Approximation numbers of Sobolev embeddings – sharp constants and tractability

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Dedicated to J.F. Traub and G.W. Wasilkowski
on the occasion of their 80th and 60th birthdays

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Abstract

We investigate optimal linear approximations (approximation numbers) in the context of periodic Sobolev spaces $H^s(\mathbb{T}^d)$ of fractional smoothness $s > 0$ for various equivalent norms including the classical one. The error is always measured in $L_2(\mathbb{T}^d)$. Particular emphasis is given to the dependence of all constants on the dimension d. We capture the exact decay rate in n and the exact decay order of the constants with respect to d, which is in fact polynomial. As a consequence we observe that none of our considered approximation problems suffers from the curse of dimensionality. Surprisingly, the square integrability of all weak derivatives up to order three (classical Sobolev norm) guarantees weak tractability of the associated multivariate approximation problem.

Keywords Approximation numbers · Sobolev embeddings · sharp constants · weak tractability · curse of dimensionality

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1 Introduction

In the present paper we investigate the asymptotic behavior of the approximation numbers of the embeddings

$$I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\,, \quad s > 0\,, \quad d \in \mathbb{N}\,, \tag{1.1}$$

where $H^s(\mathbb{T}^d)$ is the periodic Sobolev space of fractional smoothness $s > 0$ on the d-torus. The approximation numbers of a bounded linear operator $T : X \to Y$ between two Banach spaces are defined as

$$\begin{aligned} a_n(T:\, X \to Y) &:= \inf_{\operatorname{rank} A < n} \sup_{\|x|X\| \le 1} \|\, Tx - Ax|Y\| \\ &= \inf_{\operatorname{rank} A < n} \|T - A : X \to Y\|, \qquad n \in \mathbb{N}\,. \end{aligned} \tag{1.2}$$

They describe the best approximation of T by finite rank operators. If X and Y are Hilbert spaces and T is compact, then $a_n(T)$ is the nth singular number of T.

The first result on the approximation of Sobolev embeddings is due to Kolmogorov [6]. He showed already in 1936 that in the univariate (homogeneous) case with integer smoothness $m \in \mathbb{N}$ the approximation numbers $a_n(I_d : \dot{H}^m(\mathbb{T}) \to L_2(\mathbb{T}))$ decay exactly like n^{-m}. Here we are interested in the multivariate (inhomogeneous) situation, where d is large, and investigate the approximation numbers $a_n(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for arbitrary smoothness parameters $s > 0$.

In fact, there is an increasing interest in the approximation of multivariate functions since many problems from, e.g., finance or quantum chemistry, are modeled in associated function spaces on high-dimensional domains. So far, many authors have contributed to the subject, see for instance the monographs by Temlyakov [16], Tikhomirov [17] and the references therein. In [16, Chapt. 2, Thm. 4.1, 4.2] the following two-sided estimate can be found

$$c_s(d)\, n^{-s/d} \le a_n(I_d :\, H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le C_s(d)\, n^{-s/d}\,, \qquad n \in \mathbb{N}\,,$$

where the constants $c_s(d)$ and $C_s(d)$, only depending on d and s, were not explicitly determined. Our main focus is to clarify, for arbitrary but fixed $s > 0$, the dependence of these constants on d. Surprisingly, it turns out that the optimal constants decay polynomially in d, i.e.,

$$c_s(d) \sim C_s(d) \sim d^{-\alpha} \tag{1.3}$$

for some $\alpha > 0$ which depends on the chosen norm in $H^s(\mathbb{T}^d)$ and the value of the smoothness parameter $s > 0$. We give exact values of α in at least two important situations.

As a consequence of these precise estimates for the approximation numbers we obtain weak tractability results for the approximation problem of the Sobolev embeddings (1.1) . Basically, we consider three different (but of course) equivalent norms on $H^s(\mathbb{T}^d)$, see (2.6), (2.7), and (2.8) below. The first two norms are the most common natural norms obtained by taking distributional derivatives in the case s being an integer. It turns out that all the associated approximation problems do not suffer from the curse of dimensionality. In fact, we even obtain weak tractability in some of the important cases, i.e., if the smoothness s is larger than one or two, respectively, depending on the used norm. This is a quite surprising fact when taking the famous negative result into account that the approximation of infinitely differentiable functions is intractable [10]. See Remark 5.8 below for a more detailed comparison. In the case of Sobolev smoothness and L_2-approximation it seems that already less smoothness restrictions guarantee weak tractability in the worst case setting. Furthermore, our results illustrate that the notion of tractability is sensitive with respect to the choice of the equivalent norms.

The paper is organized as follows. In Section 2 we recall the definition of periodic Sobolev spaces $H^s(\mathbb{T}^d)$ and discuss various equivalent norms. In addition, we will recall some facts on Hilbert spaces, diagonal operators and associated approximation numbers. Section 3 is devoted to provide some useful combinatorial identities and related inequalities. Section 4 is the heart of this paper. Here we prove estimates of the approximation numbers as indicated above. In the final Section 5 we apply the obtained results to establish results on weak tractability.

Notation. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. With $\mathbb{T}$ we denote the torus represented by the interval $[0, 2\pi]$. For a real number a we put $a_+ := \max\{a, 0\}$. The symbol d is always reserved for the dimension in $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{N}^d$, and $\mathbb{T}^d$. For $0 < p \le \infty$ and $x \in \mathbb{R}^d$ we denote $|x|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ with the usual modification in the case $p=\infty$. If X and Y are two Banach spaces, the norm of an element x in X will be denoted by $\|x|X\|$ and the norm of an operator $A : X \to Y$ is denoted by $\|A : X \to Y\|$. The symbol $X \hookrightarrow Y$ indicates that the embedding operator is continuous.

2 Preliminaries

2.1 Sobolev spaces on the d-torus

All the results in this paper are stated for function spaces on the d-torus $\mathbb{T}^d$, which is represented in the Euclidean space $\mathbb{R}^d$ by the cube $\mathbb{T}^d = [0, 2\pi]^d$ where opposite points are identified. In particular, for a function f on $\mathbb{T}^d$ we have $f(x) = f(y)$, if $x - y = 2\pi k$ for some $k \in \mathbb{Z}^d$. Those functions can be viewed as 2π-periodic in each component.

The space $L_2(\mathbb{T}^d)$ consists of all (equivalence classes of) measurable functions f on $\mathbb{T}^d$ where the norm

$$\|f|L_2(\mathbb{T}^d)\| := \Big(\int_{\mathbb{T}^d} |f(x)|^2\, dx\Big)^{1/2}$$

is finite. All the information of a function $f \in L_2(\mathbb{T}^d)$ is contained in the sequence $(c_k(f))_k$ of its Fourier coefficients, given by

$$c_k(f) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx\,, \qquad k \in \mathbb{Z}^d\,.$$

Indeed, we have Parseval's identity

$$\|f|L_2(\mathbb{T}^d)\|^2 = \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \tag{2.1}$$

as well as

$$f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} c_k(f)\, e^{ikx}$$

with convergence in $L_2(\mathbb{T}^d)$.

Definition 2.1. *Let $m \in \mathbb{N}$. The classical Sobolev space $H^m(\mathbb{T}^d)$ is the collection of all $f \in L_2(\mathbb{T}^d)$ such that all distributional derivatives $D^\alpha f$ of order $|\alpha|_1 \le m$ belong to $L_2(\mathbb{T}^d)$. The classical norm is defined by*

$$\|\, f\,|H^m(\mathbb{T}^d)\| := \Big(\sum_{|\alpha|_1\le m} \|\, D^\alpha f\,|L_2(\mathbb{T}^d)\|^2\Big)^{1/2}.$$

One can rewrite this definition in terms of Fourier coefficients. Taking $c_k(D^\alpha f) = (ik)^\alpha c_k(f)$ into account, (2.1) implies

$$\begin{aligned}
\|\, f\,|H^m(\mathbb{T}^d)\|^2 &= \sum_{|\alpha|_1\le m} \Big\| \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} c_k(f)\,(ik)^\alpha e^{ikx} \Big| L_2(\mathbb{T}^d)\Big\|^2 \\
&= \sum_{k\in\mathbb{Z}^d} \Big(\sum_{|\alpha|_1\le m} \prod_{\ell=1}^d |k_\ell|^{2\alpha_\ell}\Big) |c_k(f)|^2
\end{aligned} \tag{2.2}$$

(using the convention $0^0 = 1$). By the multinomial identity

$$\Big(1+\sum_{i=1}^d |k_i|^2\Big)^m = \sum_{\substack{\alpha\in\mathbb{N}_0^{d+1}\\ |\alpha|_1=m}} \frac{m!}{\alpha_1!\cdot\ldots\cdot\alpha_{d+1}!}\prod_{\ell=1}^d |k_\ell|^{2\alpha_\ell}$$

we get

$$\sum_{|\alpha|_1\le m}\prod_{\ell=1}^d |k_\ell|^{2\alpha_\ell} \le \Big(1+\sum_{i=1}^d |k_i|^2\Big)^m \le m!\sum_{|\alpha|_1\le m}\prod_{\ell=1}^d |k_\ell|^{2\alpha_\ell}\,.$$

Hence, (2.2) can be bounded from above and below by

$$\begin{aligned}\frac{1}{\sqrt{m!}}\Big[\sum_{k\in\mathbb{Z}^d}\Big(1+\sum_{i=1}^d |k_i|^2\Big)^m |c_k(f)|^2\Big]^{1/2} &\le \|f|H^m(\mathbb{T}^d)\|\\ &\le \Big[\sum_{k\in\mathbb{Z}^d}\Big(1+\sum_{i=1}^d |k_i|^2\Big)^m |c_k(f)|^2\Big]^{1/2}.\end{aligned}\tag{2.3}$$

Note that the equivalence constants do not depend on d. In general, the change from one to another equivalent norm might produce equivalence constants which badly depend on d. In the sequel we are interested in situations where d is large or even $d\to\infty$. This issue is therefore of particular importance for us. In fact, we could have also started with the equivalent norm

$$\|\, f\,|H^m(\mathbb{T}^d)\|^* := \Big(\|\, f\,|L_2(\mathbb{T}^d)\|^2 + \sum_{j=1}^d \Big\|\,\frac{\partial^m f}{\partial x_j^m}\,\Big|L_2(\mathbb{T}^d)\Big\|^2\Big)^{1/2}. \tag{2.4}$$

It is easy to check that the equivalence constants between (2.2) and (2.4) depend polynomially on d. In other words, the unit balls of the respective norms differ significantly. With a similar calculation as above, we obtain

$$\|\, f\,|H^m(\mathbb{T}^d)\|^* = \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2\Big(1+\sum_{j=1}^d |k_j|^{2\,m}\Big)\Big]^{1/2}. \tag{2.5}$$

The representations (2.3) and (2.5) enable us to extend the definition of the Sobolev space to fractional smoothness $s>0$.

Definition 2.2. *Let $s>0$. The periodic Sobolev space $H^s(\mathbb{T}^d)$ is the collection of all $f\in L_2(\mathbb{T}^d)$ such that $\|f|H^s(\mathbb{T}^d)\|<\infty$, where*
(i) *the natural norm $\|f|H^s(\mathbb{T}^d)\|^+$ is defined as*

$$\|\, f\,|H^s(\mathbb{T}^d)\|^+ := \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2\Big(1+\sum_{j=1}^d |k_j|^2\Big)^s\Big]^{1/2}. \tag{2.6}$$

(ii) *The modified natural norm $\|f|H^s(\mathbb{T}^d)\|^*$ is defined as*

$$\|\, f\,|H^s(\mathbb{T}^d)\|^* := \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2\Big(1+\sum_{j=1}^d |k_j|^{2\,s}\Big)\Big]^{1/2}, \tag{2.7}$$

(iii) *whereas the norm* $\|f|H^s(\mathbb{T}^d)\|^\#$ *is a further modification defined by*

$$\| f \,|H^s(\mathbb{T}^d)\|^\# := \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \Big(1 + \sum_{j=1}^d |k_j|\Big)^{2s}\Big]^{1/2} . \tag{2.8}$$

Of course, all three norms are equivalent on $H^s(\mathbb{T}^d)$. As already mentioned, there might be equivalence constants involved which in general depend on d. However, in some special cases we even have equality of the norms, namely $\|\cdot|H^1(\mathbb{T}^d)\|^* = \|\cdot|H^1(\mathbb{T}^d)\| = \|\cdot|H^1(\mathbb{T}^d)\|^+$ and $\|\cdot|H^{1/2}(\mathbb{T}^d)\|^* = \|\cdot|H^{1/2}(\mathbb{T}^d)\|^\#$. We are interested in further norm one embeddings between different norms.

In the sequel we will often use the notation $H^{s,+}(\mathbb{T}^d)$, $H^{s,*}(\mathbb{T}^d)$, and $H^{s,\#}(\mathbb{T}^d)$ to indicate which norm we use in $H^s(\mathbb{T}^d)$. The following useful embeddings are due the monotonicity of the norms $|\cdot|_p$, where $0 < p < \infty$, except (v) which is a simple consequence of the fact that the square of an integer is larger than its absolute value.

Lemma 2.3. *Let $s > 0$. The following embeddings have norm one.*
(i) If $s \geq 1$ then

$$H^{s,\#}(\mathbb{T}^d) \hookrightarrow H^{s,+}(\mathbb{T}^d) \hookrightarrow H^{s,*}(\mathbb{T}^d) ,$$

(ii) if $1/2 \leq s \leq 1$ then

$$H^{s,\#}(\mathbb{T}^d) \hookrightarrow H^{s,*}(\mathbb{T}^d) \hookrightarrow H^{s,+}(\mathbb{T}^d) ,$$

(iii) and if $s \leq 1/2$ then

$$H^{s,*}(\mathbb{T}^d) \hookrightarrow H^{s,\#}(\mathbb{T}^d) \hookrightarrow H^{s,+}(\mathbb{T}^d) .$$

(iv) If $s > t$ then

$$H^{s,+}(\mathbb{T}^d) \hookrightarrow H^{t,+}(\mathbb{T}^d) \quad , \quad H^{s,*}(\mathbb{T}^d) \hookrightarrow H^{t,*}(\mathbb{T}^d) \quad , \quad H^{s,\#}(\mathbb{T}^d) \hookrightarrow H^{t,\#}(\mathbb{T}^d) ,$$

(v) and finally

$$H^{s,+}(\mathbb{T}^d) \hookrightarrow H^{s/2,\#}(\mathbb{T}^d) .$$

2.2 Approximation numbers

If $\tau = (\tau_n)_{n=1}^\infty$ is a sequence of real numbers with $\tau_1 \geq \tau_2 \geq ... \geq 0$, we define the diagonal operator $D_\tau : \ell_2 \to \ell_2$ by $D_\tau(\xi) = (\tau_n\xi_n)_{n=1}^\infty$. Recall the notion of the approximation numbers (1.2) already given in the introduction. The following fact concerning approximation numbers of diagonal operators is well-known, see e.g. König [5, Sect. 1.b], Pinkus [14, Thm. IV.2.2], and Novak and Wozniakowski [9, Cor. 4.12]. Comments on the history may be found in Pietsch [13, 6.2.1.3].

Lemma 2.4. *Let τ and D_τ be as above. Then*

$$a_n(D_\tau :\, \ell_2 \to \ell_2) = \tau_n , \qquad n \in \mathbb{N} .$$

Here the index set of ℓ_2 is $\mathbb{N}$. We need a modification to arbitrary countable index sets J. Then the space $\ell_2(J)$ is the collection of all $\xi = (\xi_j)_{j\in J}$ such that the norm

$$\|\xi|\ell_2(J)\| := \Big(\sum_{j\in J} |\xi_j|^2\Big)^{1/2}$$

is finite. If $w = (w_j)_{j\in J}$ is such that for every $\delta > 0$ there are only finitely many $j \in J$ with $|w_j| \geq \delta$, then the non-increasing rearrangement $(\tau_n)_{n\in\mathbb{N}}$ of $(|w_j|)_{j\in J}$ exists and we have $\lim_{n\to\infty} \tau_n = 0$. Defining $D_w : \ell_2(J) \to \ell_2(J)$ by $D_w(\xi) = (w_j \xi_j)_{j\in J}$ for $\xi \in \ell_2(J)$, Lemma 2.4 gives

$$a_n(D_w : \ell_2(J) \to \ell_2(J)) = \tau_n \,.$$

The preceding identity is scalable in the following sense.

Lemma 2.5. *Let J be a countable index set and $(w_j)_{j\in J}$ and $(\tau_n)_{n\in\mathbb{N}}$ be as above. If $s > 0$ then*

$$a_n(D_{|w|^s} : \ell_2(J) \to \ell_2(J)) = a_n(D_w : \ell_2(J) \to \ell_2(J))^s = \tau_n^s \,.$$

Next, we reduce our function space problem to the simpler context of sequence spaces and diagonal operators. The index set is now $J = \mathbb{Z}^d$. To this end, we consider the operators

$$A_s : H^{s,+}(\mathbb{T}^d) \to \ell_2(\mathbb{Z}^d) \qquad \text{and} \qquad B_s : \ell_2(\mathbb{Z}^d) \to H^{s,+}(\mathbb{T}^d)$$

defined as

$$A_s f = (w_{s,+}(k) c_k(f))_{k\in\mathbb{Z}^d} \qquad \text{and} \qquad B_s \xi = (2\pi)^{-d/2} \sum_{k\in\mathbb{Z}^d} \frac{\xi_k}{w_{s,+}(k)} e^{ikx} \,,$$

where the weights are $w_{s,+}(k) = \left(1 + \sum_{j=1}^d |k_j|^2\right)^{s/2}$. Note the semigroup property of these weights, i.e., $w_{s,+}(k) w_{t,+}(k) = w_{s+t,+}(k)$. The following commutative diagram illustrates the situation in the case $s_0 > s_1 \geq 0$:

$$\begin{tikzcd}
H^{s_0,+}(\mathbb{T}^d) \arrow[r, "I_d"] \arrow[d, "A_{s_0}"] & H^{s_1,+}(\mathbb{T}^d) \\
\ell_2(\mathbb{Z}^d) \arrow[r, "D_w"] & \ell_2(\mathbb{Z}^d) \arrow[u, "B_{s_1}"']
\end{tikzcd}$$

Here, we put for $k \in \mathbb{Z}^d$

$$w(k) = \frac{w_{s_1,+}(k)}{w_{s_0,+}(k)} \,.$$

By the definition of the norm $\|\cdot|H^{s,+}(\mathbb{T}^d)\|$ it is clear that A_s and B_s are isometries and $B_s = A_s^{-1}$. For the embedding $I_d : H^{s_0,+}(\mathbb{T}^d) \to H^{s_1,+}(\mathbb{T}^d)$ if $s_0 > s_1 \geq 0$ we obtain the factorization

$$I_d = B_{s_1} \circ D_w \circ A_{s_0} \,. \tag{2.9}$$

The multiplicativity of the approximation numbers applied to (2.9) implies

$$a_n(I_d) \leq \|A_{s_0}\| a_n(D_w) \|B_{s_1}\| = a_n(D_w) = \tau_n \,,$$

where $(\tau_n)_{n=1}^\infty$ is the non-increasing rearrangement of $(w(k))_{k\in\mathbb{Z}^d}$. The reverse inequality can be shown analogously. This gives the important identity

$$a_n(I_d) = a_n(D_w) = \tau_n \,. \tag{2.10}$$

Of course, (2.10) also holds for $I_d : H^{s_0,\#}(\mathbb{T}^d) \to H^{s_1,\#}(\mathbb{T}^d)$ and for $I_d : H^{s_0,*}(\mathbb{T}^d) \to H^{s_1,*}(\mathbb{T}^d)$ with the obvious adaption of the weights. Due to the semigroup property mentioned above and Lemma 2.5 we have in particular the nice properties

$$\begin{aligned} a_n(I_d : H^{s_0,+}(\mathbb{T}^d) \to H^{s_1,+}(\mathbb{T}^d)) &= a_n(I_d : H^{s_0-s_1,+}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \\ &= a_n(I_d : H^{1,+}1(\mathbb{T}^d) \to L_2(\mathbb{T}^d))^{s_0-s_1} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} a_n(I_d : H^{s_0,\#}(\mathbb{T}^d) \to H^{s_1,\#}(\mathbb{T}^d)) &= a_n(I_d : H^{s_0-s_1,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \\ &= a_n(I_d : H^{1,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))^{s_0-s_1} . \end{aligned}$$

For the norm $\|\cdot\|^*$ the weights are

$$w_{s,*}(k) = \Big(1 + \sum_{j=1}^d |k_j|^{2s}\Big)^{1/2}$$

Note that (2.11) does not hold due to the missing semigroup property.

3 Some combinatorics

In most of the considerations below a crucial role will be played by the cardinality $C(m,d)$ of the set

$$\mathcal{N}(m,d) := \Big\{k \in \mathbb{Z}^d : \quad \sum_{j=1}^d |k_j| \le m\Big\} , \qquad m \in \mathbb{N}_0 .$$

To obtain suitable estimates for the numbers $C(m,d)$ we need some preparation. For $d \in \mathbb{N}$, $m \in \mathbb{N}_0$ we consider the set $S_0(m,d) := \{k \in \mathbb{N}_0^d : |k|_1 = m\} \subset \mathbb{N}_0^d$ and for $m \ge d$ we put $S(m,d) = S_0(m,d) \cap \mathbb{N}^d$.

Lemma 3.1. (i) *Let $m,d \in \mathbb{N}$ with $m \ge d$. Then we have*

$$\#S(m,d) = \binom{m-1}{d-1} . \tag{3.1}$$

(ii) *Let $m,d \in \mathbb{N}$ then*

$$\#S_0(m,d) = \sum_{\ell=1}^{\min\{m,d\}} \binom{d}{\ell}\binom{m-1}{\ell-1} = \binom{m+d-1}{d-1} \tag{3.2}$$

(iii) *For $n \ge \ell$ it holds*

$$\sum_{m=\ell}^n \binom{m-1}{\ell-1} = \binom{n}{\ell} .$$

Proof. We refer to [1, Thm. 3.3,3.4]. In fact, (i) and (iii) can be easily proven by induction. As for (ii), the first identity in (3.2) is a consequence of (i). Note that the second identity is due to the fact $(k_1,...,k_d) \in S_0(m,d)$ if and only if $(k_1+1,...,k_d+1) \in S(m+d,d)$ and thus $\#S_0(m,d) = \#S(m+d,d)$. ■

Lemma 3.2. *Let $n, d \in \mathbb{N}$ be given. Then we have*

$$\begin{aligned} C(n,d) &= 1 + \sum_{m=1}^{n} \sum_{\ell=1}^{\min\{m,d\}} 2^\ell \binom{d}{\ell} \binom{m-1}{\ell-1} \\ &= \sum_{\ell=0}^{\min\{d,n\}} 2^\ell \binom{d}{\ell} \binom{n}{\ell} . \end{aligned} \tag{3.3}$$

Proof. The crucial observation is the representation

$$\mathcal{N}(n,d) = \{0\} \cup \bigcup_{m=1}^{n} \bigcup_{\ell=1}^{\min\{m,d\}} \Big\{ k \in \mathbb{Z}^d : |k|_0 = \ell, |k|_1 = m \Big\} .$$

As a consequence, the first identity in (3.3) follows directly from Lemma 3.1/(i). The second relation in (3.3) can be found in the book by Polya and Szegö as an exercise, see [15, Problem 29, pp. 4]. In fact, it is a consequence of the first relation by interchanging the two sums and using Lemma 3.1/(iii) afterwards. ■

Before dealing with inequalities we state a useful identity first, see [4, p. 5, 0.156/1]. For the reader's convenience we will give a proof.

Lemma 3.3. *Let $M, m \in \mathbb{N}$ with $m \le M$. Then*

$$\sum_{\ell=0}^{m} \binom{M}{\ell} \binom{m}{\ell} = \binom{M+m}{m} = \binom{M+m}{M} .$$

Proof. By Lemma 3.1/(iii) we see

$$\begin{aligned} \sum_{\ell=0}^{m} \binom{M}{\ell} \binom{m}{\ell} = 1 + \sum_{\ell=1}^{m} \binom{M}{\ell} \binom{m}{\ell} &= 1 + \sum_{\ell=1}^{m} \binom{M}{\ell} \sum_{k=\ell}^{m} \binom{k-1}{\ell-1} \\ &= 1 + \sum_{k=1}^{m} \sum_{\ell=1}^{k} \binom{M}{\ell} \binom{k-1}{\ell-1} \\ &= 1 + \sum_{k=1}^{m} \binom{M+k-1}{M-1} , \end{aligned}$$

where we used the second identity in (3.2) in the last step. Applying again Lemma 3.1/(iii) we find

$$1 + \sum_{k=1}^{m} \binom{M+k-1}{M-1} = \sum_{k=0}^{m} \binom{M+k-1}{M-1} = \binom{M+m}{M} .$$

The proof is complete. ■

Let us now deal with some useful estimates. The upper bound in relation (3.5) below can be found in [8, p. 195, 3.1.30]. Nevertheless, we will give the proofs of the assertions in the Lemma below.

Lemma 3.4. (i) *For all m and all d we have the estimates*

$$\binom{m+d}{d} \quad \le \quad C(m,d) \le 2^{\min(d,m)} \binom{m+d}{d}, \tag{3.4}$$

$$\max\left\{\left(1+\frac{m}{d}\right)^d, \left(1+\frac{d}{m}\right)^m\right\} \quad \le \quad \binom{m+d}{d} \le \left(1+\frac{m}{d}\right)^d \left(1+\frac{d}{m}\right)^m. \tag{3.5}$$

In particular

$$\binom{m+d}{d} \le e^{d-1} \left(1+\frac{m}{d}\right)^d. \tag{3.6}$$

(ii) For all $m = d-1 \in \mathbb{N}$ we have

$$C(d-1,d) \le \frac{6^d}{3}. \tag{3.7}$$

Proof. In (3.4) both estimates follow from (3.3) combined with Lemma 3.3. Let us now deal with (3.5). Because of

$$\prod_{j=1}^{d} \left(1+\frac{m}{j}\right) = \binom{m+d}{d} = \binom{m+d}{m} = \prod_{j=1}^{m} \left(1+\frac{d}{j}\right)$$

the estimate from below is obvious. We further obtain

$$\begin{aligned}
\log \binom{m+d}{d} \quad &= \quad \sum_{j=1}^{d} \log\left(1+\frac{m}{j}\right) \\
&\le \quad \log(1+m) + \int_1^d \log\left(1+\frac{m}{x}\right) dx \\
&= \quad \log(1+m) + x \, \log\left(1+\frac{m}{x}\right)\Big|_1^d + m \int_1^d \frac{1}{x+m}\, dx \\
&= \quad d \, \log\left(1+\frac{m}{d}\right) + m \left(\log(d+m) - \log(1+m)\right) \\
&= \quad d \, \log\left(1+\frac{m}{d}\right) + m \, \log\frac{d+m}{1+m}.
\end{aligned}$$

Consequently,

$$\binom{m+d}{d} \le \left(1+\frac{m}{d}\right)^d \left(1+\frac{d-1}{m+1}\right)^m. \tag{3.8}$$

Hence, we proved a slightly better estimate than stated in (3.5). The estimate in (3.6) is a direct consequence of (3.8) together with the relation $(1+(d-1)/m)^m \le e^{d-1}$. To prove (3.7) we observe

$$C(d-1,d) = \sum_{\ell=0}^{d-1} 2^\ell \binom{d-1}{\ell} \binom{d}{\ell} \le 2^d \sum_{\ell=0}^{d-1} 2^\ell \binom{d-1}{\ell} = 2^d (2+1)^{d-1} = \frac{6^d}{3}.$$

$\blacksquare$

4 The approximation numbers of Sobolev embeddings

In this section, we will compute and estimate the approximation numbers of $I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$. First we deal with the norm $\| \cdot |H^s(\mathbb{T}^d)\|^{\#}$ where we use combinatorial arguments. Afterwards we consider the norm $\| \cdot |H^s(\mathbb{T}^d)\|^{*}$ where volume arguments are applied. Based on these results and Lemma 2.5 we obtain results for the natural norm $\| \cdot |H^{s,+}(\mathbb{T}^d)\|$.

4.1 The approximation numbers of $H^{s,\#}(\mathbb{T}^d)$ in $L_2(\mathbb{T}^d)$

For $s > 0$ we put

$$w_{s,\#}(k) := \Big(1 + \sum_{j=1}^{d} |k_j|\Big)^s, \qquad k \in \mathbb{Z}^d .$$

By $(\sigma_j)_{j=1}^\infty$ we denote the decreasing rearrangement of $(1/w(k))_{k\in\mathbb{Z}^d}$. Employing Lemma 2.4 and (2.10) we conclude

$$a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \sigma_n , \qquad n \in \mathbb{N} .$$

Furthermore, for any $m \in \mathbb{N}_0$,

$$\#\Big\{k \in \mathbb{Z}^d : (m+1)^{-s} \le \frac{1}{w_{s,\#}(k)}\Big\} = \#\Big\{k \in \mathbb{Z}^d : w_{s,\#}(k) \le (m+1)^s\Big\} = C(m,d) .$$

Lemma 4.1. *Let $s > 0$. Then, for all $m \in \mathbb{N}$,*

$$a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \frac{1}{(m+1)^s} , \qquad \textit{if} \quad C(m-1,d) < n \le C(m,d) .$$

Remark 4.2. (i) Note that we have complete knowledge of the sequence $a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$. In particular, $a_1 = 1$ and $(a_n)_n$ is piecewise constant. However, in the given form it is of limited use.
(ii) Of course, the optimal linear operator which realizes $a_n = (m+1)^{-s}$ for $C(m-1,d) < n \le C(m,d)$ is the orthogonal projection

$$S_m f(x) := \frac{1}{(2\pi)^{d/2}} \sum_{\substack{k\in\mathbb{Z}^d \\ w_{s,\#}(k)\le m^s}} c_k(f)\, e^{ikx} .$$

Now we are able to present a first result on the asymptotic behaviour of the constants $c_s(d)$ and $C_s(d)$ which we already announced in the Introduction (1.3).

Theorem 4.3. *Let $s > 0$ and $d \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} n^{s/d}\, a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \Big(\frac{2}{\sqrt[d]{d!}}\Big)^s .$$

Proof. In the case $C(m-1,d) < n \le C(m,d)$ we have by Lemma 4.1

$$\frac{C(m-1,d)^{s/d}}{(m+1)^s} \le n^{s/d}\, a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \frac{C(m,d)^{s/d}}{(m+1)^s} . \tag{4.1}$$

Employing Lemma 3.2 for m large enough we see that

$$\begin{aligned}
\lim_{m\to\infty}\frac{(C(m,d))^{s/d}}{(m+1)^s} &= \Big[\sum_{\ell=0}^{d}2^\ell\binom{d}{\ell}\lim_{m\to\infty}\frac{\binom{m}{\ell}}{(m+1)^d}\Big]^{s/d}\\
&= \Big[2^d\binom{d}{d}\lim_{m\to\infty}\frac{\binom{m}{d}}{(m+1)^d}\Big]^{s/d}\\
&= \Big(\frac{1}{d!}\,2^d\Big)^{s/d}.
\end{aligned}$$

This proves the claim. ■

Remark 4.4. As a consequence of Stirling's formula we get that

$$\Big(\frac{2}{\sqrt[d]{d!}}\Big)^s \asymp \Big(\frac{2e}{d}\Big)^s$$

where $a_n \asymp b_n$ means here $\lim_{n\to\infty} a_n/b_n = 1$. That means we have a polynomial decay of the constants in d, which might be surprising at a first glance, especially when we compare this fact with the tractability results in Section 5. However, it is less surprising if we keep in mind that asymptotic and tractability analysis are not really comparable and represent totally different viewpoints on the same problem.

We will now give sharp bounds in the case that the index n of a_n is large, say $n \geq 6^d/3$.

Theorem 4.5. *Let $s>0$ and $d\in\mathbb{N}$.*
(i) Then

$$a_n(I_d:\, H^{s,\#}(\mathbb{T}^d)\to L_2(\mathbb{T}^d)) \leq \Big(\frac{4e}{d}\Big)^s n^{-s/d} \qquad \textit{if} \quad n > 6^d/3\,.$$

(ii) In addition,

$$a_n(I_d:\, H^{s,\#}(\mathbb{T}^d)\to L_2(\mathbb{T}^d)) \geq \frac{1}{\max\{d^s,2^s\}}\, n^{-s/d} \qquad \textit{if} \quad n \geq 2\,.$$

Proof. Let $m\geq d$ and $C(m-1,d) < n \leq C(m,d)$. For the upper bound in (i) we estimate using (3.4) and (3.6)

$$\begin{aligned}
n^{s/d}\, a_n(I_d:\, H^{s,\#}(\mathbb{T}^d)\to L_2(\mathbb{T}^d)) &\leq \frac{C(m,d)^{s/d}}{(m+1)^s} \leq \frac{\Big(2^d\binom{m+d}{d}\Big)^{s/d}}{(m+1)^s} \leq \frac{2^s\, e^s\Big(1+\frac{m}{d}\Big)^s}{(m+1)^s}\\
&\leq \Big(\frac{2e}{d}\Big)^s \sup_{m\geq d}\Big(\frac{m+d}{m+1}\Big)^s\\
&\leq \Big(\frac{4e}{d}\Big)^s.
\end{aligned}$$

This holds for all $n > C(d-1,d)$, and due to (3.7) the relation in (i) is proved.
For the proof of (ii) we again use (4.1). The left-hand side implies

$$n^{s/d}\, a_n(I_d:\, H^{s,\#}(\mathbb{T}^d)\to L_2(\mathbb{T}^d)) \geq \frac{C(m-1,d)^{s/d}}{(m+1)^s} \geq \frac{\binom{m-1+d}{d}^{s/d}}{(m+1)^s}\,, \tag{4.2}$$

valid for all $m = 1, 2, \ldots$ and

$$1 = C(0,d) \le C(m-1,d) < n \le C(m,d)\,.$$

Using the lower bound in (3.5) we conclude

$$n^{s/d}\, a_n(I_d:\, H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge d^{-s}\left(\frac{m-1+d}{m+1}\right)^s \ge d^{-s}$$

in the case $d \ge 2$. In the case $d = 1$ we return to (4.2) and obtain the lower bound 2^{-s}. ■

Let us now turn to estimates for a_n if n is rather small.

Theorem 4.6. *Let $s > 0$, $d \in \mathbb{N}$ and $2 \le n \le 2^d$. Then one has*

$$\left(\frac{\log 2}{\log(4n)}\right)^s \le a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(\frac{\log(2d+1)}{\log n}\right)^s.$$

Proof. For n in the above range there exists $1 \le m \le d$ such that $C(m-1,d) < n \le C(m,d)$ and $a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \frac{1}{(m+1)^s}$. Identity (3.3) gives

$$n \le C(m,d) = \sum_{\ell=0}^{m} 2^\ell \binom{d}{\ell}\binom{m}{\ell} \le \sum_{\ell=0}^{m} 2^\ell d^\ell \binom{m}{\ell} = (2d+1)^m\,.$$

This implies $\log n \le m \log(2d+1)$ and, moreover,

$$a_n = \frac{1}{(m+1)^s} \le \frac{1}{m^s} \le \left(\frac{\log(2d+1)}{\log n}\right)^s$$

which gives the upper bound. On the other hand,

$$n > C(m-1,d) = \sum_{\ell=0}^{m-1} 2^\ell \binom{d}{\ell}\binom{m-1}{\ell} \ge 2^{m-1} = \frac{1}{4}2^{m+1}$$

which gives $m+1 < \log(4n)/\log 2$. This implies the lower bound. ■

Remark 4.7. Note that there is a minor logarithmic gap in the previous result. Lemma 4.1 applied to $m = 1$ gives $a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = 2^{-s}$ if $2 \le n \le 2d+1$. We conjecture that for $2d+2 < n \le 2^d$ the correct behavior is

$$c_s\left(\frac{\log(d/\log n)}{\log n}\right)^s \le a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le C_s\left(\frac{\log(d/\log n)}{\log n}\right)^s.$$

4.2 The approximation numbers of $H^{s,*}(\mathbb{T}^d)$ in $L_2(\mathbb{T}^d)$

For $s > 0$ we define

$$w_{s,*}(k) := \Big(1 + \sum_{j=1}^{d} |k_j|^{2s}\Big)^{1/2}, \qquad k \in \mathbb{Z}^d\,.$$

Let us mention once more, that these weights do not have the semigroup property, see Paragraph 2.2. By $(\tau_j)_{j=1}^{\infty}$ we denote the decreasing rearrangement of the sequence $(1/w_{s,*}(k))_{k\in\mathbb{Z}^d}$. Furthermore, let $W_s := \{w_{s,*}(k) :\ k \in \mathbb{Z}^d\}$. For $r > 0$ we put

$$C(r,d,s) := \#\Big\{k \in \mathbb{Z}^d : \quad w_{s,*}(k) \le r\Big\} = \#\Big\{\ell \in \mathbb{N} : \quad \tau_\ell \ge \frac{1}{r}\Big\} . \tag{4.3}$$

Concerning approximation numbers of $I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ Lemma 2.4 and (2.10) imply the following.

Lemma 4.8. *For $r \in W_s$ we have*

$$a_{C(r,d,s)}(I_d :\ H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \frac{1}{r} .$$

In the present form the result does not give much information. Let us consider some examples. For small r it is possible to compute $C(r,d,s)$ precisely. Indeed, in case $r = \sqrt{2}$ we have for all $s > 0$

$$C(\sqrt{2},d,s) = \#\Big\{k \in \mathbb{Z}^d : \quad \sum_{j=1}^{d} |k_j|^s \le 1\Big\} = 2d+1 ,$$

whereas for $r = \sqrt{3}$ it holds in the case $s = 1/2$

$$C(\sqrt{3},d,1/2) = \#\Big\{k \in \mathbb{Z}^d : \quad \sum_{j=1}^{d} |k_j| \le 2\Big\} = 1+2d+2d^2 .$$

This implies

$$a_{2d+1}(I_d :\ H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \frac{1}{\sqrt{2}} ,$$

and

$$a_{1+2d+2d^2}(I_d :\ H^{1/2,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \frac{1}{\sqrt{3}} .$$

In the sequel we aim at a more systematic study. To begin with, we deal with the counterpart of Theorem 4.3. The crucial ingredient is the volume of the unit ball B^d_{2s} in the metric $|\cdot|_{2s}$ in $\mathbb{R}^d$, i.e.,

$$\mathrm{vol}(B^d_{2s}) := \mathrm{vol}\Big\{x \in \mathbb{R}^d :\ \sum_{j=1}^{d} |x_j|^{2s} \le 1\Big\} = 2^d\, \frac{\Gamma(1+1/(2s))^d}{\Gamma(1+d/(2s))} , \tag{4.4}$$

see Wang [20], where for $0 < x < \infty$

$$\Gamma(1+x) = \int_0^\infty t^x e^{-t}\, dt$$

denotes the Gamma-function. For the convenience of the reader we will prove upper and lower estimates of this function. Those are important to study the asymptotic behavior of the quantity $\mathrm{vol}(B^d_{2s})^{s/d}$, which will frequently appear in the sequel.

Lemma 4.9. (i) *If* $0 \le x \le 1$ *then* $\Gamma(1+x) \le 1$. *In particular* $\Gamma(1) = \Gamma(2) = 1$.
(ii) *For* $0 \le x < \infty$ *it holds*

$$\left(\frac{x}{e}\right)^x \le \Gamma(1+x) \le (1+x)^x .$$

Proof. To prove (i) it is sufficient to recall that Γ is convex and $\Gamma(1) = \Gamma(2) = 1$. The lower bound in (ii) is a consequence of

$$\Gamma(1+x) \ge \int_x^\infty t^x e^{-t}\, dt \ge x^x e^{-x} .$$

The upper bound in (ii) is clear for $0 \le x \le 1$ using (i). Let now $x > 1$, say $k \le x < k+1$ for some $k \in \mathbb{N}$. This gives

$$\Gamma(1+x) = x(x-1) \cdot \ldots \cdot (x-k+2)\Gamma(x-k+1) . \tag{4.5}$$

Observe, that $x-k+2 \in [1,2]$ and therefore $\Gamma(x-k+1) \le 1$ by (i). Thus, (4.5) yields

$$\Gamma(1+x) \le x^{k-1} \le x^x \le (x+1)^x .$$

The proof is complete. ∎

Now we will turn to the quantity $\mathrm{vol}(B_{2s}^d)^{s/d}$.

Lemma 4.10. *Let* $0 < s < \infty$ *and* $d \in \mathbb{N}$. *Then we have*

$$2^s \sqrt{\frac{1}{e(d+2s)}} \le \mathrm{vol}(B_{2s}^d)^{s/d} \le \begin{cases} 2^s \sqrt{\frac{2es}{d}} & : \quad s \ge 1/2 \,, \\ 2^s \sqrt{\frac{e(2s+1)}{d}} & : \quad s > 0 \,. \end{cases} \tag{4.6}$$

Proof. From (4.4) we obtain the identity

$$\mathrm{vol}(B_{2s}^d)^{s/d} = 2^s \sqrt{\frac{\Gamma(1+x)^{1/x}}{\Gamma(1+y)^{1/y}}}$$

with $x = 1/(2s)$ and $y = d/(2s)$. We apply Lemma 4.9/(ii) and obtain in any case on the one hand

$$\mathrm{vol}(B_{2s}^d)^{s/d} \le 2^s \sqrt{\frac{1+1/(2s)}{d/(2se)}} = 2^s \sqrt{\frac{e(2s+1)}{d}}$$

and on the other hand

$$\mathrm{vol}(B_{2s}^d)^{s/d} \ge 2^s \sqrt{\frac{1/(2se)}{1+d/(2s)}} = 2^s \sqrt{\frac{1}{e(2s+d)}} .$$

In the case $x \le 1$, i.e., $2s \ge 1$, we can slightly refine the upper estimate in (4.6) by using Lemma 4.9/(i). The proof is complete. ∎

The following main result is a counterpart of Theorem 4.3. Note that the behavior of the right-hand side in (4.7) is given in Lemma 4.10. In fact, it scales like $d^{-1/2}$.

Theorem 4.11. *For all $s > 0$ and all $d \in \mathbb{N}$ (both fixed) we have*

$$\lim_{n\to\infty} n^{s/d}\, a_n(I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \mathrm{vol}(B^d_{2s})^{s/d} . \tag{4.7}$$

Proof. *Step 1.* Preparations. We start with some volume estimates. First of all, observe, for any $m \in \mathbb{N}$,

$$\Big(1 + \sum_{j=1}^d |k_j|^{2\,s}\Big)^{1/2} \le (1+m^{2s})^{1/2} \qquad \Longleftrightarrow \qquad |k|_{2s} \le m \, .$$

To each $k \in \mathbb{Z}^d$ we associate a cube Q_k with centre k, sides parallel to the axes and side-length 1. By $\lambda\, B^d_{2s}$ we denote the ball in $\mathbb{R}^d$ with radius λ in the metric $|\cdot|_{2s}$. For $2s \ge 1$ the triangle inequality yields

$$\Big(m - \frac{d^{1/(2s)}}{2}\Big)_+ B^d_{2s} \subset \bigcup_{\substack{k\in\mathbb{Z}^d \\ |\,k\,|_{2s}\le m}} Q_k \subset \Big(m + \frac{d^{1/(2s)}}{2}\Big) B^d_{2s}$$

for all $m \in \mathbb{N}$. If $2s < 1$ the modification reads as follows

$$\Big(m^{2s} - \frac{d}{2^{2s}}\Big)_+^{1/(2s)} B^d_{2s} \subset \bigcup_{\substack{k\in\mathbb{Z}^d \\ |\,k\,|_{2s}\le m}} Q_k \subset \Big(m^{2s} + \frac{d}{2^{2s}}\Big)^{1/(2s)} B^d_{2s} \, .$$

Hence, for $2s \ge 1$ and $2s < 1$ we obtain, respectively,

$$\begin{aligned} \Big(m - \frac{d^{1/(2s)}}{2}\Big)_+^d \mathrm{vol}(B^d_{2s}) \quad &\le \quad C(m,d,s) \le \Big(m + \frac{d^{1/(2s)}}{2}\Big)^d \mathrm{vol}(B^d_{2s})\, , \\ \Big(m^{2s} - \frac{d}{2^{2s}}\Big)_+^{d/(2s)} \mathrm{vol}(B^d_{2s}) \quad &\le \quad C(m,d,s) \le \Big(m^{2s} + \frac{d}{2^{2s}}\Big)^{d/(2s)} \mathrm{vol}(B^d_{2s})\, , \end{aligned} \tag{4.8}$$

where the quantity $C(m,d,s)$ is defined as in (4.3). Let us treat the case $2s \ge 1$ in detail. The remaining case follows by obvious modifications. Relation (4.8) and the monotonicity of the a_n yield for $n \ge A(m,d)$

$$a_n(I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le (1+m^{2s})^{-1/2} \tag{4.9}$$

and for $n \le B(m,d)$

$$a_n(I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge (1+m^{2s})^{-1/2}\, , \tag{4.10}$$

where $A(m,d) := \Big(m + \frac{d^{1/(2s)}}{2}\Big)^d \mathrm{vol}(B^d_{2s})$ and $B(m,d) := \Big(m - \frac{d^{1/(2s)}}{2}\Big)^d \mathrm{vol}(B^d_{2s})\, .$

The first inequality holds for all $m \in \mathbb{N}$, the second one for all $m \ge \frac{d^{1/2s}}{2} + 1$.

Step 2. Let $A(m,d) \le n \le A(m+1,d)$. Then (4.9) leads to

$$n^{s/d}\, a_n(I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big(m + 1 + \frac{d^{1/(2s)}}{2}\Big)^s \mathrm{vol}(B^d_{2s})^{s/d}\, (1+m^{2s})^{-1/2} \, .$$

Obviously

$$\lim_{m\to\infty} \frac{\Big(m+1+\frac{d^{1/(2s)}}{2}\Big)^s}{(1+m^{2s})^{1/2}} = 1 \, .$$

Let $m \geq 2\, d^{1/2s} + 1$. For $B(m,d) \leq n \leq B(m+1,d)$ inequality (4.10) yields

$$n^{s/d}\, a_n(I_d :\, H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \geq \Big(m - \frac{d^{1/(2s)}}{2}\Big)^s \operatorname{vol}(B_{2s}^d)^{s/d}\, (1 + (m+1)^{2s})^{-1/2} .$$

Again we have

$$\lim_{m\to\infty} \Big(m - \frac{d^{1/(2s)}}{2}\Big)^s (1 + (m+1)^{2s})^{-1/2} = 1.$$

This proves (4.7) in the case $s \geq 1/2$. The modifications for $0 < s < 1/2$ are straightforward. ■

Switching from the asymptotic behavior to estimates where d is large but fixed, we get another main result of the paper, a counterpart of Theorem 4.5.

Theorem 4.12. *Let $s \geq 1/2, d \in \mathbb{N}$ and*

$$a_n := a_n(I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) .$$

Then we have for $n \geq 9^d e^{d/(2s)}$

$$a_n \leq \frac{4^s}{\sqrt{d}} \sqrt{2es}\, n^{-s/d} \tag{4.11}$$

and for $n \geq 11^d e^{d/(2s)}$ it holds

$$a_n \geq \frac{1}{\sqrt{e(d+2s)}} n^{-s/d} . \tag{4.12}$$

Proof. To prove (4.11) we fix $m_0 \in \mathbb{N}$ such that

$$1 + \frac{d^{1/(2s)}}{2} \leq m_0 < 2 + \frac{d^{1/(2s)}}{2} . \tag{4.13}$$

For all $m \geq m_0$ we have in the case $A(m,d) \leq n \leq A(m+1,d)$ the relation, cf. (4.9),

$$n^{s/d} a_n \leq \frac{A(m+1,d)^{s/d}}{(1+m^{2s})^{1/2}} \leq \frac{\Big(m + 1 + \frac{d^{1/(2s)}}{2}\Big)^s}{m^s} \cdot \operatorname{vol}(B_{2s}^d)^{s/d} . \tag{4.14}$$

By $m \geq m_0$ and the choice of m_0 in (4.13) the first factor in (4.14) does not exceed 2^s. By Lemma 4.10 we can bound $\operatorname{vol}(B_{2s}^d)^{s/d} \leq \sqrt{2es/d}$. Combining both estimates yields

$$n^{s/d} a_n \leq 4^s \sqrt{\frac{2es}{d}}$$

which proves (4.11) for large enough $n \geq A(m_0,d)$. Using again (4.13) gives

$$\begin{aligned} A(m_0,d) &= \Big(m_0 + \frac{d^{1/(2s)}}{2}\Big)^d \operatorname{vol}(B_{2s}^d) \\ &\leq (2 + d^{1/(2s)})^d 2^d \operatorname{vol}(B_{2s}^d) \\ &\leq 2^d (2 + d^{1/(2s)})^d \Big(\frac{2es}{d}\Big)^{d/(2s)} , \end{aligned}$$

where we used Lemma 4.10 in the last step. Therefore, we obtain

$$\begin{aligned} A(m_0,d) &\le 2^d\Big(\frac{2+d^{1/(2s)}}{d^{1/(2s)}}\Big)^d(2es)^{d/(2s)} \\ &\le 2^d\cdot 3^d[(2s)^{1/(2s)}]^d e^{d/(2s)}\,. \end{aligned}$$

Using $x^{1/x}\le e^{1/e}\le 3/2$ for $x>0$ gives the estimate

$$A(m_0,d)\le [6e^{1/e}]^d e^{d/(2s)}\,.$$

Hence, (4.11) is true for $n\ge 9^d e^{d/(2s)}$.

We turn to the lower estimate in (4.12). Let m_1 be chosen such that

$$2+d^{1/(2s)}\le m_1\le 3+d^{1/(2s)}\,. \tag{4.15}$$

For $m\ge m_1$ and $B(m,d)\le n\le B(m+1,d)$ we find, see (4.10),

$$n^{s/d}a_n\ge \frac{B(m,d)^{s/d}}{(1+(m+1)^{2s})^{1/2}}\ge \frac{\Big(m-\frac{d^{1/(2s)}}{2}\Big)^s}{(m+2)^s}\,\mathrm{vol}(B^d_{2s})^{s/d}, \tag{4.16}$$

where we used $((m+1)^{2s}+1)^{1/(2s)}\le (m+1)$ since $2s\ge 1$. The choice of m_1 in (4.15) and $m\ge m_1$ implies that the first factor in (4.16) is bounded from below by 2^{-s}, whereas $\mathrm{vol}(B^d_{2s})^{s/d}\ge 2^s/\sqrt{e(d+2s)}$, see Lemma 4.10. Putting both bounds into (4.16) gives (4.12) for $n\ge B(m_1,d)$. By the choice of m_1 and Lemma 4.10 in connection with $2s\ge 1$ we see

$$\begin{aligned} B(m_1,d) &= \Big(m_1-\frac{d^{1/(2s)}}{2}\Big)^d\mathrm{vol}(B^d_{2s}) \\ &\le \Big(3+\frac{d^{1/(2s)}}{2}\Big)^s 2^d\Big(\frac{2es}{d}\Big)^{d/(2s)} \\ &= \Big(\frac{6+d^{1/(2s)}}{d^{1/(2s)}}\Big)^d(2es)^{d/(2s)} \\ &= (7e^{1/e})^d e^{d/(2s)}\le 11^d e^{d/(2s)}\,. \end{aligned}$$

Hence, (4.12) is true for $n\ge 11^d e^{d/(2s)}$. This finishes the proof. ∎

Remark 4.13. (i) In the case of small smoothness $0<s<1/2$, one can prove analogous results: There exist constants $c_1(s), c_2(s)$ depending only on s, and absolute constants $C_1, C_2>1$, such that

$$a_n\le \frac{c_1(s)}{\sqrt{d}}n^{-s/d}\,,\quad \text{if}\quad n\ge C_1^{d/(2s)}\,,$$

and

$$a_n\ge \frac{c_2(s)}{\sqrt{d}}n^{-s/d}\,,\quad \text{if}\quad n\ge C_2^{d/(2s)}\,.$$

Note that for $0<s<1/2$ the exponent $d/(2s)$ can become arbitrarily large. In contrast, for $s\ge 1/2$ one has $d/(2s)\le d$.

(ii) Similar as done in Theorem 4.6 it remains to discuss the behavior of the approximation numbers if n is small. Unfortunately, we do not yet have the proper tools available to do this in a satisfactory way. However, using (4.12) and the monotonicity of the approximation numbers, we can state the following. If $1\le n<11^d e^{d/(2s)} := M$, then it holds

$$1\ge a_n\ge \frac{1}{\sqrt{e(d+2s)}}M^{-s/d}=\frac{1}{\sqrt{(d+2s)}}\frac{1}{11^s\cdot e}\,.$$

4.3 The approximation numbers of $H^{s,+}(\mathbb{T}^d)$ in $L_2(\mathbb{T}^d)$

Recall the weight sequence $w_{s,+}(k) = (1+\sum_{i=1}^d |k_i|^2)^{s/2}$, $k \in \mathbb{Z}^d$, used for defining $\|\cdot|H^s(\mathbb{T}^d)\|^+$, see Definition 2.2/(i). Clearly, for every $s > 0$ we have $w_{s,+}(k) = w_{1,*}(k)^s$, $k \in \mathbb{Z}^d$. By Lemma 2.5 we can benefit from the results in the previous section. In fact, using that

$$a_n(I_d : H^{s,+}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = a_n(I_d : H^{1,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))^s \tag{4.17}$$

for all $n \in \mathbb{N}$, we obtain as a direct consequence of Theorem 4.11 the following.

Theorem 4.14. *For all $s > 0$ and all $d \in \mathbb{N}$ (both fixed) we have*

$$\lim_{n\to\infty} n^{s/d}\, a_n(I_d : H^{s,+}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \mathrm{vol}(B_2^d)^{s/d}\,.$$

Moreover, Theorem 4.12 in connection with (4.17) implies the following result.

Theorem 4.15. *Let $s > 0, d \in \mathbb{N}$ and*

$$a_n := a_n(I_d : H^{s,+}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))\,.$$

Then we have for $n \geq 9^d e^{d/2}$

$$a_n \leq 4^s \Big(\frac{2e}{d}\Big)^{s/2} n^{-s/d}$$

and for $n \geq 11^d e^{d/2}$ it holds

$$a_n \geq \Big(\frac{1}{e(d+2)}\Big)^{s/2} n^{-s/d}\,.$$

Due to to the multiplicativity of approximation numbers in connection with (2.3) and Theorem 4.15 we obtain the following result for the embedding $I_d : H^m(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$, see Definition 2.1.

Corollary 4.16. *Let $m, d \in \mathbb{N}$ and $a_n := a_n(I_d : H^m(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$. Then for $n \geq 9^d e^{d/2}$ we have*

$$a_n \leq 4^m \sqrt{m!}\Big(\frac{2e}{d}\Big)^{m/2} n^{-m/d}$$

and for $n \geq 11^d e^{d/2}$ we get

$$a_n \geq \Big(\frac{1}{e(d+2)}\Big)^{m/2} n^{-m/d}\,.$$

Remark 4.17. All three Theorems 4.5, 4.12, and 4.15 (as well as Corollary 4.16) are satisfactory in the following sense. We proved the two-sided inequalities for sufficiently large n

$$\frac{a_s}{d^s}\, n^{-s/d} \leq a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \frac{A_s}{d^s}\, n^{-s/d}\,, \tag{4.18}$$

$$\frac{b_s}{\sqrt{d}}\, n^{-s/d} \leq a_n(I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \frac{B_s}{\sqrt{d}}\, n^{-s/d}\,, \tag{4.19}$$

and

$$\frac{c_s}{d^{s/2}}\, n^{-s/d} \leq a_n(I_d : H^{s,+}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \frac{C_s}{d^{s/2}}\, n^{-s/d}\,, \tag{4.20}$$

where $a_s, b_s, c_s, A_s, B_s, C_s$ depend only on s. Note that we captured the exact decay rate in n and the exact order of the constants with respect to d. Surprisingly, the constants decay polynomially in d and their decay order differs according to the chosen norm $\|\cdot|H^s(\mathbb{T}^d)\|^*$, $\|\cdot|H^s(\mathbb{T}^d)\|^\#$, and $\|\cdot|H^s(\mathbb{T}^d)\|^+$, respectively. Of course, if the space is smaller one would expect a better decay rate. This fact is reflected by the embeddings in Lemma 2.3.

Let us emphasize, that (4.19), (4.20) and Corollary 4.16 are of certain interest, since we deal with natural/classical Sobolev norms obtained by taking derivatives in the case of s being an integer, see (2.2), (2.3), (2.4), and (2.5). The norm $\|\cdot|H^s(\mathbb{T}^d)\|^\#$ seems to be artificial. However, it is very useful since we have powerful tools from combinatorics available. The results in (4.18) are not just interesting for its own sake. Some further assertions, mainly from the next Section 5, can be reduced to this situation.

Remark 4.18. In this paper we exclusively deal with isotropic Sobolev spaces $H^s(\mathbb{T}^d)$. The recent paper [3] considers a similar framework for mixed smoothness Sobolev spaces and further generalizations thereof but with completely different methods. Tractability issues have not been considered there. In this context we refer to [9, p. 33]. There it has been shown that the approximation in the mixed smoothness Sobolev space $S^1H(\mathbb{T}^d)$ of order one is weakly tractable in L_2. In a forthcoming paper [7] the authors will improve this result to quasi-polynomial tractability by exploiting similar methods as developed in the present paper. Moreover, the exact behavior of the approximation numbers in n and d will be of main interest.

5 (In)Tractable approximation of Sobolev embeddings

5.1 General notions of tractability

Various concepts of tractability are discussed in the recent monographs by Novak and Woźniakowski [9, 11, 12]. For arbitrary $s > 0$ and all $d \in \mathbb{N}$ we consider the embedding operators (formal identities)

$$I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\,,$$

where the Sobolev spaces are equipped with the norms $\|\cdot|H^s(\mathbb{T}^d)\|^*$, $\|\cdot|H^s(\mathbb{T}^d)\|^\#$, and $\|\cdot|H^s(\mathbb{T}^d)\|^+$. In all three cases we have $\|I_d\| = 1$ for all $s > 0$ and $d \in \mathbb{N}$. In other words, the *normalized error criterion* is satisfied. In this context, a *linear algorithm* that uses *arbitrary information* (Λ^{all}) is of the form

$$A_{n,d}(f) = \sum_{j=1}^{n} L_j(f) g_j\,, \tag{5.1}$$

where $g_j \in L_2(\mathbb{T}^d)$ and L_j are continuous linear functionals. If the error is measured in the norm of $L_2(\mathbb{T}^d)$ we can identify the algorithm $A_{n,d}$ with a bounded linear operator $A_{n,d} : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ of rank at most n. The worst-case error of $A_{n,d}$ with respect to the unit ball (respective norms) in $H^s(\mathbb{T}^d)$

$$\sup_{\|f|H^s(\mathbb{T}^d)\|\le 1} \|f - A_{n,d}(f)|L_2(\mathbb{T}^d)\|$$

clearly coincides with the operator norm $\|I_d - A_{n,d} : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\|$, and the *nth minimal worst-case error* with respect to *linear algorithms and general information*

$$\inf_{\operatorname{rank} A_{n,d} \le n} \|I - A_{n,d} : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\|$$

is just the *approximation number* $a_{n+1}(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$, see (1.2). Since we deal with a Hilbert space setting (source and target space) it is sufficient to restrict to *linear algorithms* with *non-adaptive* information (5.1), see [18], [2] as well as [9, Thm. 4.5, 4.8, 4.11].

Finally, the *information complexity* of the d-variate approximation problem is measured by the quantity $n(\varepsilon, d)$ defined by

$$n(\varepsilon, d) := \inf\{n \in \mathbb{N} : a_n(I_d) \le \varepsilon\} \tag{5.2}$$

as $\varepsilon \to 0$ and $d \to \infty$. The approximation problem is *quasi-polynomially tractable* if there exist two constants $C, t > 0$ such that

$$n(\varepsilon, d) \le C\exp(t(1 + \ln(\varepsilon^{-1}))(1 + \ln d)) . \tag{5.3}$$

It is called *weakly tractable*, if

$$\lim_{1/\varepsilon + d \to \infty} \frac{\log n(\varepsilon, d)}{1/\varepsilon + d} = 0 , \tag{5.4}$$

i.e., $n(\varepsilon, d)$ neither depends exponentially on $1/\varepsilon$ nor on d. Clearly, a problem is weakly tractable if it is quasi-polynomially tractable. The problem is called *intractable*, if (5.4) does not hold, see the definition [9, p. 7]. If for some $0 < \varepsilon < 1$ the number $n(\varepsilon, d)$ is an exponential function in d then we say that the approximation problem suffers from *the curse of dimensionality.* In other words, if there exist positive numbers c, ε_0, γ such that

$$n(\varepsilon, d) \ge c(1 + \gamma)^d , \quad \text{for all } 0 < \varepsilon \le \varepsilon_0 \text{ and infinitely many } d \in \mathbb{N} , \tag{5.5}$$

then the problem suffers from *the curse of dimensionality*.

5.2 Tractability results for $H^s(\mathbb{T}^d)$

In this section we study tractability issues of the approximation problem for the Sobolev embeddings

$$I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d) , \quad d \in \mathbb{N} ,$$

with respect to the norms $\| \cdot |H^s(\mathbb{T}^d)\|^{\#}$, $\| \cdot |H^s(\mathbb{T}^d)\|^{*}$, and $\| \cdot |H^s(\mathbb{T}^d)\|^{+}$. If $s = m \in \mathbb{N}$ we also consider the embedding with respect to the classical norm $H^m(\mathbb{T}^d)$, see Definition 2.1.

We will mainly deal with weak tractability. The following Proposition shows that this is the best we can hope for.

Proposition 5.1. *For every $s > 0$ none of the above mentioned approximation problems is quasi-polynomially tractable.*

Proof. Let us prove the statement for the problem $I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ for any $s > 0$. Recall the definition of the set $\mathcal{N}(m, d)$ and its cardinality $C(m, d)$ from Section 3. Clearly, every $k \in \{-1, 0, 1\}^d$ belongs to $\mathcal{N}(d, d)$ and therefore $C(d, d) \ge 3^d$. By Lemma 4.1 we get immediately $a_{3^d}(I_d) \ge a_{C(d,d)}(I_d) = (d+1)^{-s}$. Putting $\varepsilon_d = (d+2)^{-s}$ we obtain $n(\varepsilon_d, d) \ge 3^d$, and this contradicts (5.3). What remains is a matter of embeddings. In fact, if $s \ge 1$ Lemma 2.3/(i) implies the same statement for $H^{s,*}(\mathbb{T}^d)$ and $H^{s,+}(\mathbb{T}^d)$. For smaller s we argue with Lemma 2.3/(iv). Let us finally deal with $H^m(\mathbb{T}^d)$ if $s = m \in \mathbb{N}$. By (2.3) we have the norm one embedding $H^{m,+}(\mathbb{T}^d) \hookrightarrow H^m(\mathbb{T}^d)$. Since we know from the above arguments that quasi-polynomial tractability does not hold for $H^{m,+}(\mathbb{T}^d)$ the proof is complete. ■

The following result concerns weak tractability of the approximation problem for Sobolev embeddings $I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$. We give a positive answer for "large" s and a negative answer for the remaining "small" values of s.

Theorem 5.2. *Let $s > 0$. Then the approximation problem for the embeddings*

$$I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\,, \quad d \in \mathbb{N}\,,$$

(i) is weakly tractable, if $s > 1$,
(ii) and intractable, if $0 < s \leq 1$.

Proof. To prove part (ii), it is enough to consider the case $s = 1$ since we have the embedding $H^{1,\#}(\mathbb{T}^d) \hookrightarrow H^{s,\#}(\mathbb{T}^d)$ with norm one for all $0 < s < 1$. From Lemma 4.1 we know that

$$a_{C(m-1,d)}(I_d : H^{1,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \frac{1}{m}$$

This gives for all $m \in \mathbb{N}$

$$n\Big(\frac{1}{m+1}, d\Big) \geq C(m-1,d) + 1 > C(m-1,d)\,.$$

Choosing $m = d+1$ and $\varepsilon = \varepsilon_d = \frac{1}{d+1}$ this yields

$$\frac{\log n(\varepsilon_d, d)}{1/\varepsilon_d + d} \geq \frac{\log C(d,d)}{2d+1} \geq \frac{d \log 2}{d+1} \xrightarrow[d\to\infty]{} \frac{\log 2}{2}\,,$$

where we took $C(d,d) \geq \binom{2d}{d} \geq 2^d$ into account, see (3.5). That means, that we have shown intractability in the case $0 < s \leq 1$.

We proceed with the proof of (i). Assume $s > 1$. Let $0 < \varepsilon \leq 1$ be given and select $m \in \mathbb{N}$ such that $(m+1)^{-s} < \varepsilon \leq m^{-s}$. From Lemma 4.1 we get

$$a_n(I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \frac{1}{(m+1)^s}$$

if $C(m-1,d) < n \leq C(m,d)$, which implies

$$n(\varepsilon, d) \leq n\Big(\frac{1}{(m+1)^s}, d\Big) \leq C(m,d) \leq 2^{\min\{m,d\}} \binom{m+d}{d}\,,$$

where the last inequality is given in Lemma 3.4. Using $\binom{m+d}{m} = \binom{m+d}{d}$ we see that

$$2^{\min\{m,d\}} \binom{m+d}{d} \leq [2(m+d)]^{\min\{m,d\}}$$

and hence $n(\varepsilon, d) \leq [2(m+d)]^{\min\{m,d\}}$. We put $x = 1/\varepsilon + d$ and obtain, taking $s > 1$ into account,

$$m + d \leq m^s + d \leq x\,.$$

Additionally, we have the slightly better estimate $\min\{m,d\} \leq m \leq (m^s + d)^{1/s} \leq x^{1/s}$. Consequently,

$$\frac{\log(\varepsilon, d)}{1/\varepsilon + d} \leq \frac{x^{1/s} \log(2x)}{x} \xrightarrow[x\to\infty]{} 0\,,$$

since $s > 1$. ■

The next theorem gives a negative answer for the approximation problem $I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ for all $s > 0$.

Theorem 5.3. *For all $s > 0$ the approximation problem*

$$I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\,, \quad d \in \mathbb{N}\,,$$

is intractable.

Proof. Due to the embedding relations with norm one it is enough to consider large s, say $s \geq 1$. We will use the estimates from (4.10), i.e.,

$$a_n(I_d : H^{s,*}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \geq \frac{1}{\sqrt{1+m^{2s}}} > \frac{1}{2m^s} =: \varepsilon$$

if $n \leq B(m,d)$ and $m > 1 + \frac{d^{1/(2s)}}{2}$. For the complexity this implies

$$n(\varepsilon, d) \geq B(m,d)\,.$$

Choose now, for all $d \in \mathbb{N}$, the number $m_d \in \mathbb{N}$ such that $d^{1/s} \leq m_d < d^{1/s} + 1$ and set $\varepsilon_d = 1/(2(m_d)^s)$. This gives

$$n(\varepsilon_d, d) \geq B(m_d, d) = \Big(m_d - \frac{d^{1/(2s)}}{2}\Big)^d \mathrm{vol}(B^d_{2s})\,.$$

Again we use Lemma 4.10 to bound $\mathrm{vol}(B^d_{2s})$ from below. This gives

$$n(\varepsilon_d, d) \geq \frac{\Big(2m_d - d^{1/(2s)}\Big)^d}{[e(d+2s)]^{d/2s}} \geq \frac{(m_d)^d}{[e(d+2s)]^{d/2s}} \geq \Big(\frac{d^2}{e(2s+d)}\Big)^{d/(2s)}\,,$$

where the last two inequalities are due to the choice of m_d. This gives

$$\frac{\log n(\varepsilon_d, d)}{1/\varepsilon_d + d} \geq \frac{d}{2s} \cdot \frac{\log\Big(\frac{d^2}{e(d+s)}\Big)}{1/\varepsilon_d + d} \geq \frac{d}{2s}\,\frac{\log\Big(\frac{d^2}{e(d+s)}\Big)}{2^{s+1}d + d}\,,$$

where the inequality $1/\varepsilon_d = 2(m_d)^s \leq 2^{s+1}d$ has been used in the last step. Finally, we obtain

$$\frac{\log n(\varepsilon_d, d)}{1/\varepsilon_d + d} \geq \frac{\log\Big(\frac{d^2}{e(d+s)}\Big)}{2^{s+2}s + 2s} \xrightarrow[d\to\infty]{} \infty\,.$$

The proof is complete. ■

Remark 5.4. In the proof of Theorem 5.3 we showed more than actually needed for intractability, namely

$$\limsup_{1/\varepsilon + d \to \infty} \frac{\log n(\varepsilon, d)}{1/\varepsilon + d} = \infty.$$

This might indicate that the problem is "very" intractable. However, we will see in Theorem 5.6 below that it still does not suffer from the curse of dimensionality.

The next result on weak tractability deals with the natural norm on $H^s(\mathbb{T}^d)$, see Definition 2.2/(i). Surprisingly, we obtain a positive result in the case $s > 2$ and a negative result for $0 < s \le 1$. The situation in between is so far unclear.

Theorem 5.5. *Let $s > 0$. Then the approximation problem for the embeddings*

$$I_d : H^{s,+}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\,, \quad d \in \mathbb{N}\,,$$

(i) is weakly tractable, if $s > 2$,
(ii) and intractable, if $0 < s \le 1$.

Proof. The assertion (i) follows directly from Theorem 5.2 together with the embedding in Lemma 2.3/(v). As for (ii) we combine Theorem 5.3 with Lemma 2.3/(ii),(iii). ■

Theorem 5.6. *For every $s > 0$ none of the above considered approximation problems suffers from the curse of dimensionality.*

Proof. Let us first prove the assertion for $I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ for all s. Let $0 < \varepsilon \le 1$ be given and fixed. Select $m \in \mathbb{N}$ such that $(m+1)^{-s} < \varepsilon \le m^{-s}$. By proceeding literally as in the proof of Theorem 5.2/(i) we obtain $n(\varepsilon, d) \le [2(m+d)]^{\min\{m,d\}}$. Hence, for all $d > m$ we obtain

$$n(\varepsilon, d) \le (4d)^m\,.$$

This contradicts (5.5), and therefore we have no curse of dimensionality for the situation $I_d : H^{s,\#}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$.

What remains follows from the norm one embeddings in Lemma 2.3. In fact, it is enough to consider small values of s, say $0 < s \le 1$. In this case we always have

$$H^{s,*}(\mathbb{T}^d) \hookrightarrow H^{s,+}(\mathbb{T}^d) \hookrightarrow H^{s/2,\#}(\mathbb{T}^d)$$

as a consequence of Lemma 2.3/(ii),(iii),(v). This concludes the proof. ■

We finish the paper with tractability statements concerning the embeddings $I_d : H^m(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$, $m \in \mathbb{N}$, using the classical norm, see Definition 2.1. In (2.3) we have already seen that the norms $\|\cdot|H^m(\mathbb{T}^d)\|$ and $\|\cdot|H^m(\mathbb{T}^d)\|^+$ are equivalent with equivalence constants only depending on m. Hence, if m is fixed, both problems have the same behavior with respect to weak tractability and curse of dimensionality. As a direct consequence of Theorems 5.5 and 5.6 we obtain the following final result.

Corollary 5.7. *Let $m \in \mathbb{N}$. Then the approximation problem for the embeddings*

$$I_d : H^m(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\,, \quad d \in \mathbb{N}\,,$$

(i) *does not suffer from the curse of dimensionality,*
(ii) *is intractable if $m = 1$, and*
(iii) *weakly tractable if $m \ge 3$.*

Remark 5.8. In other words, the preceding corollary states that the boundedness of first order derivatives is not sufficient for weak tractability. For bounded second order derivatives it is so far not clear what happens. However, the L_2-boundedness of all weak derivatives up to order three guarantees weak tractability. This is a remarkable fact, since in [10] the authors showed that the approximation of infinitely differentiable multivariate functions is intractable and even suffers from the curse of dimensionality. However, this is no contradiction since both settings are not really comparable. Note that in [10] the error is measured in L_∞ and the quantity defining the class of functions F_d contains a sup over the order of the derivatives and not a sum like in our setting. We have already seen in Theorems 5.2, 5.3 that the notion of weak tractability is sensitive with respect to the used norms. Let us also mention the recent paper [19] in this context. There it has been shown that a renorming of the class of infinitely differentiable functions leads to weak tractability even for standard information.

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