar{\ell}} * f \,. \end{aligned}$$

Applied to the function $\varphi_{\bar{\ell}}(x - \cdot) \in D(\Omega)$ we obtain pointwise

$$(\varphi_{\bar{\ell}} * f)(x) = \sum_{\bar{m} \in \mathbb{N}_0^d} (\varphi_{\bar{\ell}} * \varphi_{\bar{m}}^{\bar{\ell}} * \psi_{\bar{m}}^{\bar{\ell}} * f)(x) \quad , \quad x \in \Omega \,. \tag{28}$$

Let us finally define for $k, \ell \in \mathbb{N}_0$

$$\sigma_{m,\ell}^i(t) = \begin{cases} 2^\ell \varphi_0^i(2^\ell t) & : \quad m = 0 \\ \varphi_\ell^i(t) & : \quad m > 0 \end{cases} \quad , \quad t \in \mathbb{R} \,,$$

and $\sigma_{\bar{m},\bar{\ell}}(x) = \prod_{i=1}^d \sigma_{m_i,\ell_i}^i(x_i)$ for $\bar{\ell}, \bar{m} \in \mathbb{N}_0^d$. Clearly, we have $\sigma_{\bar{m},\bar{\ell}} \in D(-\mathcal{Q}_{\bar{b}})$ and

$$\varphi_{\bar{\ell}} * \varphi_{\bar{m}}^{\bar{\ell}} = \sigma_{\bar{m},\bar{\ell}} * \varphi_{\bar{m}+\bar{\ell}}.$$

Plugging this into (28) we end up with the pointwise representation

$$\begin{aligned} (\varphi_{\bar{\ell}} * f)(y) &= \sum_{\bar{m} \in \mathbb{N}_0^d} (\psi_{\bar{m}}^{\bar{\ell}} * \sigma_{\bar{m},\bar{\ell}} * \varphi_{\bar{m}+\bar{\ell}} * f)(y) \\ &= \sum_{\bar{m} \in \mathbb{N}_0^d} [(\psi_{\bar{m}}^{\bar{\ell}} * \sigma_{\bar{m},\bar{\ell}}) * (\varphi_{\bar{m}+\bar{\ell}} * f)](y) \\ &= \sum_{\bar{m} \in \mathbb{N}_0^d} \int_\Omega (\psi_{\bar{m}}^{\bar{\ell}} * \sigma_{\bar{m},\bar{\ell}})(y - z) \cdot (\varphi_{\bar{m}+\bar{\ell}} * f)(z) \, dz \end{aligned} \tag{29}$$

for all $y \in \Omega$.

Step 2. Let us prove the following important inequality first. For every $s > 0$ and every vector $\bar{N} \in \mathbb{N}_0^d$ we have

$$|(\varphi_{\bar{\ell}} * f)(x)|^s \leq c \sum_{\bar{k} \in \mathbb{N}_0^d} 2^{-\bar{k}\bar{N}s} 2^{|\bar{k}+\bar{\ell}|_1} \int_\Omega \frac{|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)|^s}{\prod_{i=1}^d (1 + 2^{\ell_i}|x_i - y_i|)^{N_i s}} \, dy \,, \tag{30}$$

where c is independent of $f \in S'(\Omega)$, $x \in \Omega$ and $\bar{\ell} \in \mathbb{N}_0^d$.
The representation (29) will be the starting point to prove (30). Namely, we have for $y \in \Omega$

$$
\begin{aligned}
|(\varphi_{\bar{\ell}} * f)(y)| &\leq \sum_{\bar{m} \in \mathbb{N}_0^d} \int_\Omega |(\psi_{\bar{m}}^{\bar{\ell}} * \sigma_{\bar{m},\bar{\ell}})(y-z)| \cdot |(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|\, dz \\
&\leq \sum_{\bar{m} \in \mathbb{N}_0^d} \int_\Omega \frac{|(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|}{\prod_{i=1}^d (1+2^{\ell_i}|y_i - z_i|)^{N_i}}\, dz \prod_{i=1}^d S^i_{m_i,\ell_i}\,, \qquad (31)
\end{aligned}
$$

where

$$
S^i_{m_i,\ell_i} = \sup_{t\in\mathbb{R}} |[(\psi^i_{m_i})_{\ell_i} * \sigma^i_{m_i,\ell_i}](t)| \cdot (1+2^{\ell_i}|t|)^{N_i} \quad , \quad i = 1,...,d\,.
$$

Elementary properties of the convolution yield (compare with (22))

$$
\begin{aligned}
S^i_{m_i,\ell_i} &= 2^{\ell_i} \sup_{t\in\mathbb{R}} |(\psi^i_{m_i} * \eta^i_{m_i,\ell_i})(2^{\ell_i}t)| \cdot (1+2^{\ell_i}|t|)^{N_i} \\
&= 2^{\ell_i} \sup_{t\in\mathbb{R}} |(\psi^i_{m_i} * \eta^i_{m_i,\ell_i})(t)| \cdot (1+|t|)^{N_i}\,.
\end{aligned}
$$

where

$$
\eta^i_{m,\ell}(t) = \begin{cases} \varphi^i(t) & : \quad \ell > 0, m > 0 \\ \varphi^i_0(t) & : \quad \text{otherwise}\,. \end{cases}
$$

Lemma 2.3 yields

$$
S^i_{m_i,\ell_i} \leq c_{N_i} 2^{\ell_i} 2^{-m_i N_i}
$$

which we put into (31) to obtain

$$
|(\varphi_{\bar{\ell}} * f)(y)| \leq C_{\bar{N}} \sum_{\bar{m}\in\mathbb{N}_0^d} 2^{-\bar{m}\bar{N}} \int_\Omega \frac{2^{|\bar{m}+\bar{\ell}|_1} |(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|}{\prod_{i=1}^d (1+2^{\ell_i}|y_i - z_i|)^{N_i}}\, dz\,. \qquad (32)
$$

At this point we modify the proof of Theorem 1.22 in [22] essentially. For further explanation see Remark 4.4 below. We prefer the strategy used by Rychkov in [9, Thm. 3.2] and [10, Lem. 2.9], which is a variant of the Strömberg/Torchinsky technique introduced in [17, Chapt. V].
Let us continue by replacing $\bar{\ell}$ by $\bar{k} + \bar{\ell}$ in (32) and multiply on both sides with $2^{-\bar{k}\bar{N}}$.

Then we can estimate

$$2^{-\bar{k}\bar{N}}|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)| \leq C_{\bar{N}} \sum_{\bar{m}\in\mathbb{N}_0^d} 2^{-\bar{k}\bar{N}} 2^{-\bar{m}\bar{N}} \int_\Omega \frac{2^{|\bar{m}+\bar{k}+\bar{\ell}|_1}|(\varphi_{\bar{m}+\bar{k}+\bar{\ell}} * f)(z)|}{\prod_{i=1}^d (1+2^{k_i+\ell_i}|y_i-z_i|)^{N_i}}\, dz \quad (33)$$

$$\leq C_{\bar{N}} \sum_{\bar{m}\in\mathbb{N}_0^d} 2^{-(\bar{m}+\bar{k})\bar{N}} \int_\Omega \frac{2^{|\bar{m}+\bar{k}+\bar{\ell}|_1}|(\varphi_{\bar{m}+\bar{k}+\bar{\ell}} * f)(z)|}{\prod_{i=1}^d (1+2^{\ell_i}|y_i-z_i|)^{N_i}}\, dz$$

$$= C_{\bar{N}} \sum_{\bar{m}\in\bar{k}+\mathbb{N}_0^d} 2^{-\bar{m}\bar{N}} \int_\Omega \frac{2^{|\bar{m}+\bar{\ell}|_1}|(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|}{\prod_{i=1}^d (1+2^{\ell_i}|y_i-z_i|)^{N_i}}\, dz$$

$$\leq C_{\bar{N}} \sum_{\bar{m}\in\mathbb{N}_0^d} 2^{-\bar{m}\bar{N}} \int_\Omega \frac{2^{|\bar{m}+\bar{\ell}|_1}|(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|}{\prod_{i=1}^d (1+2^{\ell_i}|y_i-z_i|)^{N_i}}\, dz\,. \quad (34)$$

Next, we apply the elementary inequalities

$$(1+2^{\ell_i}|y_i-z_i|)\cdot(1+2^{\ell_i}|x_i-y_i|) \geq (1+2^{\ell_i}|x_i-z_i|)\,, \quad (35)$$

$$|(\varphi_{\bar{m}+\bar{\ell}} * f)(z)| \leq |(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|^s \prod_{i=1}^d (1+2^{\ell_i}|x_i-z_i|)^{N_i(1-s)} \times \sup_{y\in\Omega} \frac{|(\varphi_{\bar{m}+\bar{\ell}} * f)(y)|^{1-s}}{\prod_{i=1}^d (1+2^{\ell_i}|x_i-y_i|)^{N_i(1-s)}}\,,$$

where $0 < s \leq 1$. The use of the maximal function

$$M_{\bar{\ell},\bar{N}}(x) = \sup_{\bar{k}\in\mathbb{N}_0^d} \sup_{y\in\Omega} 2^{-\bar{k}\bar{N}} \frac{|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)|}{\prod_{i=1}^d (1+2^{\ell_i}|x_i-y_i|)^{N_i}}\,, \quad x\in\Omega\,, \quad (36)$$

gives the estimates

$$M_{\bar{\ell},\bar{N}}(x) \leq C_{\bar{N}} \sum_{\bar{m}\in\mathbb{N}_0^d} 2^{-\bar{m}\bar{N}} \int_\Omega \frac{2^{|\bar{m}+\bar{\ell}|_1}|(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|}{\prod_{i=1}^d (1+2^{\ell_i}|x_i-z_i|)^{N_i}}\, dz \quad (37)$$

$$\leq C_{\bar{N}} \sum_{\bar{m}\in\mathbb{N}_0^d} 2^{-\bar{m}\bar{N}s} \Big(2^{-\bar{m}\bar{N}} \sup_{y\in\Omega} \frac{|(\varphi_{\bar{m}+\bar{\ell}} * f)(y)|}{\prod_{i=1}^d (1+2^{\ell_i}|x_i-y_i|)^{N_i}}\Big)^{1-s} \quad (38)$$

$$\int_\Omega \frac{2^{|\bar{m}+\bar{\ell}|_1}|(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|^s}{\prod_{i=1}^d (1+2^{\ell_i}|x_i-z_i|)^{N_i s}}\, dz\,.$$

Observe that we can estimate the term $(...)^{1-s}$ in the right-hand side of (38) by $M_{\bar{\ell},\bar{N}}(x)^{1-s}$. Hence, if $M_{\bar{\ell},\bar{N}}(x) < \infty$ we obtain from (38)

$$M_{\bar{\ell},\bar{N}}(x)^s \leq C_{\bar{N}} \sum_{\bar{m}\in\mathbb{N}_0^d} 2^{-\bar{m}\bar{N}s} \int_\Omega \frac{2^{|\bar{m}+\bar{\ell}|_1}|(\varphi_{\bar{m}+\bar{\ell}} * f)(z)|^s}{\prod_{i=1}^d (1+2^{\ell_i}|x_i-z_i|)^{N_i s}}\, dz\,, \quad (39)$$

where $C_{\bar{N}}$ is independent of x, f and $\bar{\ell}$. Recall the definition of $M_{\bar{\ell},\bar{N}}(x)$ in (36). We claim that there exists $\bar{N}^f = (N_1^f, ..., N_d^f) \in \mathbb{N}_0^d$ such that $M_{\bar{\ell},\bar{N}}(x) < \infty$ for all $\bar{N} \geq \bar{N}^f$. Indeed, we use that $f \in S'(\Omega)$, i.e. there is an $M \in \mathbb{N}_0$ and $c_f > 0$ such that

$$|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)| \leq c_f \sup_{|\bar{\alpha}|_1 \leq M} \sup_{z \in \mathbb{R}^d} |D^{\bar{\alpha}} \varphi_{\bar{k}+\bar{\ell}}(z)| \cdot (1 + |y-z|)^M .$$

Assuming $\min\limits_{n=1,...,d} N_n > M$ we estimate as follows

$$\begin{aligned} |(\varphi_{\bar{\ell}} * f)(x)| &\leq M_{\bar{\ell},\bar{N}}(x) && (40)\\ &\leq c \sup_{\bar{k} \in \mathbb{N}_0^d} \sup_{y \in \Omega} 2^{-\bar{k}\bar{N}} \frac{|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)|}{(\prod_{i=1}^d (1+|x_i-y_i|^2))^{\frac{\min N_n}{2}}} \\ &\leq c \sup_{\bar{k} \in \mathbb{N}_0^d} \sup_{y \in \Omega} 2^{-\bar{k}\bar{N}} \frac{|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)|}{(1+|x-y|^2)^{\frac{\min N_n}{2}}} \\ &\leq c \sup_{\bar{k} \in \mathbb{N}_0^d} \sup_{y \in \Omega} 2^{-\bar{k}\bar{N}} 2^{|\bar{k}+\bar{\ell}|_1(M+1)} \sup_{z \in \mathbb{R}^d} \sup_{|\bar{\alpha}|_1 \leq M} \frac{|D^{\bar{\alpha}} \gamma_{\bar{k}+\bar{\ell}}(z)| \cdot (1+|y-z|)^M}{(1+|x-y|)^{\min N_n}} \\ &\leq c 2^{|\bar{\ell}|_1(M+1)} \sup_{\bar{k} \in \mathbb{N}_0^d} \sup_{z \in \mathbb{R}^d} \sup_{|\bar{\alpha}|_1 \leq M} |D^{\bar{\alpha}} \gamma_{\bar{k}+\bar{\ell}}(z)| (1+|x-z|)^{\min N_n}, && (41) \end{aligned}$$

where we again used the inequality (compare with (35))

$$1 + |y-z| \leq (1+|x-y|)(1+|x-z|)$$

and put $\gamma_{\bar{k}+\bar{\ell}}(x_1, ..., x_d) = \prod_{i=1}^d \gamma^i_{k_i+\ell_i}(x_i)$ and

$$\gamma^i_\ell(t) = \begin{cases} \varphi^i_0(t) & : \ \ell = 0 \\ \varphi^i(t) & : \ \ell > 0 \end{cases} .$$

Hence, the occurrence of only 2^d different compactly supported smooth functions $\gamma_{\bar{k}+\bar{\ell}}$ in (41) implies the boundedness of $M_{\bar{\ell},\bar{N}}(x)$ for $x \in \Omega$ if $N_i > M = N_i^f$, $i = 1, ..., d$. Therefore, (39) together with (40) yield (30) with $c = C_{\bar{N}}$, independent of x, f and $\bar{\ell}$, for all $\bar{N} \geq \bar{N}^f$. But this is not yet what we want. The condition $\bar{N} \geq \bar{N}^f$ is disturbing the picture. Let us define $\bar{N}^*$ by $N_i^* = \max\{N_i^f, N_i\}$, $i = 1, ..., d$, which implies $\bar{N} \leq \bar{N}^*$ as well as $\bar{N}^f \leq \bar{N}^*$. Consequently, if $\bar{N} \not\geq \bar{N}^f$ then (30) holds with $\bar{N}^*$ instead of $\bar{N}$ and $c = C_{\bar{N}^*}$. Observe that the right-hand side of (30) decreases as the components of $\bar{N}$ increase. Therefore, we have (30) for all $\bar{N} \in \mathbb{N}_0^d$ but with $c = c(f) = C_{\bar{N}^*}$ depending on f. This is still not yet what we want. Now we argue as follows: Starting with (30) where $c = c(f)$ and $\bar{N} \in \mathbb{N}_0^d$ arbitrary, we apply the same arguments as used from (33) to (34), change to the maximal function (36) with the help of (35) and finish with (39) instead of (37) but with a constant that depends on

f. But this does not matter now. Important is, that a finite right-hand side of (39) (which is the same as rhs(30)) implies $M_{\bar{\ell},\bar{N}}(x) < \infty$.
We assume rhs(30) $< \infty$. Otherwise there is nothing to prove in (30). Returning to (38) and having in mind that now $M_{\bar{\ell},\bar{N}}(x) < \infty$, we end up with (39) for all $\bar{N}$ and $C_{\bar{N}}$ independent of f. Finally, from (39) we obtain (30) and are done in case $0 < s \leq 1$.
Of course, (30) also holds true for $s > 1$ with a much simpler proof. In that case, we use (32) with $\bar{N}+1$ instead of $\bar{N}$ and apply Hölder's inequality with respect to $1/s + 1/s' = 1$ first for integrals and then for sums.

Step 3. The inequality (30) implies immediately a stronger version of itself. Using (35) again we obtain for $\bar{a} \leq \bar{N}$

$$(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}(x)^s \leq c \sum_{\bar{k}\in\mathbb{N}_0^d} 2^{-\bar{k}\bar{N}s} 2^{|\bar{k}+\bar{\ell}|_1} \int_{\Omega} \frac{|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)|^s}{\prod_{i=1}^{d}(1+2^{\ell_i}|x_i-y_i|)^{a_i s}}\, dy\,.$$

If $a_i s > 1$ then we have

$$g_\ell^i(t) = \frac{2^\ell}{(1+2^\ell|t|)^{a_i s}} \in L_1(\mathbb{R})$$

for $i = 1, ..., d$. Consider now the integral on the right-hand side extended to $\mathbb{R}^d$ with $|\varphi_{\bar{k}+\bar{\ell}} * f|_{\Omega}^s(y)$ instead of $|(\varphi_{\bar{k}+\bar{\ell}} * f)(y)|^s$. Then it is the componentwise convolution of the functions $g_{\ell_i}^i$ sequentially with $|\varphi_{\bar{k}+\bar{\ell}} * f|_{\Omega}^s$. Now we use the majorant property of the Hardy-Littlewood maximal operator (see Paragraph 2.5 and [16, Chapt. 2]) and estimate

$$\begin{aligned}
|2^{\bar{\ell}\bar{r}}(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}(x)|^s &\leq c \sum_{\bar{k}\in\mathbb{N}_0^d} 2^{-\bar{k}\bar{N}s} 2^{|\bar{k}|_1} 2^{\bar{\ell}\bar{r}s} (g_{\ell_1}^1 * (\ldots * (g_{\ell_d}^d * |\varphi_{\bar{k}+\bar{\ell}} * f|_{\Omega}^s)\ldots))(x) \\
&\leq c \sum_{\bar{k}\in\mathbb{N}_0^d} 2^{\bar{\ell}\bar{r}s} 2^{\bar{k}(-\bar{N}s+1)} \bar{M}(|\varphi_{\bar{k}+\bar{\ell}} * f|_{\Omega}^s)(x) \\
&= c \sum_{\bar{k}\in\bar{\ell}+\mathbb{N}_0^d} 2^{\bar{\ell}\bar{r}s} 2^{(\bar{k}-\bar{\ell})(-\bar{N}s+1)} \bar{M}(|\varphi_{\bar{k}} * f|_{\Omega}^s)(x) \\
&\leq c \sum_{\bar{k}\in\bar{\ell}+\mathbb{N}_0^d} 2^{(\bar{\ell}-\bar{k})(\bar{N}s-1+\bar{r}s)} 2^{\bar{k}\bar{r}s} \bar{M}(|\varphi_{\bar{k}} * f|_{\Omega}^s)(x)\,.
\end{aligned}$$

Choosing $1/a_i < s < \min\{p,q\}$ and $N_i > \max\{0,-r_i\} + a_i$ and putting

$$\delta = \min\{N_i + r_i - 1/s \ :\ i = 1, ..., d\}$$

we obtain

$$|2^{\bar{\ell}\bar{r}}(\varphi_{\bar{\ell}}^{\Omega} f)_{\bar{a}}(x)|^s \leq c \sum_{\bar{k}\in\mathbb{N}_0^d} 2^{-\delta s|\bar{\ell}-\bar{k}|_1} 2^{\bar{k}\bar{r}s} \bar{M}(|\varphi_{\bar{k}} * f|_{\Omega}^s)(x)\,.$$

Here we apply Lemma 2.4 in $L_{p/s}(\ell_{q/s},\Omega)$ and extend to $\mathbb{R}^d$ on the right-hand side. Consequently,

$$\|2^{\bar{\ell}\bar{r}s}(\varphi_{\bar{\ell}}^{\Omega}f)_{\bar{a}}(x)^s|L_{p/s}(\ell_{q/s},\Omega)\| \le c\|2^{\bar{k}\bar{r}s}\bar{M}(|\varphi_{\bar{k}}*f|_{\Omega}^s)(x)|L_{p/s}(\ell_{q/s})\|\,. \tag{42}$$

Now we apply the Fefferman-Stein inequality (see Paragraph 2.5/Theorem 2.2, having in mind that $p/s, q/s > 1$) and get

$$\begin{aligned}\|2^{\bar{k}\bar{r}s}\bar{M}(|\varphi_{\bar{k}}*f|_{\Omega}^s)(x)|L_{p/s}(\ell_{q/s})\| &\le c\|2^{\bar{k}\bar{r}s}|\varphi_{\bar{k}}*f|_{\Omega}^s(x)|L_{p/s}(\ell_{q/s})\| \\ &= \|2^{\bar{k}\bar{r}s}|\varphi_{\bar{k}}*f|^s(x)|L_{p/s}(\ell_{q/s},\Omega)\|\,. \end{aligned} \tag{43}$$

Rewriting (43) and the left-hand side of (42) to $\|\cdot|L_p(\ell_q)\|^s$ we obtain the desired inequality. Theorem 3.4 then gives the "$\le$"-inequality in (ii). The modifications for proving (i) are obvious.

Step 4. The "$\ge$"-inequalities in (i) and (ii) are immediate consequences of Definition 2.5. ■

Remark 4.3 *The given characterization also holds for $\Omega \in \mathcal{D}_{\bar{b}} + y$. See Remark 3.5.*

Remark 4.4 *Markus Hansen (Jena) observed that the proofs of Theorem 1.22 in [22] and Proposition 3 in [2] are not correct for the same reason. The critical arguments are based on a corresponding one in the proof of (23), (23') in [8]. The finiteness of the Peetre maximal function is assumed. But this is not true in general under the stated assumptions. Consider in dimension $d = 1$ the functions*

$$\Psi_0(t) = \Psi_1(t) = e^{-t^2}$$

and, if $a > 0$ is given, the tempered distribution $f(t) = |t|^n$ with $a < n \in \mathbb{N}$. Anyhow, the stated results (1.47) and (1.48) in [22] hold true. The necessary modifications for the proof are given in the proof of our Theorem 4.2 for $\Omega = \mathbb{R}^d$ in Step 2 and 3 after (32).

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