On an extreme class of real interpolation spaces

To the memory of Professor Pedro Matos

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Abstract

We investigate the limit class of interpolation spaces that comes up by the choice $\theta = 0$ in the definition of the real method. These spaces arise naturally interpolating by the J-method associated to the unit square. Their duals coincide with the other extreme spaces obtained by the choice $\theta = 1$. We also study the behaviour of compact operators under these two extreme interpolation methods. Moreover, we establish some interpolation formulae for function spaces and for spaces of operators.

Key words:
Extreme interpolation spaces, real interpolation, J-functional, K-functional, interpolation methods associated to polygons, compact operators, Lorentz-Zygmund function spaces, spaces of operators.
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1 Introduction

Interpolation theory is a very useful tool not only in functional analysis, operator theory, harmonic analysis and partial differential equations, but also in some other more distant areas of mathematics. Monographs by Butzer and Berens [6], Bergh and Löfström [5], Triebel [36], Beauzamy [2], Bennett and Sharpley [4], Connes [18] and Amrein, Boutet de Monvel and Georgescu [1] illustrate this fact. The real interpolation method $(A_0, A_1)_{\theta,q}$ is particularly useful due to its flexibility. It has several equivalent definitions, being the more important those given by Peetre's K- and J-functional, which allow to use it in different contexts.

Let A_0, A_1 be Banach spaces with A_0 continuously embedded in A_1 and $1 \leq q \leq \infty$. When θ runs in $(0, 1)$, spaces $(A_0, A_1)_{\theta,q}$ form a "continuous scale" of spaces joining A_0 with A_1. If we imagine A_0 and A_1 sitting on the endpoints of the segment $[0, 1]$ then we can think of $(A_0, A_1)_{\theta,q}$ as the space located at the point θ (see Fig. 1.1).

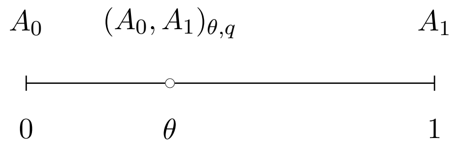

Figure 1.1

This picture is also connected with definitions of K- and J-functionals: Having in mind that A_j is sitting on the point j, we modify the norm of $A_0 + A_1 = A_1$ by inserting the weight t^j in front of $\|\cdot\|_{A_j}$ and the outcome is the K-functional (see Section 2 for the precise definition). The J-functional is generated similarly but working with the intersection $A_0 \cap A_1 = A_0$.

The geometrical elements involved in the construction of the real interpolation space are more visible when we consider its extensions to finite families (N-

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tuples) of Banach spaces. So, in the extension proposed by Cobos and Peetre [17], spaces of the N-tuple $\{A_1, \ldots, A_N\}$ are thought of as sitting on the vertices of a convex polygon $\Pi = \overline{P_1 \cdots P_N}$ in the plane $\mathbb{R}^2$. K- and J-functionals with two parameters t, s are defined by inserting the weight $t^{x_j} s^{y_j}$ in front of the norm of A_j, where $P_j = (x_j, y_j)$ is the vertex on which A_j is sat. Then, for any point (α, β) in the interior of Π, K- and J-spaces are introduced by using an (α, β)-weighted L_q-norm. For the special choice of Π as the simplex, these constructions give back (the first nontrivial case of) spaces introduced by Sparr [35], and if Π coincides with the unit square they recover spaces studied by Fernandez [23].

Developing the theory of interpolation methods associated to polygons, there is a case that sometimes is harder and may give rise to unexpected results. This is the case when the interior point (α, β) is in any diagonal of Π (see [22,16]). A recent results on this matter of Cobos, Fernández-Cabrera and Martín [11] shows that if $A_0 \hookrightarrow A_1$ and we interpolate using the unit square the 4-tuple obtained by sitting A_0 on the vertices $(0,0)$ and $(1,1)$, and A_1 on $(1,0)$ and $(0,1)$ then when (α, β) lies on the diagonal $\beta = 1 - \alpha$, K-spaces coincide with limit interpolation spaces $(A_0, A_1)_{1,q;K}$. The case of the J-spaces was left open in [11].

Accordingly, we investigate here spaces that arise using the J-method when (α, β) lies on the diagonals. It turns out that if $\beta = \alpha$ then they correspond to the extreme choice $\theta = 0$ in the construction of the real interpolation space realized as a J-space. This new class of interpolation spaces, that we call $(0, q; J)$-spaces, are not far from A_0. In fact, if $0 < \theta_0 < \theta_1 < 1$, $X = (A_0, A_1)_{\theta_0, q}$ and $Y = (A_0, A_1)_{\theta_1, q}$, then $(X, Y)_{0,q;J}$ has a similar description to X but instead of multiplying the K-functional by $t^{-\theta_0}$, we have to multiply by $t^{-\theta_0}(1 + \log t)^{-1/q'}$ where $1/q + 1/q' = 1$ (see Section 3 for the precise result). We show that $(0, q; J)$-spaces can be equivalently described by means of the K-functional and we identify some concrete $(0, q; J)$-spaces generated by couples of function spaces and of spaces of operators.

Results that we establish on $(0, q; J)$-spaces exhibit a number of important changes in comparison with the theory of the real method. For example, referring to norm estimates for interpolated operators, instead of the well-known inequality for the real method

$$\|T\|_{(A_0,A_1)_{\theta,q},(B_0,B_1)_{\theta,q}} \leq \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta},$$

interpolated operators by the $(0, q; J)$-method satisfy

$$\|T\|_{(A_0,A_1)_{0,q;J},(B_0,B_1)_{0,qJ}} \leq \|T\|_{A_0,B_0} \Big[1 + \Big(\log \frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}} \Big)_+ \Big].$$

As for compact operators, a result of Cwikel [19] and Cobos, Kühn and Schonbek [15] shows that if any restriction of the operator is compact, then the interpolated operator by the real method is compact as well. However, this is not the case for the $(0, q; J)$-method as we show in Section 6. It turns out that compactness of $T : A_1 \to B_1$ is not enough to imply that the interpolated operator is compact. Nevertheless, compactness of $T : A_0 \to B_0$ does it.

We also establish here new results on $(1, q; K)$-spaces. Among others, we show that on the contrary to the case of the $(0, q; J)$-method, if $T : A_0 \to B_0$ is compact then the interpolated operator by the $(1, q; K)$-method might fail to be compact, but compactness of $T : A_1 \to B_1$ implies that the interpolated operator is also compact. Furthermore, we prove that if $1 < q < \infty$ and $1/q + 1/q' = 1$ then the dual of a $(0, q; J)$-space coincides with a $(1, q'; K)$-space and, conversely, the dual of a $(1, q; K)$-space is a $(0, q'; J)$-space.

The paper is organized as follows. In Section 2 we review some basic concepts from real interpolation and we recall definitions of function spaces and spaces of operators that we shall need latter. In Section 3 we introduce $(0, q; J)$-spaces and we show some basic properties and examples. The description of $(0, q; J)$-spaces in terms of the K-functional is given in Section 4, where we also determine some more concrete $(0, q; J)$-spaces. In Section 5 we show that $(0, q; J)$-spaces arise by interpolation using the unit square and the point (α, β) in the diagonal $\beta = \alpha$. Compactness results for interpolated operators are established in Section 6. Section 7 is devoted to $(1, q; K)$-spaces and in the final Section 8 we prove the duality theorems.

2 Preliminaries

Let $\bar{A} = (A_0, A_1)$ be a couple of Banach spaces with $A_0 \hookrightarrow A_1$, where the symbol $\hookrightarrow$ means continuous inclusion. For each $t > 0$, Peetre's K- and J-functionals are defined by

$$\begin{aligned} K(t, a) &= K(t, a; A_0, A_1) \\ &= \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, \ a \in A_1 \,, \end{aligned}$$

and

$$J(t, a) = J(t, a; A_0, A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \qquad a \in A_0 \,.$$

For $0 < \theta < 1$ and $1 \le q \le \infty$, the *real interpolation spaces* $(A_0, A_1)_{\theta,q}$ realized as a K-space consists of all elements $a \in A_1$ having a finite norm

$$\|a\|_{\theta,q;K} = \begin{cases} \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{1/q} & \text{if } 1 \le q < \infty, \\ \sup_{t>0} \{t^{-\theta} K(t, a)\} & \text{if } q = \infty, \end{cases}$$

(see [6,5,36,4]). According to the equivalence theorem (see [5, Theorem 3.3.1]), the space $(A_0, A_1)_{\theta,q}$ can be equivalently described by means of the J-functional as the collection of all those $a \in A_1$ which can be represented as $a = \int_0^\infty u(t)dt/t$ (convergence in A_1), where $u(t)$ is a strongly measurable function with values in A_0 and

$$\left(\int_0^\infty \left(t^{-\theta} J(t, u(t)) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

(the integral should be replaced by the supremum when $q = \infty$). Moreover, $\| \cdot \|_{\theta,q;K}$ is equivalent to the norm

$$\|a\|_{\theta,q;J} = \inf \left\{ \left(\int_0^\infty \left(t^{-\theta} J(t, u(t)) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} : a = \int_0^\infty u(t) \frac{dt}{t} \right\}.$$

Since we are working with an ordered couple, that is $A_0 \hookrightarrow A_1$, it is not difficult to check that the top norms are equivalent to those obtained replacing the interval $(0, \infty)$ by the smaller interval $(1, \infty)$. Namely,

$$\left(\int_1^\infty \left(t^{-\theta} K(t, a) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

and

$$\inf \left\{ \left(\int_1^\infty \left(t^{-\theta} J(t, u(t)) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} : a = \int_1^\infty u(t) \frac{dt}{t} \right\}. \quad (2.1)$$

This observation will be important to define extreme interpolation spaces.

Let $\bar{B} = (B_0, B_1)$ be another couple of Banach spaces with $B_0 \hookrightarrow B_1$. We put $T \in \mathcal{L}(\bar{A}, \bar{B})$ to mean that T is a bounded linear operator from A_1 into B_1, whose restriction to A_0 defines a bounded linear operator from A_0 into B_0. Let $\|T\|_{A_j,B_j}$ be the norm of T acting from A_j into B_j $(j = 0, 1)$. It is well-known that the restriction $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is bounded with norm less than or equal to $\|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}$.

If we replace the function t^θ in the definition of $(A_0, A_1)_{\theta,q}$ by a more general function parameter $\varrho(t)$, then we obtain spaces $(A_0, A_1)_{\varrho,q}$ (see, for example, [28,29]).

Next we consider some concrete cases. Let (Ω, μ) be a finite measure space. For any $1 \leq p \leq \infty$, we let L_p be the usual *Lebesgue space*. Given any measurable function f on Ω, we denote by f^* its non-increasing rearrangement

$$f^*(t) = \inf\{s > 0 : \mu(\{x \in \Omega : \ |f(x)| > s\}) \leq t\},$$

and we put $f^{**}(t) = (1/t) \int_0^t f^*(s)ds$. For $1 < p < \infty, 1 \le q \le \infty$ and $b \in \mathbb{R}$ or $p = \infty, 1 \le q \le \infty$ and $b < -1/q$, the *Lorentz-Zygmund function space* $L_{p,q}(\log L)_b$ is defined to be the set of all (equivalence classes of) measurable functions f on Ω which have a finite norm

$$\|f\|_{L_{p,q}(\log L)_b} = \left(\int_0^{\mu(\Omega)} \left[t^{\frac{1}{p}} (1 + |\log t|)^b f^{**}(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

(with the obvious modification if $q = \infty$). We refer to [3,4,20,33] for basic properties of Lorentz-Zygmund function spaces. Note that $L_{p,p}(\log L)_b = L_p(\log L)_b$ are the *Zygmund spaces* and $L_{p,q}(\log L)_0 = L_{p,q}$ are the *Lorentz spaces* with the Lebesgue spaces $L_p = L_{p,p}$ as special case.

It is well-known that

$$K(t, f; L_\infty, L_1) = t \int_0^{1/t} f^*(s)ds = f^{**}(1/t).$$

This formula yields

$$(L_\infty, L_1)_{\theta,q} = L_{p,q} \tag{2.2}$$

provided that $0 < \theta = \frac{1}{p} < 1$ and $1 \le q \le \infty$. On the other hand, interpolating with the function parameter $\varrho_{\theta,b}(t) = t^\theta (1 + |\log t|)^{-b}, t > 0$, we get

$$(L_\infty, L_1)_{\varrho_{\theta,b},q} = L_{p,q}(\log L)_b$$

where $0 < \theta = \frac{1}{p} < 1, 1 \le q \le \infty$ and $b \in \mathbb{R}$. Using this last formula one can describe Lorentz-Zygmund spaces in terms of Lorentz spaces (see [12,30,10]).

We shall also work with weighted spaces. Given any σ-finite measure space (Ω, μ), by a weight $w(x)$ on Ω we mean any positive measurable function on Ω. The *weighted L_p-space* $L_p(w)$ consists of all (equivalence classes of) measurable functions f on Ω such that $\|f\|_{L_p(w)} = \|wf\|_{L_p} < \infty$.

For our latter considerations, we shall also need some spaces of operators. Let H be a Hilbert space and let $\mathcal{L}(H)$ be the Banach space of all bounded linear operators acting from H into H. For $T \in \mathcal{L}(H)$, the *singular numbers* of T are

$$s_n(T) = \inf\{\|T - R\|_{H,H} : R \in \mathcal{L}(H) \text{ with rank } R < n\}, \ n \in \mathbb{N}.$$

For $1 \le p \le \infty$, the *Schatten p-class* $\mathcal{L}_p(H)$ is formed by all those $T \in \mathcal{L}(H)$ having a finite norm

$$\|T\|_{\mathcal{L}_p(H)} = \Big(\sum_{n=1}^{\infty} s_n(T)^p \Big)^{1/p}.$$

See [26,31].

The so-called *Macaev ideal* $\mathcal{L}_{\mathcal{M}}(H)$ consists of all $T \in \mathcal{L}(H)$ such that

$$\|T\|_{\mathcal{L}_{\mathcal{M}}(H)} = \sup_{n\in\mathbb{N}} \left\{ (1+\log n)^{-1} \sum_{k=1}^{n} s_k(T) \right\} < \infty.$$

See [26,18].

Some remarks concerning notation. For a real number a we put $a_+ = \max\{a, 0\}$. As usual, given two quantities X, Y we write $X \lesssim Y$ whenever there is a constant $c > 0$ independent of the elements involved in X and Y, such that $X \le cY$. Notation $X \backsim Y$ means $X \lesssim Y$ and $Y \lesssim X$. Given two sequences $(b_n), (d_n)$ of non-negative real numbers, notation $b_n \lesssim d_n$ has a similar meaning: there is $c > 0$ such that $b_n \le c d_n$ for all $n \in \mathbb{N}$. If $b_n \lesssim d_n$ and $d_n \lesssim b_n$, we write $b_n \backsim d_n$.

3 A class of extreme interpolation spaces

In Section 2 we have defined spaces $(A_0, A_1)_{\theta,q}$ for $0 < \theta < 1$ and $1 \le q \le \infty$. As one can see in [6, Proposition 3.2.7], if we take $\theta = 0$ in the definition then the J-spaces are meaningful only if $q = 1$. However, as we shall show in this section, working with ordered couples, the norm given in (2.1) still makes sense for $\theta = 0$ and any $1 \le q \le \infty$, and it leads to interesting spaces.

Definition 3.1 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \le q \le \infty$. The space $(A_0, A_1)_{0,q;J}$ is formed by all those elements $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that*

$$a = \int_1^\infty u(t) \frac{dt}{t} \quad (\textit{convergence in } A_1) \tag{3.1}$$

and

$$\left(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \right)^{1/q} < \infty \tag{3.2}$$

(with the usual modification if $q = \infty$). We set

$$\|a\|_{0,q;J} = \inf \left\{ \left(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \right)^{1/q} \right\}$$

where the infimum is taken over all representations u satisfying (3.1) and (3.2).

The next proposition shows some basic properties of this construction.

Lemma 3.2 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \le q \le \infty$. We have*

$$A_0 \hookrightarrow (A_0, A_1)_{0,q;J} \hookrightarrow A_1. \tag{3.3}$$

For q=1, we get $(A_0, A_1)_{0,1;J} = A_0$.

Proof. Without loss of generality, we may assume that $\|a\|_{A_1} \le \|a\|_{A_0}$ for any $a \in A_0$. Since $a = \int_1^\infty a\chi_{(1,e)}(t)dt/t$, we have

$$\begin{aligned}\|a\|_{0,q;J} &\le \Big(\int_1^e J(t, a\chi_{(1,e)}(t))^q \frac{dt}{t}\Big)^{1/q} \\ &\le \Big(\int_1^e t^q \frac{dt}{t}\Big)^{1/q} \|a\|_{A_0} \le \Big(\frac{e^q - 1}{q}\Big)^{1/q} \|a\|_{A_0}.\end{aligned}$$

Therefore, $A_0 \hookrightarrow (A_0, A_1)_{0,q;J}$.

On the other hand, if $a \in (A_0, A_1)_{0,q;J}$ and $a = \int_1^\infty u(t)dt/t$ then we obtain by Hölder inequality

$$\begin{aligned}\|a\|_{A_1} &\le \int_1^\infty t^{-1} J(t, u(t)) \frac{dt}{t} \\ &\le \Big(\int_1^\infty t^{-q'} \frac{dt}{t}\Big)^{1/q'} \Big(\int_1^\infty J(t, u(t))^q \frac{dt}{t}\Big)^{1/q} \\ &= (\frac{1}{q'})^{1/q'} \Big(\int_1^\infty J(t, u(t))^q \frac{dt}{t}\Big)^{1/q}\end{aligned}$$

where $1/q + 1/q' = 1$. This yields that $(A_0, A_1)_{0,q;J} \hookrightarrow A_1$.

Finally, if $q = 1$, given any $a \in (A_0, A_1)_{0,1;J}$ and any representation $a = \int_1^\infty u(t)dt/t$ satisfying (3.2), it turns out that the integral is absolutely convergent in A_0 since

$$\int_1^\infty \|u(t)\|_{A_0} \frac{dt}{t} \le \int_1^\infty J(t, u(t)) \frac{dt}{t} < \infty.$$

Hence, a belongs to A_0 and $\|a\|_{A_0} \le \|a\|_{0,q;J}$. This completes the proof. □

Let B_0, B_1 be further Banach spaces with $B_0 \hookrightarrow B_1$ and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. Obviously, if $a \in A_0$ we have

$$J(t, Ta; B_0, B_1) \le \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\} J(t, a; A_0, A_1).$$

This yields that the restriction

$$T : (A_0, A_1)_{0,q;J} \to (B_0, B_1)_{0,q;J}$$

is bounded with norm less than or equal to $\max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$. We shall establish a better estimate in Theorem 4.9.

In order to show some further properties of these spaces, we remark that $(A_0, A_1)_{0,q;J}$ coincides with the collection of all those $a \in A_1$ such that $a = \sum_{n=1}^{\infty} u_n$ (convergence in A_1), with $(u_n)_n \subset A_0$ and $(\sum_{n=1}^{\infty} J(2^n, u_n)^q)^{1/q} < \infty$. Moreover, $\|\cdot\|_{0,q;J}$ is equivalent to

$$\|a\|_{0,q} = \inf\Big\{\Big(\sum_{n=1}^{\infty} J(2^n, u_n)^q\Big)^{1/q} : a = \sum_{n=1}^{\infty} u_n\Big\}$$

Of course, the number 2 does not play any special role here. It can be replaced by any $\lambda > 1$. The discretization $t = \lambda^n$ does not change the space but produces an equivalent norm. It is also worth mentioning that the condition $\sum_{n=1}^{\infty} J(2^n, u_n)^q < \infty$ implies absolute convergence of the series $\sum_{n=1}^{\infty} u_n$ in A_1. This can be checked easily by using Hölder's inequality.
Given any Banach space A and $s > 0$, we denote by sA the space A with the norm $s\|\cdot\|_A$.

Lemma 3.3 *Let A be a Banach space, let $t > 0$ and $1 \le q \le \infty$. Put*

$$\eta_q(t) = \sup\Big\{\sum_{n=1}^{\infty} \min\{1, t/2^n\}|\xi_n| : \Big(\sum_{n=1}^{\infty} |\xi_n|^q\Big)^{1/q}\Big\}.$$

Then

$$\Big(A, \frac{1}{t}A\Big)_{0,q;J} = \frac{1}{\eta_q(t)}A$$

and the norm of $\frac{1}{\eta_q(t)}A$ is equal to the discrete norm $\|\cdot\|_{0,q}$.

Proof. The argument is based on an idea used in [9] to determine the characteristic function of the abstract J-method. Take any $a \in A$. It is easy to check that $K(t, a; A, 1/tA) = \|a\|_A$. Hence, given any discrete representation $a = \sum_{n=1}^{\infty} u_n$ we have

$$\begin{aligned} \|a\|_A = K(t, a) &\le \sum_{n=1}^{\infty} K(t, u_n) \\ &\le \|(J(2^n, u_n))\|_{\ell_q} \sum_{n=1}^{\infty} \min\{1, t/2^n\} \frac{J(2^n, u_n)}{\|(J(2^n, u_n))\|_{\ell_q}} \\ &\le \|(J(2^n, u_n))\|_{\ell_q}\, \eta_q(t)\,. \end{aligned}$$

Therefore, $(1/\eta_q(t))\|a\|_A \le \|a\|_{0,q}$.

To establish the converse inequality, take any $\varepsilon > 0$ and find $(\xi_n)_n \in \ell_q$ such that $\|(\xi_n)_n\|_{\ell_q} = 1$ and $\eta_q(t) - \varepsilon \le \sum_{n=1}^{\infty} \min\{1, t/2^n\}|\xi_n|$. Given any $a \in A$, we can represent a by

$$a = \sum_{n=1}^{\infty} \min\{1, t/2^n\} \frac{|\xi_n|}{D} a \qquad \text{where} \qquad D = \sum_{n=1}^{\infty} \min\{1, t/2^n\}|\xi_n| \,.$$

We have for every $n \in \mathbb{N}$

$$\begin{aligned} J\Big(2^n, \sum_{n=1}^{\infty} \min\Big\{1, \frac{t}{2^n}\Big\} \frac{|\xi_n|}{D} a; A, \frac{1}{t} A\Big) &= \min\Big\{1, \frac{t}{2^n}\Big\} \max\Big\{1, \frac{2^n}{t}\Big\} \frac{|\xi_n|}{D} \|a\|_A \\ &= \frac{|\xi_n|}{D} \|a\|_A \,. \end{aligned}$$

Whence $\|a\|_{0,q} \le (1/D)\|a\|_A$ and therefore $(\eta_q(t) - \varepsilon)\|a\|_{0,q} \le \|a\|_A$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\|a\|_{0,q} \le (1/\eta_q(t))\|a\|_A$. $\square$

Next we compute the function η_q.

Lemma 3.4 *Let $1 \le q \le \infty$ and $1/q + 1/q' = 1$. For $t \ge 2$, we have*

$$\eta_q(t) \sim (\log t)^{1/q'} \,.$$

Proof. Let $n = [\log_2 t]$, where $[\cdot]$ is the greatest integer function. Put $\mu_m = (\log_2 t)^{-1/q}$ if $1 \le m \le n$ and $\mu_m = 0$ otherwise. Then $\|(\mu_m)\|_{\ell_q} \le 1$, so

$$\eta_q(t) \ge \sum_{m=1}^{\infty} \min\{1, t/2^m\}|\mu_m| = \sum_{m=1}^{n} (\log_2 t)^{-1/q} \gtrsim (\log_2 t)^{1/q'} \,.$$

Conversely, given any $(\xi_m) \in \ell_q$ with $\|(\xi_m)\|_{\ell_q} \le 1$ we have by Hölder's inequality

$$\begin{aligned} \sum_{m=1}^{\infty} \min\{1, t/2^m\}|\xi_m| &\le \Big(\sum_{m=1}^{\infty} \min\{1, t/2^m\}^{q'}\Big)^{1/q'} \\ &\le (\sum_{m=1}^{n} 1)^{1/q'} + t\Big(\sum_{m=n+1}^{\infty} 2^{-mq'}\Big)^{1/q'} \\ &\lesssim n^{1/q'} + t2^{-n} \\ &\lesssim (\log t)^{1/q'} \,. \qquad \square \end{aligned}$$

The following result refers to interpolation of vector-valued sequence spaces. Given any sequence (G_n) of Banach spaces, any sequence (γ_n) of positive numbers and $1 \leq q \leq \infty$, we put

$$\ell_q(\gamma_n G_n) = \Big\{ x = (x_n) : x_n \in G_n \text{ and } \|x\|_{\ell_q(\gamma_n G_n)} = \Big\| (\gamma_n \|x_n\|_{G_n}) \Big\|_{\ell_q} < \infty \Big\} .$$

When $\gamma_n = 1$ for any $n \in \mathbb{N}$, we write simply $\ell_q(G_n)$. If all G_n coincide with the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) then we get a weighted ℓ_q space that we denote by $\ell_q(\gamma_n)$

Theorem 3.5 *Let $1 \leq q \leq \infty$ and let (A_n), (B_n) be two sequences of Banach spaces with $A_n \hookrightarrow B_n$ for every $n \in \mathbb{N}$ and $\sup\{\|I_n\|_{A_n,B_n} : n \in \mathbb{N}\} < \infty$, so $\ell_q(A_n) \hookrightarrow \ell_q(B_n)$. We have with equivalent norms*

$$(\ell_q(A_n), \ell_q(B_n))_{0,q;J} = \ell_q((A_n, B_n)_{0,q;J}) .$$

Proof. Take any $x = (x_n)$ in $(\ell_q(A_n), \ell_q(B_n))_{0,q;J}$. We write x as a sum $x = \sum_{m=1}^{\infty} u^m$ with $u^m = (u_n^m) \in \ell_q(A_n)$ where the series converges in $\ell_q(B_n)$. Thus, we have $x_n = \sum_{m=1}^{\infty} u_n^m$ in B_n. This gives

$$\begin{aligned} \|x_n\|_{(A_n,B_n)_{0,q;J}} &\lesssim \Big(\sum_{m=1}^{\infty} J(2^m, u_n^m; A_n, B_n)^q \Big)^{1/q} \\ &\leq \Big(\sum_{m=1}^{\infty} (\|u_n^m\|_{A_n}^q + 2^{mq} \|u_n^m\|_{B_n}^q) \Big)^{1/q} . \end{aligned}$$

Hence,

$$\begin{aligned} \Big(\sum_{n=1}^{\infty} \|x_n\|_{(A_n,B_n)_{0,q;J}}^q \Big)^{1/q} &\lesssim \Big(\sum_{m,n=1}^{\infty} (\|u_n^m\|_{A_n}^q + 2^{mq} \|u_n^m\|_{B_n}^q) \Big)^{1/q} \\ &\lesssim \Big(\sum_{m=1}^{\infty} J(2^m, u^m; \ell_q(A_n), \ell_q(B_n))^q \Big)^{1/q} . \end{aligned}$$

This shows that

$$(\ell_q(A_n), \ell_q(B_n))_{0,q;J} \hookrightarrow \ell_q((A_n, B_n)_{0,q;J}) .$$

Conversely, take $x = (x_n) \in \ell_q((A_n, B_n)_{0,q;J})$. For any $\varepsilon > 0$ we can find a representation $x_n = \sum_{m=1}^{\infty} u_n^m$ with

$$\Big(\sum_{m=1}^{\infty} J(2^m, u_n^m; A_n, B_n)^q \Big)^{1/q} \lesssim (1 + \varepsilon) \|x_n\|_{(A_n,B_n)_{0,q;J}} .$$

Put $u^m = (u_n^m)_{n=1}^{\infty}$. We have

$$\Big(\sum_{m=1}^{\infty} J(2^m, u^m; \ell_q(A_n), \ell_q(B_n))^q\Big)^{1/q} \sim \Big(\sum_{n,m=1}^{\infty} (\|u_n^m\|_{A_n}^q + 2^{mq}\|u_n^m\|_{B_n}^q)\Big)^{1/q}$$
$$\sim \Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J(2^m, u_n^m; A_n, B_n)^q\Big)^{1/q}$$
$$\lesssim (1+\varepsilon)\Big(\sum_{n=1}^{\infty} \|x_n\|_{(A_n,B_n)_{0,q;J}}^q\Big)^{1/q}.$$

So, the series $\sum_{m=1}^{\infty} u^m$ is convergent in $\ell_q(B_n)$. Since each coordinate of x coincides with the corresponding coordinate of $\sum_{m=1}^{\infty} u^m$, we obtain that $x = \sum_{m=1}^{\infty} u^m$. Therefore, x belongs to $(\ell_q(A_n), \ell_q(B_n))_{0,q;J}$ and

$$\|x\|_{(\ell_q(A_n),\ell_q(B_n))_{0,q;J}} \lesssim \|x\|_{\ell_q((A_n,B_n)_{0,q;J})}.$$

This completes the proof. □

As a direct consequence of Theorem 3.5 and Lemmata 3.3 and 3.4 we obtain the following result.

Corollary 3.6 *Let (G_n) be a sequence of Banach spaces, let $\lambda > 1$, $1 \le q \le \infty$ and $1/q + 1/q' = 1$. Then we have with equivalent norms*

$$(\ell_q(G_n), \ell_q(\lambda^{-n} G_n))_{0,q;J} = \ell_q(n^{-1/q'} G_n).$$

Next we determine the spaces that arise applying the $(0, q; J)$-method to two spaces obtained by using the real method.

Theorem 3.7 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$, let $0 < \theta_0 < \theta_1 < 1$, $1 \le q \le \infty$ and $1/q + 1/q' = 1$. Then*

$$((A_0, A_1)_{\theta_0,q}, (A_0, A_1)_{\theta_1,q})_{0,q;J} = \Big\{a \in A_1 : \|a\| = \Big(\sum_{n=1}^{\infty} [n^{-1/q'} 2^{-\theta_0 n} K(2^n, a; A_0, A_1)]^q\Big)^{1/q} < \infty\Big\}.$$

Moreover, $\|\cdot\|$ is an equivalent norm to $\|\cdot\|_{0,q;J}$.

Proof. Write $Z_{0,q}$ for the space $((A_0, A_1)_{\theta_0,q}, (A_0, A_1)_{\theta_1,q})_{0,q;J}$. For each $n \in \mathbb{N}$, let F_n be the space A_1 endowed with the norm $2^{-\theta_0 n} K(2^n, \cdot; A_0, A_1)$ and let $j\,a = (a, a, a, \cdots)$. Realizing the space $(A_0, A_1)_{\theta_k,q}$ as a K-space in discrete norm, it is easily checked that

$$j : (A_0, A_1)_{\theta_k,q} \longrightarrow \ell_q(2^{-(\theta_k - \theta_0)n} F_n)$$

is bounded for $k = 0, 1$. Interpolating the operator j and using Corollary 3.6, we derive that

$$j : Z_{0,q} \longrightarrow (\ell_q(F_n), \ell_q(2^{-(\theta_1-\theta_0)n} F_n))_{0,q;J} = \ell_q(n^{-1/q'} F_n)$$

is bounded. Consequently, there is a constant $M > 0$ such that for any $a \in Z_{0,q}$

$$\|a\| = \|j\, a\|_{\ell_q(n^{-1/q'} F_n)} \leq M \|a\|_{0,q;J} \, .$$

Conversely, for each $n \in \mathbb{N}$, let G_n be the space A_0 normed by $2^{-\theta_0 n} J(2^n, \cdot; A_0, A_1)$ and let π be the operator that associates to each sequence (u_n) its sum $\pi(u_n) = \sum_{n=1}^{\infty} u_n$ in A_1. According to the discrete characterization of $(A_0, A_1)_{\theta_k,q}$ viewed as a J-space, we have that

$$\pi : \ell_q(2^{-(\theta_k-\theta_0)n} G_n) \longrightarrow (A_0, A_1)_{\theta_k,q}$$

is bounded for $k = 0, 1$. Interpolating and using Corollary 3.6 we get that

$$\pi : \ell_q(n^{-1/q'} G_n) \longrightarrow Z_{0,q}$$

is bounded. Therefore, there is a constant $M_1 > 0$ such that for any $(u_n) \in \ell_q(n^{-1/q'} G_n)$ we have that $a = \sum_{n=1}^{\infty} u_n$ belongs to $Z_{0,q}$ with

$$\|a\|_{0,q} \leq M_1 \|(u_n)\|_{\ell_q(n^{-1/q'} G_n)}. \tag{3.4}$$

Now take any $a \in Z_{0,q}$, then $a \in (A_0, A_1)_{\theta_1,q}$ and so $t^{-1} K(t, a; A_0, A_1) \to 0$ as $t \to \infty$. Adapting the proof of the so-called fundamental lemma (see [5, Lemma 3.3.2]), we can find a representation $a = \sum_{n=1}^{\infty} u_n$ with $(u_n) \subset A_0$ and $J(2^n, u_n; A_0, A_1) \leq M_2 K(2^n, a; A_0, A_1)$ for any $n \in \mathbb{N}$. By (3.4), we deduce that

$$\begin{aligned}
\|a\|_{0,q} &\leq M_1 \Big(\sum_{n=1}^{\infty} [n^{-1/q'} 2^{-\theta_0 n} J(2^n, u_n; A_0, A_1)]^q \Big)^{1/q} \\
&\leq M_1 M_2 \Big(\sum_{n=1}^{\infty} [n^{-1/q'} 2^{-\theta_0 n} K(2^n, a; A_0, A_1)]^q \Big)^{1/q} \\
&= M_1 M_2 \|a\|
\end{aligned}$$

and the result follows. $\Box$

Note that the space that comes out in Theorem 3.7 does not depend on θ_1. Next we write down a concrete case for function spaces.

Corollary 3.8 *Let (Ω, μ) be a finite measure space, let $1 < p_1 < p_0 < \infty$, $1 \leq q \leq \infty$ and $1/q + 1/q' = 1$. Then*

$$(L_{p_0,q}, L_{p_1,q})_{0,q;J} = L_{p_0,q}(\log L)_{-1/q'}$$

with equivalence of norms.

Proof. As we pointed out in (2.2), we have that

$$K(t, f; L_\infty, L_1) = t \int_0^{1/t} f^*(s)\, ds = f^{**}(1/t)$$

and $L_{p_j,q} = (L_\infty, L_1)_{\theta_j,q}$ for $\theta_j = 1/p_j$, $j = 0, 1$. Then, by Theorem 3.7, we obtain

$$\begin{aligned}
\|f\|_{(L_{p_0,q},L_{p_1,q})_{0,q;J}} &\sim \Big(\int_1^\infty [(1+\log t)^{-1/q'} t^{-1/p_0} f^{**}(1/t)]^q \,\frac{dt}{t}\Big)^{1/q} \\
&= \Big(\int_0^1 [(1+|\log t|)^{-1/q'} t^{1/p_0} f^{**}(t)]^q \,\frac{dt}{t}\Big)^{1/q} \\
&\sim \|f\|_{L_{p_0,q}(\log L)_{-1/q'}} . \qquad \square
\end{aligned}$$

4 Description of the extreme spaces using the K-functional

In this section we shall show that $(0, q; J)$-spaces have an equivalent description by using the K-functional. The K-representation makes more easy the characterization of some important extreme spaces and yields a good norm estimate for interpolated operators. Since $(A_0, A_1)_{0,1;J} = A_0$, we only pay attention to the case $1 < q \le \infty$.

Definition 4.1 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$. For $1 < q \le \infty$, the space $(A_0, A_1)_{\log,q;K}$ is formed by all elements $a \in A_1$ having a finite norm*

$$\|a\|_{\log,q;K} = \begin{cases} \Big(\int_1^\infty \Big[\frac{K(t,a)}{1+\log t}\Big]^q \frac{dt}{t}\Big)^{1/q} & \textit{if } 1 < q < \infty \\ \sup_{t>1} \Big\{\frac{K(t,a)}{1+\log t}\Big\} & \textit{if } q = \infty\,. \end{cases}$$

It is not hard to check that the norm $\|\cdot\|_{\log,q;K}$ is equivalent to

$$\|a\|_{\log,q} = \begin{cases} \Big(\sum_{n=1}^\infty \Big[\frac{K(2^n,a)}{n}\Big]^q\Big)^{1/q} & \text{if } 1 < q < \infty \\ \sup_{n\ge 1} \Big\{\frac{K(2^n,a)}{n}\Big\} & \text{if } q = \infty\,. \end{cases}$$

Theorem 4.2 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 < q \leq \infty$. Then*

$$(A_0, A_1)_{0,q;J} = (A_0, A_1)_{\log,q;K}$$

with equivalence of norms.

Proof. We start with the case $q = \infty$. Let $a \in (A_0, A_1)_{0,\infty;J}$ with $a = \int_1^\infty u(t)\, dt/t$. We have

$$\begin{aligned} K(s,a) &\leq \int_1^\infty K(s,u(t))\,\frac{dt}{t} \leq \int_1^\infty \min\{1, s/t\} J(t,u(t))\,\frac{dt}{t} \\ &\leq \sup_{1<t<\infty} \{J(t,u(t))\} \int_1^\infty \min\{1, s/t\}\,\frac{dt}{t} \\ &= (1+\log s) \sup_{1<t<\infty} \{J(t,u(t))\}\,. \end{aligned}$$

This implies that $a \in (A_0, A_1)_{\log,\infty;K}$ with $\|a\|_{\log,\infty;K} \leq \|a\|_{0,\infty;J}$.

In order to establish the remaining inequality, take $a \in (A_0, A_1)_{\log,\infty;K}$. We can decompose $a = a_{0,0} + a_{1,0}$ with $a_{j,0} \in A_j$ and

$$\|a_{0,0}\|_{A_0} + \|a_{1,0}\|_{A_1} \leq 2K(1,a) \tag{4.1}$$

and for $\nu = 1, 2, \ldots$ we can also find $a_{j,\nu} \in A_j$ such that $a = a_{0,\nu} + a_{1,\nu}$ and

$$\|a_{0,\nu}\|_{A_0} + \lambda_{\nu+1}\|a_{1,\nu}\|_{A_1} \leq 2K(\lambda_{\nu+1}, a)\,, \tag{4.2}$$

where $\lambda_\nu = 2^{2^\nu}$. Then

$$\begin{aligned} \|a_{1,\nu}\|_{A_1} &\leq 2\frac{K(\lambda_{\nu+1},a)}{1+\log\lambda_{\nu+1}} \cdot \frac{1+\log\lambda_{\nu+1}}{\lambda_{\nu+1}} \\ &\leq 2\|a\|_{\log,\infty;K} \cdot \frac{1+\log\lambda_{\nu+1}}{\lambda_{\nu+1}} \longrightarrow 0 \quad \text{as} \quad \nu \to \infty\,. \end{aligned}$$

Put

$$u_0 = a_{0,0}\ ,\ u_1 = a_{0,1} - a_{0,0}\ ,\ \cdots\ ,\ u_\nu = a_{0,\nu} - a_{0,\nu-1}\ , \cdots$$

Obviously, we have $(u_\nu)_\nu \subset A_0$ and moreover

$$\Big\| a - \sum_{\nu=0}^N u_\nu \Big\|_{A_1} = \|a - a_{0,N}\|_{A_1} = \|a_{1,N}\|_{A_1} \longrightarrow 0 \quad \text{as} \quad N \to \infty\,.$$

We write $I_0 = [1,2)$ and $I_\nu = [\lambda_{\nu-1}, \lambda_\nu)$ for $\nu = 1, 2, \dots$.
Furthermore we define

$$u(t) = \begin{cases} \frac{1}{\log 2} u_0 & \text{if } t \in I_0 \\ \frac{1}{2^{\nu-1}\log 2}\, u_\nu & \text{if } t \in I_\nu, \nu = 1, 2, \dots \end{cases} \tag{4.3}$$

Then, working in A_1, we have

$$\int_1^\infty u(t)\, \frac{dt}{t} = \sum_{\nu=0}^{\infty} \int_{I_\nu} u(t)\, \frac{dt}{t} = \sum_{\nu=0}^{\infty} u_\nu = a\,.$$

Furthermore, for $t \in I_0$, we get

$$J(t, u(t)) \lesssim J(2, u_0) \leq \|a_{0,0}\|_{A_0} + 2\|a - a_{1,0}\|_{A_1} \lesssim K(1, a) \lesssim \|a\|_{\log,\infty;K}$$

For $\nu = 1, 2, \dots$ and $t \in I_\nu$, we obtain

$$\begin{aligned} J(t, u(t)) &\leq \frac{1}{2^{\nu-1}\log 2} J(\lambda_\nu, u_\nu) \\ &\leq \frac{1}{2^{\nu-1}\log 2} (\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0} + \lambda_\nu \|a_{1,\nu-1}\|_{A_1} + \lambda_{\nu+1}\|a_{1,\nu}\|_{A_1}) \\ &\leq \frac{1}{2^{\nu-1}\log 2} (2K(\lambda_{\nu+1}, a) + 2K(\lambda_\nu, a)) \\ &\lesssim \frac{K(\lambda_{\nu+1}, a)}{2^{\nu+1}} \lesssim \|a\|_{\log,\infty;K}\,. \end{aligned}$$

This yields that $a \in (A_0, A_1)_{0,\infty;J}$ with $\|a\|_{0,\infty;J} \lesssim \|a\|_{\log,\infty;K}$, which completes the proof in case $q = \infty$.

Next we consider the remaining case $1 < q < \infty$. Assume that $a \in (A_0, A_1)_{0,q;J}$ and let $a = \sum_{n=1}^\infty u_n$ be any discrete J-representation of a. We have

$$\begin{aligned} K(2^n, a) &\leq \Big\| \sum_{k=1}^{n} u_k \Big\|_{A_0} + 2^n \Big\| \sum_{k=n+1}^{\infty} u_k \Big\|_{A_1} \\ &\leq \sum_{k=1}^{n} J(2^k, u_k) + 2^n \sum_{k=n+1}^{\infty} 2^{-k} J(2^k, u_k)\,. \end{aligned}$$

Using Hölder's inequality, we get for the second sum

$$2^n \sum_{k=n+1}^{\infty} 2^{-k} J(2^k, u_k) \leq 2^n \Big(\sum_{k=n+1}^{\infty} 2^{-kq'} \Big)^{1/q'} \Big(\sum_{k=n+1}^{\infty} J(2^k, u_k)^q \Big)^{1/q}$$
$$\lesssim \Big(\sum_{k=1}^{\infty} J(2^k, u_k)^q \Big)^{1/q} .$$

This gives

$$\|a\|_{\log,q} = \Big(\sum_{n=1}^{\infty} \Big[\frac{K(2^n, a)}{n} \Big]^q \Big)^{1/q}$$
$$\lesssim \Big(\sum_{n=1}^{\infty} \Big[\frac{1}{n} \sum_{k=1}^{n} J(2^k, u_k) \Big]^q \Big)^{1/q} + \Big(\sum_{n=1}^{\infty} \frac{1}{n^q} \Big)^{1/q} \Big(\sum_{k=1}^{\infty} J(2^k, u_k)^q \Big)^{1/q} .$$

Here $\sum_{n=1}^{\infty} 1/n^q < \infty$ because of $1 < q < \infty$. Let us estimate the first sum. We put $\alpha_k = J(2^k, u_k)$ and denote the non-increasing rearrangement of (α_k) by (α_k^*) . Using Hardy's inequality (see, for instance, [31, page 52]) we have

$$\Big(\sum_{n=1}^{\infty} \Big[\frac{1}{n} \sum_{k=1}^{n} \alpha_k \Big]^q \Big)^{1/q} \leq \Big(\sum_{n=1}^{\infty} \Big[\frac{1}{n} \sum_{k=1}^{n} \alpha_k^* \Big]^q \Big)^{1/q} \lesssim \Big(\sum_{n=1}^{\infty} (\alpha_n^*)^q \Big)^{1/q} .$$

Consequently, we get

$$\|a\|_{\log,q} \lesssim \Big(\sum_{n=1}^{\infty} J(2^n, u_n)^q \Big)^{1/q}$$

This implies that $(A_0, A_1)_{0,q;J} \hookrightarrow (A_0, A_1)_{\log,q;K}$.

To prove the converse embedding, let $a \in (A_0, A_1)_{\log,q;K}$. Choose decompositions $a = a_{0,\nu} + a_{1,\nu}$ as given in (4.1) and (4.2) and define the function $u(t)$ by (4.3). As we have seen before this yields $a = \int_1^{\infty} u(t)\, dt/t$ with

$$J(t, u(t)) \lesssim K(1, a) \qquad \text{if} \qquad t \in J_0$$

and

$$J(t, u(t)) \lesssim \frac{K(\lambda_{\nu+1}, a)}{2^{\nu+1}} \qquad \text{if} \qquad t \in I_\nu \ , \nu = 1, 2, ...$$

We have

$$\int_{I_0} J(t, u(t))^q \frac{dt}{t} \lesssim K(1, a)^q \lesssim \int_{I_0} \Big(\frac{K(t, a)}{1 + \log t} \Big)^q \frac{dt}{t}$$

and for $\nu = 1, 2, ...$

$$\int_{I_\nu} J(t,u(t))^q \frac{dt}{t} \lesssim \Big(\frac{K(\lambda_{\nu+1},a)}{2^{\nu+1}}\Big)^q \int_{I_\nu} \frac{dt}{t}$$
$$= \Big(\frac{K(\lambda_{\nu+1},a)}{2^{\nu+1}}\Big)^q 2^{\nu-1}\log 2$$
$$\lesssim \int_{I_{\nu+2}} \Big(\frac{K(t,a)}{1+\log t}\Big)^q \frac{dt}{t}\,.$$

Consequently,

$$\|a\|_{0,q;J} \leq \Big(\int_1^\infty J(t,u(t))^q \,\frac{dt}{t}\Big)^{1/q} = \Big(\sum_{\nu=0}^{\infty}\int_{I_\nu} J(t,u(t))^q \,\frac{dt}{t}\Big)^{1/q}$$
$$\lesssim \Big(\sum_{\nu=0}^{\infty}\int_{I_\nu} \Big(\frac{K(t,a)}{1+\log t}\Big)^q \,\frac{dt}{t}\Big)^{1/q} = \|a\|_{\log,q;K}\,.$$

This completes the proof. $\square$

Next we identify some other interesting $(0,q;J)$-spaces.

Corollary 4.3 *Let (Ω,μ) be a finite measure space and let $1 < q \leq \infty$. Then*

$$(L_\infty, L_1)_{0,q;J} = L_{\infty,q}(\log L)_{-1}$$

with equivalence of norms.

Proof. Using Theorem 4.2 we have

$$\|f\|_{0,q;J} \sim \|f\|_{\log,q;K} = \Big(\int_1^\infty \Big(\frac{K(t,f)}{1+\log t}\Big)^q \,\frac{dt}{t}\Big)^{1/q} = \Big(\int_1^\infty \Big(\frac{f^{**}(1/t)}{1+\log t}\Big)^q \frac{dt}{t}\Big)^{1/q}.$$

A change of variable shows that the last expression is equivalent to

$$\|f\|_{L_{\infty,q}(\log L)_{-1}} = \Big(\int_0^{\mu(\Omega)} [(1+|\log t|)^{-1} f^{**}(t)]^q \,\frac{dt}{t}\Big)^{1/q}. \qquad \square$$

Note that if $q=\infty$ we obtain the space $L_{\infty,\infty}(\log L)_{-1}$ which coincides with the Zygmund space L_{exp}.

The corresponding result to Corollary 4.3 for spaces of operators in a Hilbert space H reads as follows.

Corollary 4.4 *We have*

$$(\mathcal{L}_1(H), \mathcal{L}(H))_{0,\infty;J} = \mathcal{L}_{\mathcal{M}}(H)$$

and for $1 < q < \infty$

$$(\mathcal{L}_1(H), \mathcal{L}(H))_{0,q;J} = \left\{T \in \mathcal{L}(H) : \|T\| = \left(\sum_{n=1}^{\infty} \left[\frac{1}{n}\sum_{j=1}^{2^n} s_j(T)\right]^q\right)^{1/q} < \infty\right\}.$$

Proof. The result is a direct consequence of the K-description of the $(0, q; J)$-space and the fact that

$$K(n, T; \mathcal{L}_1(H), \mathcal{L}(H)) = \sum_{j=1}^{n} s_j(T) \quad , \quad n \in \mathbb{N}$$

(see [36] and the references given there). □

In the literature one can find some results on $(\log, q; K)$-spaces. We refer, for example, to the papers by Gogatishvili, Opic and Trebels [25] and by Karadzhov and Milman [30]. In particular, they can be realized as $\delta^{(p)-}$ extrapolation spaces in the sense of [30]. They are also limit cases of logarithmic interpolation spaces in the terminology of [12]. The following characterization can be established by using the same arguments as in [12, Theorem 1]. We write $A_{\theta,q}$ for the real interpolation space $(A_0, A_1)_{\theta,q}$ normed by the discrete K-norm $\|a\|_{\theta,q} = \left(\sum_{n=1}^{\infty}[2^{-\theta n}K(2^n, a)]^q\right)^{1/q}$.

Theorem 4.5 *Let* $A_0 \hookrightarrow A_1$ *be Banach spaces and let* $1 < q < \infty$*. Then*

$$(A_0, A_1)_{0,q;J} = \left\{a \in \bigcap_{n=1}^{\infty} A_{2^{-n},q} : \|a\| = \left(\sum_{n=1}^{\infty}[2^{-n}\|a\|_{A_{2^{-n},q}}]^q\right)^{1/q} < \infty\right\}$$

and $\|\cdot\|$ *is an equivalent norm to* $\|\cdot\|_{0,q;J}$.

Theorem 4.5 also holds true when $q = \infty$. In fact, that special case is contained in [10, Theorem 2.6].

The following reiteration result covers limit cases which are not included in Theorem 3.7 .

Theorem 4.6 *Let* A_0, A_1 *be Banach spaces with* $A_0 \hookrightarrow A_1$*, let* $0 < \theta < 1$*,* $1 < q \leq \infty$ *and* $1/q + 1/q' = 1$*. Then we have with equivalent norms*

(a)

$$((A_0,A_1)_{\theta,q},A_1)_{0,q;J} = \Big\{a\in A_1 : \Big(\int_1^\infty [t^{-\theta}(1+\log t)^{-1/q'}K(t,a;A_0,A_1)]^q\,\frac{dt}{t}\Big)^{1/q} < \infty\Big\}.$$

(b)

$$(A_0,(A_0,A_1)_{\theta,q})_{0,q;J} = (A_0,A_1)_{0,q;J}\,.$$

(c)

$$(A_0,(A_0,A_1)_{0,q;J})_{0,q;J} = \Big\{a\in A_1 : \Big(\int_1^\infty \Big[\frac{K(t,a;A_0,A_1)}{(1+\log t)^{1/q}(1+|\log\log t|)}\Big]^q\,\frac{dt}{t}\Big)^{1/q} < \infty\Big\}.$$

Proof. According to [5, Corollary 3.6.2/b], we have

$$K(t,a;(A_0,A_1)_{\theta,q},A_1) \sim \Big(\int_0^{t^{1/(1-\theta)}} [s^{-\theta}K(s,a;A_0,A_1)]^q\,\frac{ds}{s}\Big)^{1/q}.$$

Combining this fact with Theorem 4.2, we obtain

$$\begin{aligned}
\|a\|_{((A_0,A_1)_{\theta,q},A_1)_{0,q;J}} &\sim \Big(\int_1^\infty \int_0^{t^{1/(1-\theta)}} \Big[\frac{s^{-\theta}K(s,a)}{1+\log t}\Big]^q\,\frac{ds}{s}\,\frac{dt}{t}\Big)^{1/q}\\
&= \Big(\int_0^\infty s^{-\theta q}\Big(\int_{\max\{1,s^{1-\theta}\}}^\infty \frac{1}{(1+\log t)^q}\,\frac{dt}{t}\Big)K(s,a)^q\,\frac{ds}{s}\Big)^{1/q}\\
&\sim \Big(\int_0^\infty s^{-\theta q}\Big(\frac{1}{1+\log\,\max\{1,s^{1-\theta}\}}\Big)^{q-1}K(s,a)^q\,\frac{ds}{s}\Big)^{1/q}\\
&\sim \Big(\int_1^\infty s^{-\theta q}(1+\log s)^{1-q}K(s,a)^q\,\frac{ds}{s}\Big)^{1/q}
\end{aligned}$$

which gives (a).

The proof of (b) is similar but we use now the relation

$$K(t,a;A_0,(A_0,A_1)_{\theta,q}) \sim t\Big(\int_{t^{1/\theta}}^\infty [s^{-\theta}K(s,a)]^q\,\frac{ds}{s}\Big)^{1/q}$$

(see [5, Corollary 3.6.2/(a)]).

Let us establish (c). Using similar ideas as those in [27, Theorem 3.6], one can show that

$$K(t,a;A_0,(A_0,A_1)_{0,q;J}) \sim t\Big(\int_{e^{t^{q'}}}^{\infty} \Big[\frac{K(s,a;A_0,A_1)}{1+\log s}\Big]^q \frac{ds}{s}\Big)^{1/q}.$$

Consequently,

$$\begin{aligned}
\|a\|_{(A_0,(A_0,A_1)_{0,q;J})_{0,q;J}} &\sim \Big(\int_1^{\infty}\Big[\frac{t}{1+\log t}\Big]^q \int_{e^{t^{q'}}}^{\infty}\Big[\frac{K(s,a;A_0,A_1)}{1+\log s}\Big]^q \frac{ds}{s}\frac{dt}{t}\Big)^{1/q}\\
&= \Big(\int_e^{\infty}\Big[\frac{K(s,a;A_0,A_1)}{1+\log s}\Big]^q \int_1^{(\log s)^{1/q'}} \Big[\frac{t}{1+\log t}\Big]^q \frac{dt}{t}\frac{ds}{s}\Big)^{1/q}\\
&\sim \Big(\int_e^{\infty}\frac{K(s,a;A_0,A_1)^q}{(1+\log s)(1+\log\log s)^q}\frac{ds}{s}\Big)^{1/q}\\
&\sim \Big(\int_1^{\infty}\Big[\frac{K(s,a;A_0,A_1)}{(1+\log s)^{1/q}(1+|\log\log s|}\Big]^q\frac{ds}{s}\Big)^{1/q}. \qquad \square
\end{aligned}$$

As a consequence of Theorem 4.6 and equality (2.2), we derive the following.

Corollary 4.7 *Let (Ω,μ) be a finite measure space, let $1<p<\infty$, $1<q\le\infty$ and $1/q+1/q'=1$. Then we have with equivalent norms*

(a)

$$(L_{p,q},L_1)_{0,q;J} = L_{p,q}(\log L)_{-1/q'}.$$

(b)

$$(L_\infty,L_{p,q})_{0,q;J} = L_{\infty,q}(\log L)_{-1}.$$

The following formula refers to a general couple of weighted L_q-spaces.

Theorem 4.8 *Let (Ω,μ) be any σ-finite measure space, let $1<q\le\infty$ and $1/q+1/q'=1$. Assume that w_0 and w_1 are weights on Ω such that $w_0(x)\ge w_1(x)$ μ-a.e. Put*

$$w(x) = w_0(x)\Big(1+\log\frac{w_0(x)}{w_1(x)}\Big)^{-1/q'}.$$

Then we have with equivalent norms

$$(L_q(w_0),L_q(w_1))_{0,q;J} = L_q(w).$$

Proof. It is not hard to check that

$$K(t,f;L_q(w_0),L_q(w_1)) \sim \Big(\int\limits_\Omega [\min\{w_0(x),tw_1(x)\}|f(x)|\,]^q\,d\mu(x)\Big)^{1/q}. \quad (4.4)$$

Therefore,

$$\begin{aligned}
\|f\|_{0,q;J} &\sim \Big(\int\limits_1^\infty \Big[\frac{K(t,f)}{1+\log t}\Big]^q \frac{dt}{t}\Big)^{1/q} \\
&\sim \Big(\int\limits_\Omega |f(x)|^q \int\limits_1^\infty \Big[\frac{\min\{w_0(x),tw_1(x)\}}{1+\log t}\Big]^q \frac{dt}{t}\,d\mu(x)\Big)^{1/q} \\
&= \Big(\int\limits_\Omega |f(x)|^q\Big(w_1(x)^q \int\limits_1^{\frac{w_0(x)}{w_1(x)}} \Big[\frac{t}{1+\log t}\Big]^q \frac{dt}{t} + w_0(x)^q \int\limits_{\frac{w_0(x)}{w_1(x)}}^\infty \Big[\frac{1}{1+\log t}\Big]^q \frac{dt}{t}\Big)\,d\mu(x)\Big)^{1/q} \\
&\sim \Big(\int\limits_\Omega (|f(x)|w(x))^q\,d\mu(x)\Big)^{1/q}. \qquad \square
\end{aligned}$$

We close this section by improving the norm estimate for interpolated operators. Put

$$\varphi(t,s) = t(1+(\log s/t)_+)\,. \quad (4.5)$$

Theorem 4.9 *Let $\bar{A}=(A_0,A_1)$, $\bar{B}=(B_0,B_1)$ be couples of Banach spaces with $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$. Suppose that $T \in \mathcal{L}(\bar{A},\bar{B})$ and let $M_j = \|T\|_{A_j,B_j}$ for $j=0,1$. Then for $1<q\le\infty$ we have*

$$\|T\|_{(A_0,A_1)_{\log,q;K},(B_0,B_1)_{\log,q;K}} \le \varphi(M_0,M_1)\,.$$

Proof. It is clear that

$$K(t,Ta;B_0,B_1) \le M_0 K(M_1 t/M_0,a;A_0,A_1)\,, \quad t>0\,, \quad a\in A_1\,.$$

Hence, if $M_1 \le M_0$, we get

$$\|Ta\|_{\log,q;K} \le M_0\|a\|_{\log,q;K} = \varphi(M_0,M_1)\|a\|_{\log,q;K}\,.$$

If $M_1 \ge M_0$, making a change of variable, we obtain

$$
\begin{aligned}
\|Ta\|_{\log,q;K} &\le M_0\Big(\int\limits_{M_1/M_0}^{\infty}\Big[\frac{K(s,a)}{1+\log(sM_0/M_1)}\Big]^q\frac{ds}{s}\Big)^{1/q}\\
&\le M_0\sup_{M_1/M_0<s<\infty}\Big\{\frac{1+\log s}{1+\log(sM_0/M_1)}\Big\}\|a\|_{\log,q;K}\\
&= M_0\sup_{M_1/M_0<s<\infty}\Big\{1+\frac{\log(M_1/M_0)}{1+\log(sM_0/M_1)}\Big\}\|a\|_{\log,q;K}\\
&= M_0(1+\log(M_1/M_0))\|a\|_{\log,q;K}\,. \qquad \square
\end{aligned}
$$

5 Relationship with interpolation over the unit square

In this section we shall show that $(0,q;J)$-spaces arise naturally interpolating over the unit square. We start by recalling the definition of the J-method defined by means of a polygon (see [17]).

Let $\Pi = \overline{P_1\cdots P_N}$ be a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $P_j=(x_j,y_j)$ and let $\overline{A}=\{A_1,\dots,A_N\}$ be a Banach N-tuple, that is to say, N-Banach spaces A_j which are continuously embedded in a common Hausdorff topological vector space. We imagine each space A_j as sitting on the vertex P_j. For $t,s>0$ and $a\in\Delta(\overline{A})=A_1\cap\cdots\cap A_N$, we put

$$J(t,s;a)=J(t,s;a;\overline{A})=\max\Big\{t^{x_j}s^{y_j}\|a\|_{A_j}:\ 1\le j\le N\Big\}.$$

Given any interior point (α,β) of Π, $(\alpha,\beta)\in\operatorname{Int}\Pi$, and any $1\le q\le\infty$, the J-space $\overline{A}_{(\alpha,\beta),q;J}$ consists of all those $a\in\Sigma(\overline{A})=A_1+\cdots+A_N$ which can be represented by

$$a=\int\limits_0^\infty\int\limits_0^\infty v(t,s)\,\frac{dt}{t}\,\frac{ds}{s}\quad(\text{convergence in }\ \Sigma(\overline{A}))\tag{5.1}$$

where $v(t,s)$ is a strongly measurable function with values in $\Delta(\overline{A})$ and

$$\Bigg(\int\limits_0^\infty\int\limits_0^\infty\Big(t^{-\alpha}s^{-\beta}J(t,s;v(t,s))\Big)^q\,\frac{dt}{t}\,\frac{ds}{s}\Bigg)^{\frac{1}{q}}<\infty.\tag{5.2}$$

We put

$$\|a\|_{\overline{A}_{(\alpha,\beta),q;J}}=\inf_v\left\{\Bigg(\int\limits_0^\infty\int\limits_0^\infty\Big(t^{-\alpha}s^{-\beta}J(t,s;v(t,s))\Big)^q\,\frac{dt}{t}\,\frac{ds}{s}\Bigg)^{\frac{1}{q}}\right\}$$

where the infimum is taken over all v satisfying (5.1) and (5.2).

Note that if $\tilde{\Pi} = \overline{P_{j_1} \cdots P_{j_M}}$ is another convex polygon whose M vertices all belong to Π and we denote by $\tilde{A}$ the subtuple of $\overline{A}$ given by $\tilde{A} = \{A_{j_1}, \cdots, A_{j_M}\}$ then for any $a \in \Delta(\overline{A})$ we have $J(t, s; a; \tilde{A}) \leq J(t, s; a; \overline{A})$. Hence, if $(\alpha, \beta) \in$ Int $\tilde{\Pi}$, we have

$$\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \tilde{A}_{(\alpha,\beta),q;J}. \tag{5.3}$$

We refer to the papers by Cobos and Peetre [17], Cobos, Kühn and Schonbek [15] or Ericsson [22] for details on interpolation spaces associated to polygons. Sometimes when (α, β) lies on any diagonal of Π, results are more difficult and the outcome may be unexpected. Next we show that extreme spaces $(A_0, A_1)_{0,q;J}$ arise by interpolation using the unit square and (α, β) in the diagonal $\beta = \alpha$ of the 4-tuple obtained by sitting A_0 on $(0, 0)$ and $(1, 1)$ and A_1 on $(1, 0)$ and $(0, 1)$ (see Fig.5.1).

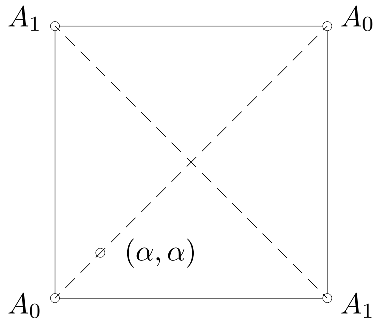

Figure 5.1

Subsequently, $\Pi = \overline{P_1 P_2 P_3 P_4}$ stand for the unit square with $P_1 = (0, 0), P_2 = (1, 0), P_3 = (0, 1)$ and $P_4 = (1, 1)$. We shall use in the arguments a number of properties of the Bochner integral which can be seen in [37] for example.

Theorem 5.1 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$, let $0 < \alpha < 1$ and $1 \leq q \leq \infty$. Interpolating the 4-tuple $\{A_0, A_1, A_1, A_0\}$ by using the unit square Π, we have with equivalent norms*

$$(A_0, A_1, A_1, A_0)_{(\alpha,1-\alpha),q;J} = \begin{cases} (A_0, A_1)_{1-2\alpha,q} & \text{if} \quad 0 < \alpha < 1/2, \\ (A_0, A_1)_{2\alpha-1,q} & \text{if} \quad 1/2 < \alpha < 1, \\ (A_0, A_1)_{0,q;J} & \text{if} \quad \alpha = 1/2, \end{cases}$$

and

$$(A_0, A_1, A_1, A_0)_{(\alpha,\alpha),q;J} = (A_0, A_1)_{0,q;J} \quad \text{for any} \quad 0 < \alpha < 1.$$

Proof. Symmetry illustrated in Fig. 5.1 shows that if we exchange spaces sitting in the edges of the diagonal $\beta = 1 - \alpha$, the picture remains the same.

This yields that for any $(\alpha, \beta) \in \operatorname{Int} \Pi$ we have

$$(A_0, A_1, A_1, A_0)_{(\alpha,\beta),q;J} = (A_0, A_1, A_1, A_0)_{(1-\alpha,1-\beta),q;J}.$$

Hence, it suffices to establish the result for $0 < \alpha \leq 1/2$. In order to make a difference between J-functionals for the 4-tuple and for the couple, we put $\bar{J}(t, a) = J(t, a; A_0, A_1)$.

Consider first the point $(\alpha, 1-\alpha)$ with $0 < \alpha < 1/2$. Let $\tilde{\Pi}$ be the triangle $\overline{P_1 P_3 P_4}$. Then $(\alpha, 1-\alpha) \in \operatorname{Int} \tilde{\Pi}$. If follows from (5.3) that

$$(A_0, A_1, A_1, A_0)_{(\alpha,1-\alpha),q;J} \hookrightarrow (A_0, A_1, A_0)_{(\alpha,1-\alpha),q;J}.$$

On the other hand, see [22, Corollary 4] or [11, (2.9)], we have

$$(A_0, A_1, A_0)_{(\alpha,1-\alpha),q;J} = (A_0, A_1)_{1-2\alpha,q}.$$

Therefore,

$$(A_0, A_1, A_1, A_0)_{(\alpha,1-\alpha),q;J} \hookrightarrow (A_0, A_1)_{1-2\alpha,q}.$$

Conversely, take $a \in (A_0, A_1)_{1-2\alpha,q}$ and let

$$a = \int_1^\infty u(t) \frac{dt}{t} \quad \text{with} \quad \Big(\int_1^\infty (t^{-1+2\alpha} \bar{J}(t, u(t)))^q \frac{dt}{t} \Big)^{1/q} < \infty.$$

Put

$$v(t, s) = \begin{cases} u(1/t) & \text{if} \quad 1/t \leq s \leq e/t, \\ 0 & \text{in} \quad \text{any other case.} \end{cases} \tag{5.4}$$

Then

$$\int_0^\infty \int_0^\infty v(t, s) \frac{dt}{t} \frac{ds}{s} = \int_0^1 \Big(\int_{1/t}^{e/t} \frac{ds}{s} \Big) u(\frac{1}{t}) \frac{dt}{t} = a. \tag{5.5}$$

Moreover, for $0 < t < 1$ and $1/t < s < e/t$, we have

$$\begin{aligned} J(t, s; v(t, s)) &= \max\{\|u(1/t)\|_{A_0}, t\|u(1/t)\|_{A_1}, s\|u(1/t)\|_{A_1}, ts\|u(1/t)\|_{A_0}\} \\ &\leq \max\{e\|u(1/t)\|_{A_0}, \frac{e}{t}\|u(1/t)\|_{A_1}\} \\ &= e\bar{J}(1/t, u(1/t)). \end{aligned}$$

Consequently,

$$\Big(\int_0^\infty\int_0^\infty \Big(t^{-\alpha}s^{-1+\alpha}J(t\ ,\ s;v(t,s))\Big)^q\frac{dt}{t}\frac{ds}{s}\Big)^{1/q}$$

$$\lesssim\Big(\int_0^1\int_{1/t}^{e/t}\Big(t^{-\alpha}s^{-1+\alpha}\bar J(\frac{1}{t},u(\frac{1}{t}))\Big)^q\frac{ds}{s}\frac{dt}{t}\Big)^{1/q}$$

$$\lesssim\Big(\int_0^1\Big(t^{1-2\alpha}\bar J(\frac{1}{t},u(\frac{1}{t}))\Big)^q\frac{dt}{t}\Big)^{1/q}$$

$$=\Big(\int_1^\infty\Big(t^{-1+2\alpha}\bar J(t,u(t))\Big)^q\frac{dt}{t}\Big)^{1/q}.$$

This yields the required embedding.

Next consider the point (α,α). Let $a\in(A_0,A_1)_{0,q;J}$ with $a=\int_1^\infty u(t)dt/t$, then we can define $v(t,s)$ by (5.4) and we still have $a=\int_0^\infty\int_0^\infty v(t,s)dt/t\,ds/s$ (see (5.5)). Furthermore,

$$\Big(\int_0^\infty\int_0^\infty \Big(t^{-\alpha}s^{-\alpha}J(t\ ,\ s;v(t,s))\Big)^q\frac{dt}{t}\frac{ds}{s}\Big)^{1/q}$$

$$\lesssim\Big(\int_0^1\int_{1/t}^{e/t}\Big(t^{-\alpha}s^{-\alpha}\bar J(\frac{1}{t},u(\frac{1}{t}))\Big)^q\frac{ds}{s}\frac{dt}{t}\Big)^{1/q}$$

$$\lesssim\Big(\int_0^1\bar J\Big(\frac{1}{t},u(\frac{1}{t})\Big)^q\frac{dt}{t}\Big)^{1/q}.$$

This implies that

$$(A_0,A_1)_{0,q;J}\hookrightarrow(A_0,A_1,A_1,A_0)_{(\alpha,\alpha),q;J}.$$

To prove the converse embedding, take $a\in(A_0,A_1,A_1,A_0)_{(\alpha,\alpha),q;J}$ and let $a=\int_0^\infty\int_0^\infty v(t,s)dt/t\,ds/s$ such that

$$\Big(\int_0^\infty\int_0^\infty\Big(t^{-\alpha}s^{-\alpha}J(t,s;v(t,s))\Big)^q\frac{dt}{t}\frac{ds}{s}\Big)^{1/q}\le 2\|a\|_{(\alpha,\alpha),q;J}.$$

Put

$$a_1=\int_1^\infty\int_1^\infty v(t,s)\frac{ds}{s}\frac{dt}{t}\quad,\quad a_2=\int_0^1\int_0^1 v(t,s)\frac{ds}{s}\frac{dt}{t}\ ,$$

$$a_3=\int_0^1\int_{1/t}^\infty v(t,s)\frac{ds}{s}\frac{dt}{t}\quad,\quad a_4=\int_0^1\int_{1/s}^\infty v(t,s)\frac{dt}{t}\frac{ds}{s}\ ,$$

$$a_5=\int_1^\infty\int_0^{1/s} v(t,s)\frac{dt}{t}\frac{ds}{s}\quad,\quad a_6=\int_1^\infty\int_0^{1/t} v(t,s)\frac{ds}{s}\frac{dt}{t}\ .$$

Clearly, $a=\sum_{j=1}^6 a_j$ (see Fig. 5.2). We are going to show that $a_j\in(A_0,A_1)_{0,q;J}$ for each $j=1,\dots,6$ with $\|a_j\|_{0,q;J}\lesssim\|a\|_{(\alpha,\alpha),q;J}$.

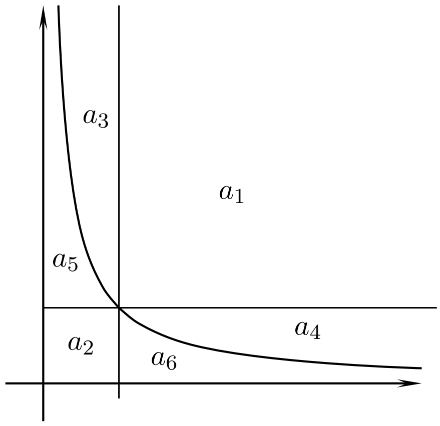

Figure 5.2

The double integral which defines a_1 is absolutely convergent in A_0 because using Hölder's inequality we get with $1/q + 1/q' = 1$

$$\int_1^\infty \int_1^\infty \|v(t,s)\|_{A_0} \frac{ds}{s}\frac{dt}{t} \le \int_1^\infty \int_1^\infty t^{-1}s^{-1} J(t,s;v(t,s)) \frac{ds}{s}\frac{dt}{t}$$
$$\le \Big(\int_1^\infty \int_1^\infty (t^{\alpha-1}s^{\alpha-1})^{q'} \frac{ds}{s}\frac{dt}{t}\Big)^{1/q'} \Big(\int_1^\infty \int_1^\infty (t^{-\alpha}s^{-\alpha} J(t,s;v(t,s)))^q \frac{ds}{s}\frac{dt}{t}\Big)^{1/q}$$
$$\lesssim \|a\|_{(\alpha,\alpha),q;J}.$$

Hence, $a_1 \in A_0$ and so $a_1 \in (A_0, A_1)_{0,q;J}$ with $\|a_1\|_{0,q;J} \lesssim \|a_1\|_{A_0} \lesssim \|a\|_{(\alpha,\alpha),q;J}$.

The case of a_2 follows with a similar argument but using now that $\|v(t,s)\|_{A_0} \le J(t,s;v(t,s))$.

In the rest of the proof we shall use freely that

$$J(t,s;w) = \max\{1,ts\} \bar{J}\Big(\frac{\max\{t,s\}}{\max\{1,ts\}}, w\Big) \quad , \quad w \in A_0.$$

In order to show that $a_3 \in (A_0, A_1)_{0,q;J}$, write $u(t) = \int_t^\infty v(1/t,s)ds/s$. This integral is absolutely convergent in A_0 because for the norm $\bar{J}(t,\cdot) \ge \|\cdot\|_{A_0}$ we have

$$\int_t^\infty \bar{J}(t\ ,\ v(\frac{1}{t},s)) \frac{ds}{s} = \int_t^\infty ts^{-1} J(\frac{1}{t},s;v(\frac{1}{t},s)) \frac{ds}{s}$$
$$\le \Big(\int_t^\infty s^{(\alpha-1)q'} \frac{ds}{s}\Big)^{1/q'} \Big(\int_t^\infty (ts^{-\alpha} J(\frac{1}{t},s;v(\frac{1}{t},s))^q \frac{ds}{s}\Big)^{1/q}$$
$$\lesssim \Big(\int_t^\infty (t^\alpha s^{-\alpha} J(\frac{1}{t},s;v(\frac{1}{t},s)))^q \frac{ds}{s}\Big)^{1/q}.$$

Therefore,

$$\begin{aligned}
\|a_3\|_{0,q;J} &\leq \Big(\int_1^\infty \bar{J}(t,u(t))^q \frac{dt}{t}\Big)^{1/q} \\
&\lesssim \Big(\int_1^\infty \int_t^\infty (t^\alpha s^{-\alpha} J(\frac{1}{t}, s; v(\frac{1}{t}, s)))^q \frac{ds}{s}\frac{dt}{t}\Big)^{1/q} \\
&\lesssim \|a\|_{(\alpha,\alpha),q;J}.
\end{aligned}$$

The case of a_4 can be checked with the same arguments but changing the role of t and s.

To deal with a_5 we put $u(s) = \int_0^{1/s} v(t,s)dt/t$. Then

$$\begin{aligned}
\bar{J}(s,u(s)) &\leq \int_0^{1/s} \bar{J}(s, v(t,s))\frac{dt}{t} = \int_0^{1/s} J(t,s;v(t,s))\frac{dt}{t} \\
&\leq \Big(\int_0^{1/s} t^{\alpha q'}\frac{dt}{t}\Big)^{1/q'} \Big(\int_0^{1/s} (t^{-\alpha} J(t,s;v(t,s)))^q \frac{dt}{t}\Big)^{1/q} \\
&\lesssim \Big(\int_0^{1/s} (t^{-\alpha} s^{-\alpha} J(t,s;v(t,s)))^q \frac{dt}{t}\Big)^{1/q}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|a_5\|_{0,q;J} &\leq \Big(\int_1^\infty \bar{J}(s,u(s))^q \frac{ds}{s}\Big)^{1/q} \\
&\lesssim \Big(\int_1^\infty \int_0^{1/s} (t^{-\alpha} s^{-\alpha} J(t,s;v(t,s)))^q \frac{dt}{t}\frac{ds}{s}\Big)^{1/q} \\
&\lesssim \|a\|_{(\alpha,\alpha),q;J}.
\end{aligned}$$

The remaining a_6 can be treated with similar arguments but changing the role of t and s.

The proof is completed. □

Note the "continuity" that Theorem 5.1 shows. Along the diagonal from $(0,1)$ to $(1,0)$, the interpolation spaces decrease when α increases for $0 < \alpha < 1/2$. The smaller possible space is attached at $\alpha = 1/2$ and it is $(A_0, A_1)_{0,q;J}$. Then, for $1/2 < \alpha < 1$, the interpolation spaces increase with α. On the other hand, along the other diagonal they are constantly equal to $(A_0, A_1)_{0,q;J}$ (see Fig. 5.3).

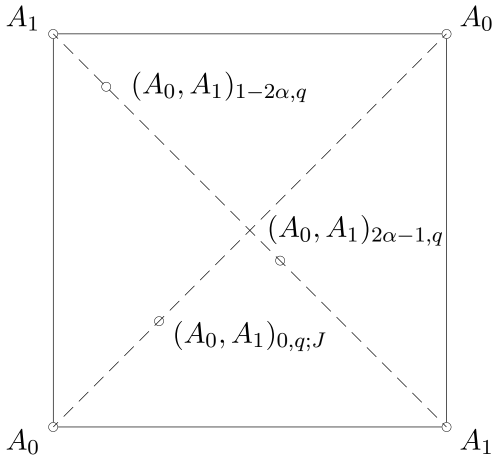

Figure 5.3

Theorem 5.1 together with [11, Theorems 3.5 and 3.1] complete the description of all interpolation spaces generated by $\{A_0, A_1, A_1, A_0\}$ using the unit square.

6 Compactness

Our aim in this section is to investigate the behavior of compact linear operators under the $(0, q; J)$-method. As we have seen in Theorem 5.1, we have

$$(A_0, A_1)_{0,q;J} = (A_0, A_1, A_1, A_0)_{(\alpha,\alpha),q;J} .$$

This formula allows to apply to the $(0, q; J)$-method a result of Cobos and Peetre on interpolation of compact operators by the methods associated to polygons (see [17, Theorem 6.1] or [24, Corollary 3.4]) and the following holds.

Corollary 6.1 *Let $\bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1)$ be couples of Banach spaces with $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, let further $T \in \mathcal{L}(\bar{A}, \bar{B})$ such that $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are compact and suppose that $1 \le q \le \infty$. Then*

$$T : (A_0, A_1)_{0,q;J} \to (B_0, B_1)_{0,q;J}$$

is compact.

For the real method, a result of Cwikel [19] and Cobos, Kühn and Schonbek [15] shows that compactness in just one of the restrictions $T : A_j \to B_j$ is enough to imply that $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is compact. However, this is not the case for the $(0, q; J)$-method as we will show next by means of an example.

Counterexample 6.2 Let $1 \le q \le \infty$ and consider the Banach spaces $\ell_q \hookrightarrow \ell_q(2^{-n})$ and $\ell_q \hookrightarrow \ell_q(e^{-n})$. Choose T as the identity map I. Then $I : \ell_q \to \ell_q$ is bounded and $I : \ell_q(2^{-n}) \to \ell_q(e^{-n})$ is compact because it is the limit of the sequence of finite rank operators given by $P_m(\xi_n) = (\xi_1, ..., \xi_m, 0, 0, ...)$. By Corollary 3.6, interpolating these spaces we have with $1/q + 1/q' = 1$

$$(\ell_q, \ell_q(2^{-n}))_{0,q;J} = \ell_q(n^{-1/q'}) \quad , \quad (\ell_q, \ell_q(e^{-n}))_{0,q;J} = \ell_q(n^{-1/q'})$$

and it is clear that $I : \ell_q(n^{-1/q'}) \to \ell_q(n^{-1/q'})$ is not compact.

Hence, compactness of $T : A_1 \to B_1$ is not enough for the interpolated operator to be compact. However, as we show next, if T acts compactly between the smaller spaces then the interpolated operator is also compact. For this aim we shall use techniques originated in the papers by Cobos, Edmunds and Potter [7] and Cobos and Fernandez [8] based on properties of the vector-valued sequence spaces related to the interpolation method under consideration. We start with an auxiliary result.

Lemma 6.3 *Let (G_n) be a sequence of Banach spaces and let $1 \le q \le \infty$. Then*

$$\ell_q(G_n) \hookrightarrow (\ell_1(G_n), \ell_1(2^{-n}G_n))_{0,q;J}\,.$$

Proof. Take $x = (x_n)_n \in \ell_q(G_n)$. Let $u^m = (u^m_n)$ be the sequence having all coordinates equal to 0 except the m-th coordinate which is equal to x_m. Clearly, $u^m \in \ell_1(G_n)$ for each $m \in \mathbb{N}$ and it is easy to see that $x = \sum_{m=1}^{\infty} u^m$ (convergence in $\ell_1(2^{-n}G_n)$). Moreover,

$$J(2^m, u^m) = \max\{\|u^m\|_{\ell_1(G_n)}, 2^m \|u^m\|_{\ell_1(2^{-n}G_n)}\} = \|x_m\|_{G_m}\,.$$

Hence,

$$\|x\|_{0,q} \le \Big(\sum_{m=1}^{\infty} J(2^m, u^m)^q\Big)^{1/q} = \|x\|_{\ell_q(G_n)}\,. \qquad \Box$$

Next we proceed with the compactness result.

Theorem 6.4 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be couples of Banach spaces with $A_0 \hookrightarrow A_1$, $B_0 \hookrightarrow B_1$, let $1 \le q \le \infty$ and $T \in \mathcal{L}(\bar{A}, \bar{B})$. If $T : A_0 \to B_0$ is compact, then*

$$T : (A_0, A_1)_{0,q;J} \to (B_0, B_1)_{0,q;J}$$

is compact as well.

Proof. If $q = 1$, the result follows from equalities $(A_0, A_1)_{0,1;J} = A_0$ and $(B_0, B_1)_{0,1;J} = B_1$. Assume then $1 < q \le \infty$. For $n \in \mathbb{N}$, put $G_n = (A_0, J(2^n, \cdot))$ and consider operators $P_1, P_2, ...$ and π, which associate to each vector-valued sequence (u_n) the sequence $P_m(u_n) = (u_1, u_2, ..., u_m, 0, 0, ...)$ and the sum

$\pi(u_n) = \sum\limits_{n=1}^{\infty} u_n$ (in A_1), respectively. We write $\hat{T} = T \circ \pi$. It is clear that operators $\hat{T} : \ell_1(G_n) \to B_0$ and $\hat{T} : \ell_1(2^{-n}G_n) \to B_1$ are bounded. The factorization

$$(\ell_1(G_n), \ell_1(2^{-n}G_n))_{0,q;J} \xrightarrow{P_m} \ell_1(G_n) \xrightarrow{\pi} A_0 \xrightarrow{T} B_0 \hookrightarrow (B_0, B_1)_{0,q;J}$$

and compactness of $T : A_0 \to B_0$ implies that

$$\hat{T}P_m : (\ell_1(G_n), \ell_1(2^{-n}G_n))_{0,q;J} \to (B_0, B_1)_{0,q;J}$$

is compact for each $m \in \mathbb{N}$. We claim that

$$\hat{T} : (\ell_1(G_n), \ell_1(2^{-n}G_n))_{0,q;J} \to (B_0, B_1)_{0,q;J}$$

is also compact. Indeed, it is enough to show that the sequence of norms of interpolated operators $(\|\hat{T} - \hat{T}P_m\|)$ converges to 0. By Theorem 4.9, this reduces to check that $\|\hat{T} - \hat{T}P_m\|_{\ell_1(G_n),B_0} \longrightarrow 0$ as $m \to \infty$. It is clear that this sequence is convergent because it is not increasing and it is bounded from below by 0. Put $\lambda = \lim_{m\to\infty} \|\hat{T} - \hat{T}P_m\|_{\ell_1(G_n),B_0}$. To show that $\lambda = 0$, find a sequence $(x_m) \subset \ell_1(G_n)$ such that $\|x_m\|_{\ell_1(G_n)} \leq 1$ for each $m \in \mathbb{N}$ and $\|\hat{T}(I - P_m)x_m\|_{B_0}$ goes to λ as m goes to infinity. We write $y_m = (I - P_m)x_m$. Then it turns out that (y_m) is bounded in $\ell_1(G_n)$. Hence, compactness of $\hat{T} : \ell_1(G_n) \to B_0$ implies that there is a subsequence $(y_{m'})$ and some $b \in B_0$ such that $\hat{T}y_{m'} \longrightarrow b$ as $m' \to \infty$. It follows that $\lambda = \|b\|_{B_0}$.

Consider now the sequence (y_m) in $\ell_1(2^{-n}G_n)$. We have

$$\|y_m\|_{\ell_1(2^{-n}G_n)} = \|(I - P_m)x_m\|_{\ell_1(2^{-n}G_n)} \leq 2^{-m}\|x_m\|_{\ell_1(G_n)} \leq 2^{-m}.$$

Hence $\hat{T}y_{m'} \longrightarrow 0$ in B_1 which implies that $b = 0$ and therefore $\lambda = 0$.

Consequently, $\hat{T} : (\ell_1(G_n), \ell_1(2^{-n}G_n))_{0,q;J} \to (B_0, B_1)_{0,q;J}$ is compact. Using Lemma 6.3, we get that $\hat{T} : \ell_q(G_n) \to (B_0, B_1)_{0,q;J}$ is compact as well. This last operator can be factorized as

$$\ell_q(G_n) \xrightarrow{\pi} (A_0, A_1)_{0,q;J} \xrightarrow{T} (B_0, B_1)_{0,q;J},$$

and $\pi : \ell_q(G_n) \to (A_0, A_1)_{0,q;J}$ is a metric surjection when we regard $(A_0, A_1)_{0,q;J}$ with the discrete J-norm, therefore compactness of $T : (A_0, A_1)_{0,q;J} \to (B_0, B_1)_{0,q;J}$ follows. □

We close this section with a quantitative result. It refers to entropy numbers of interpolated operators. Recall that for $T \in \mathcal{L}(A, B)$, where A and B are

Banach spaces, the k-th (dyadic) entropy number $e_k(T)$ of T is defined by

$$e_k(T) = \inf\left\{\varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1, ..., b_{2^{k-1}} \in B\right\},$$

where U_A and U_B are the closed unit balls of A and B, respectively (see [34] and [21]). Clearly, T is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$. The asymptotic decay of entropy numbers can be considered as a measure of the degree of compactness of the operator T. We state the result in terms of the function $\varphi(t, s) = t(1 + (\log s/t)_+)$ which we already considered in (4.5).

Theorem 6.5 *Let A_0, A_1, B be Banach spaces with $A_0 \hookrightarrow A_1$ and suppose that $T \in \mathcal{L}(A_1, B)$. Then for $1 < q \le \infty$ we have*

$$e_{n_0+n_1-1}(T : (A_0, A_1)_{0,q;J} \to B) \lesssim \varphi(e_{n_0}(T : A_0 \to B), e_{n_1}(T : A_1 \to B)).$$

Proof. It suffices to establish the inequality for $q = \infty$ because $(A_0, A_1)_{0,q;J} \hookrightarrow (A_0, A_1)_{0,\infty;J}$. Moreover, by Theorem 4.2 we may work with the norm $\|\cdot\|_{\log,\infty;K}$. Take $\varepsilon_j > e_{n_j}(T : A_j \to B)$. By definition of entropy numbers, there exist $b^j_1, ..., b^j_{s_j}$ with $s_j \le 2^{n_j-1}$ such that

$$T(U_{A_j}) \subset \bigcup_{k=1}^{s_j} (b^j_k + \varepsilon_j U_B) \quad , \quad j = 0, 1.$$

Write $t = \max\{1, e_{n_1}(T : A_1 \to B) / e_{n_0}(T : A_0 \to B)\}$. Given any $a \in (A_0, A_1)_{0,\infty;J}$ with $\|a\|_{\log,\infty;K} \le 1$ and any $\varepsilon > 0$, we can find $a_j \in A_j$ such that $a = a_0 + a_1$ and

$$\|a_0\|_{A_0} + t\|a_1\|_{A_1} \le (1 + \varepsilon)K(t, a) \le (1 + \varepsilon)(1 + \log t).$$

Put

$$r_j = (1 + \varepsilon)t^{-j}(1 + \log t) \quad , \quad j = 0, 1.$$

Then $\|a_j\|_{A_j} \le r_j$ and so we are able to choose k_j such that $\|Ta_j - r_j b^j_{k_j}\|_B \le r_j\varepsilon_j$. This implies that

$$\begin{aligned}\|Ta - (r_0 b^0_{k_0} + r_1 b^1_{k_1})\|_B &\le \|Ta_0 - r_0 b^0_{k_0}\|_B + \|Ta_1 - r_1 b^1_{k_1}\|_B \\ &\le r_0\varepsilon_0 + r_1\varepsilon_1 \\ &= (1 + \varepsilon)\varepsilon_0(1 + \log t) + (1 + \varepsilon)\varepsilon_1(1 + \log t)/t.\end{aligned}$$

Since $s_0 s_1 \le 2^{n_0+n_1-1-1}$, it follows that

$$e_{n_0+n_1-1}(T : (A_0, A_1)_{\log,\infty;K} \to B) \le (1+\varepsilon)\varepsilon_0(1+\log t)+(1+\varepsilon)\varepsilon_1(1+\log t)/t.$$

Letting $\varepsilon_j \to e_{n_j}(T : A_j \to B)$, $j = 0, 1$, and $\varepsilon \to 0$, we derive that

$$e_{n_0+n_1-1}(T : (A_0, A_1)_{\log,\infty;K} \to B) \le 2\varphi(e_{n_0}(T : A_0 \to B), e_{n_1}(T : A_1 \to B)) .$$

This completes the proof. $\square$

7 Remarks on (1,q;K)-spaces

Gomez and Milman investigated in [27] the extreme spaces $(A_0, A_1)_{1,q;K}$ which are obtained by the limit choice $\theta = 1$ in the definition of real interpolation spaces by means of the K-functional (see Definition 7.1 below). Other results on these spaces can be found in several papers in the literature (see, for example, [32] and [25]). It has been shown recently by Cobos, Fernández-Cabrera and Martín that spaces $(A_0, A_1)_{1,q;K}$ also arise by interpolating the 4-tuple $\{A_0, A_1, A_1, A_0\}$ by using the unit square, but this time we should consider the K-method and the interior point (α, β) should lie on the diagonal $\beta = 1-\alpha$ (see [11, Theorem 3.5]).

Next we establish some new results on $(1, q; K)$-spaces with the help of ideas developed in the previous sections. The results exhibit that $(1, q; K)$-spaces behave in a "dual way" than $(0, q, J)$-spaces.

Definition 7.1 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \le q < \infty$. The space $(A_0, A_1)_{1,q;K}$ consists of all $a \in A_1$ which have a finite norm*

$$\|a\|_{1,q;K} = \Big(\int_1^\infty [t^{-1}K(t, a)]^q \, \frac{dt}{t} \Big)^{1/q} .$$

Definition 7.1 also makes sense when $q = \infty$ but the resulting space is just A_1. Note also that $\|\cdot\|_{1,q;K}$ is equivalent to the discrete norm

$$\|a\|_{1,q} = \Big(\sum_{n=1}^\infty [2^{-n}K(2^n, a)]^q \Big)^{1/q} .$$

This norm is used to establish the following formula.

Lemma 7.2 *Let $1 \le q < \infty$. Then we have with equivalent norms*

$$(\ell_q, \ell_q(2^{-n}))_{1,q;K} = \ell_q(n^{1/q}2^{-n}) .$$

Proof. For any $a = (a_n) \in \ell_q(2^{-n})$ and $t > 0$, the K-functional for the couple

$(\ell_q, \ell_q(2^{-n}))$ is given by

$$K(t,a) = \Big(\sum_{n=1}^{\infty} (\min\{1, t2^{-n}\}|a_n|)^q\Big)^{1/q}.$$

Therefore

$$\begin{aligned}
\|a\|_{1,q} &= \Big(\sum_{n=1}^{\infty} 2^{-nq} \sum_{k=1}^{\infty} (\min\{1, 2^n 2^{-k}\}|a_k|)^q\Big)^{1/q} \\
&= \Big(\sum_{k=1}^{\infty} |a_k|^q \sum_{n=1}^{\infty} (\min\{2^{-n}, 2^{-k}\})^q\Big)^{1/q} \\
&\sim \Big(\sum_{k=1}^{\infty} (k^{1/q} 2^{-k}|a_k|)^q\Big)^{1/q} = \|a\|_{\ell_q(n^{1/q}2^{-n})}. \qquad \square
\end{aligned}$$

Remark 7.3 *If we choose the discretization $t = 3^n$, the same arguments yield*

$$(\ell_q, \ell_q(3^{-n}))_{1,q;K} = \ell_q(n^{1/q}3^{-n}).$$

In a more abstract way, if $\lambda > 1$ and (G_n) is a sequence of Banach spaces, then we have

$$(\ell_q(G_n), \ell_q(\lambda^{-n}G_n))_{1,q;K} = \ell_q(n^{1/q}\lambda^{-n}G_n).$$

This formula and similar arguments as those in the proof of Theorem 3.7 imply that if $A_0 \hookrightarrow A_1$, $0 < \theta_0 < \theta_1 < 1$ and $1 \le q < \infty$, then

$$\begin{aligned}
&((A_0, A_1)_{\theta_0,q}, (A_0, A_1)_{\theta_1,q})_{1,q;J} \\
&\quad = \Big\{a \in A_1 : \|a\| = \Big(\sum_{n=1}^{\infty} [2^{-\theta_1 n} n^{1/q} K(2^n, a; A_0, A_1)]^q\Big)^{1/q} < \infty\Big\}.
\end{aligned}$$

As for the characteristic function we have

$$\Big(A, \frac{1}{t}A\Big)_{1,q;K} = \frac{(\log t)^{1/q}}{t} A.$$

Next we determine the space obtained by interpolation of a general couple of weighted spaces.

Theorem 7.4 *Let (Ω, μ) be a σ-finite measure space and $1 \le q < \infty$. Assume that w_0, w_1 are weights on Ω such that $w_0(x) \ge w_1(x)$ μ-a.e. Put*

$$w(x) = w_1(x)\Big(1 + \log\frac{w_0(x)}{w_1(x)}\Big)^{1/q}.$$

Then we have with equivalent norms,

$$(L_q(w_0), L_q(w_1))_{1,q;K} = L_q(w)\,.$$

Proof. Using the estimate for the K-functional pointed out in (4.4), we obtain

$$\begin{aligned}
\|f\|_{1,q;K} &\sim \Big(\int_\Omega \int_1^\infty \Big(\frac{\min\{w_0(x), tw_1(x)\}}{t}\Big)^q |f(x)|^q \frac{dt}{t} d\mu(x)\Big)^{1/q} \\
&= \Big(\int_\Omega \Big[\int_1^{\frac{w_0(x)}{w_1(x)}} w_1(x)^q \frac{dt}{t} + \int_{\frac{w_0(x)}{w_1(x)}}^\infty (w_0(x)/t)^q \frac{dt}{t}\Big] |f(x)|^q \, d\mu(x)\Big)^{1/q} \\
&\sim \Big(\int_\Omega (w(x)|f(x)|)^q \, d\mu(x)\Big)^{1/q}. \qquad \square
\end{aligned}$$

Next we show the formula which corresponds to Theorem 4.6/(c).

Theorem 7.5 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \le q < \infty$. Then*

$$((A_0, A_1)_{1,q;K}, A_1)_{1,q;K} = \Big\{a \in A_1 : \Big(\int_1^\infty \Big[\frac{K(t, a; A_0, A_1)}{t(1 + \log t)^{1/q}}\Big]^q \frac{dt}{t}\Big)^{1/q} < \infty\Big\}.$$

Proof. Using the techniques in [27, Theorem 3.6], one can derive that

$$K(t, a; (A_0, A_1)_{1,q;K}, A_1) \sim \Big(\int_1^{e^{t^q}} \Big[\frac{K(s, a; A_0, A_1)}{s}\Big]^q \frac{ds}{s}\Big)^{1/q}.$$

Therefore,

$$\begin{aligned}
\|a\|_{((A_0,A_1)_{1,q;K},A_1)_{1,q;K}} &\sim \Big(\int_1^\infty \frac{1}{t^q} \int_1^{e^{t^q}} \Big[\frac{K(s, a; A_0, A_1)}{s}\Big]^q \frac{ds}{s} \frac{dt}{t}\Big)^{1/q} \\
&= \Big(\int_e^\infty \Big[\frac{K(s, a; A_0, A_1)}{s}\Big]^q \int_{(\log s)^{1/q}}^\infty \frac{1}{t^q} \frac{dt}{t} \frac{ds}{s}\Big)^{1/q} \\
&\sim \Big(\int_1^\infty \Big[\frac{K(s, a; A_0, A_1)}{s(1 + \log s)^{1/q}}\Big]^q \frac{ds}{s}\Big)^{1/q}. \qquad \square
\end{aligned}$$

Our next aim is to describe spaces $(A_0, A_1)_{1,q;K}$ by means of the J-functional. Put $\rho(t) = t^{-1}(1 + \log t)$ and define $(A_0, A_1)_{\rho,q;J}$ as the collection of all those $a \in A_1$ for which there is a strongly measurable function $u(t)$ with values in A_0 such that

$$a = \int_1^\infty u(t)\,\frac{dt}{t} \qquad \text{(convergence in } A_1) \tag{7.1}$$

and

$$\Big(\int_1^\infty [\rho(t)J(t,u(t))]^q\,\frac{dt}{t}\Big)^{1/q} < \infty\,. \tag{7.2}$$

We put

$$\|a\|_{\rho,q;J} = \inf\Big\{\Big(\int_1^\infty [\rho(t)J(t,u(t))]^q\,\frac{dt}{t}\Big)^{1/q}\Big\}\,,$$

where the infimum is taken over all representations u satisfying (7.1) and (7.2). The discretization $t = 2^n$ yields an equivalent norm which we denote by $\|\cdot\|_{\rho,q}$.

Theorem 7.6 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and let $1 \le q < \infty$. Then*

$$(A_0, A_1)_{1,q;K} = (A_0, A_1)_{\rho,q;J}$$

with equivalence of norms.

Proof. Take $a \in (A_0, A_1)_{\rho,1;J}$ with

$$a = \int_1^\infty u(t)\,\frac{dt}{t} \qquad \text{and} \qquad \int_1^\infty \rho(t)J(t,u(t))\,\frac{dt}{t} \le 2\|a\|_{\rho,1;J}\,.$$

Then

$$\begin{aligned}
\|a\|_{1,1;K} &= \int_1^\infty \frac{K(t,a)}{t}\,\frac{dt}{t} \le \int_1^\infty \frac{1}{t}\int_1^\infty K(t,u(s))\,\frac{ds}{s}\,\frac{dt}{t} \\
&\le \int_1^\infty J(s,u(s)) \int_1^\infty \frac{1}{t}\min\{1,t/s\}\,\frac{dt}{t}\,\frac{ds}{s} \\
&= \int_1^\infty \rho(s)J(s,u(s))\,\frac{ds}{s} \le 2\|a\|_{\rho,1;J}\,.
\end{aligned}$$

To check the embedding $(A_0, A_1)_{\rho,q;J} \hookrightarrow (A_0, A_1)_{1,q;K}$ when $1 < q < \infty$, we work with discrete representations. Suppose that

$$a = \sum_{n=1}^\infty u_n \qquad \text{with} \qquad \Big(\sum_{n=1}^\infty \Big[\frac{1+n}{2^n}J(2^n,u_n)\Big]^q\Big)^{1/q} \le 2\|a\|_{\rho,q}\,.$$

It follows from

$$\begin{aligned}2^{-n}K(2^n,a) &\le 2^{-n}\Big(\Big\|\sum_{k=1}^{n} u_k\Big\|_{A_0} + 2^n\Big\|\sum_{k=n+1}^{\infty} u_k\Big\|_{A_1}\Big)\\ &\le 2^{-n}\sum_{k=1}^{n} J(2^k,u_k) + \sum_{k=n+1}^{\infty} 2^{-k}J(2^k,u_k)\end{aligned}$$

that

$$\begin{aligned}\|a\|_{1,q} &= \Big(\sum_{n=1}^{\infty}[2^{-n}K(2^n,a)]^q\Big)^{1/q}\\ &\le \Big(\sum_{n=1}^{\infty}\Big[2^{-n}\sum_{k=1}^{n}J(2^k,u_k)\Big]^q\Big)^{1/q} + \Big(\sum_{n=1}^{\infty}\Big[\sum_{k=n+1}^{\infty}2^{-k}J(2^k,u_k)\Big]^q\Big)^{1/q}\\ &= \Big(\sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\frac{2^{k-n}}{k}\,\alpha_k\Big]^q\Big)^{1/q} + \Big(\sum_{n=1}^{\infty}\Big[\sum_{k=n+1}^{\infty}\frac{\alpha_k}{k}\Big]^q\Big)^{1/q}\\ &= S_1 + S_2\,,\end{aligned}$$

where we have written $\alpha_k = (k/2^k)J(2^k,u_k)$.

Using Hardy's inequality we obtain for S_1 that

$$S_1 \le \Big(\sum_{n=1}^{\infty}\Big[\frac{1}{n}\sum_{k=1}^{n}\alpha_k\Big]^q\Big)^{1/q} \lesssim \Big(\sum_{n=1}^{\infty}\alpha_k^q\Big)^{1/q} \le 2\|a\|_{\rho,q}\,.$$

In order to estimate S_2, choose $(\gamma_n)\in\ell_{q'}$ with

$$\|(\gamma_n)\|_{\ell_{q'}} = 1 \qquad \text{and} \qquad S_2 = \sum_{n=1}^{\infty}\gamma_n\sum_{k=n+1}^{\infty}\frac{\alpha_k}{k}\,.$$

Then Hölder's inequality and Hardy's inequality yield

$$\begin{aligned}S_2 = \sum_{k=1}^{\infty}\Big(\frac{1}{k}\sum_{n=1}^{k}\gamma_n\Big)\alpha_k &\le \Big(\sum_{k=1}^{\infty}\alpha_k^q\Big)^{1/q}\cdot\Big(\sum_{k=1}^{\infty}\Big[\frac{1}{k}\sum_{n=1}^{k}\gamma_n\Big]^{q'}\Big)^{1/q'}\\ &\lesssim \Big(\sum_{k=1}^{\infty}\alpha_k^q\Big)^{1/q}\|(\gamma_n)\|_{\ell_{q'}}\\ &\le 2\|a\|_{\rho,q}\,.\end{aligned}$$

To establish the converse inequality we first note that

$$K(t,a) = tK(1/t,a;A_1,A_0) \qquad \text{and} \qquad J(t,a) = tJ(1/t,a;A_1,A_0)\,. \quad (7.3)$$

Let $\hat{K}(t,a) = K(t,a;A_1,A_0)$ and $\hat{J}(t,a) = J(t,a;A_1,A_0)$. Using (7.3) it is not hard to check that

$$(A_0,A_1)_{1,q;K} = (A_1,A_0)_{0,q;K} \qquad \text{and} \qquad (A_0,A_1)_{\rho,q;J} = (A_1,A_0)_{\psi,q;J} \ ,$$

where we put $\psi(t) = 1 - \log t$ for $0 < t < 1$ and spaces on the couple (A_1, A_0) are defined in the natural way

$$(A_1,A_0)_{0,q;K} = \left\{ a \in A_1 : \|a\|_{0,q;K} = \Big(\int_0^1 \hat{K}(t,a)^q \, \frac{dt}{t} \Big)^{1/q} < \infty \right\}$$

and

$$(A_1,A_0)_{\psi,q;J} = \left\{ a = \int_0^1 w(t) \, \frac{dt}{t} : \Big(\int_0^1 [\psi(t)\hat{J}(t,w(t))]^q \, \frac{dt}{t} \Big)^{1/q} < \infty \right\}$$

with the norm in $(A_1,A_0)_{\psi,q;J}$ given by

$$\|a\|_{\psi,q;J} = \inf \left\{ \Big(\int_0^1 [\psi(t)\hat{J}(t,w(t))]^q \, \frac{dt}{t} \Big)^{1/q} \right\} .$$

To complete the proof, we should show that $(A_1,A_0)_{0,q;K} \hookrightarrow (A_1,A_0)_{\psi,q;J}$.

Take $a \in (A_1,A_0)_{0,q;K}$ and let $\eta_\nu = 2^{-2^\nu}$, $\nu = 0,1,2,...$ Find decompositions $a = a_{1,\nu} + a_{0,\nu}$ with $a_{j,\nu} \in A_j$ such that

$$\|a_{1,0}\|_{A_1} + \|a_{0,0}\|_{A_0} \le 2\hat{K}(1,a)$$

and

$$\|a_{1,\nu}\|_{A_1} + \eta_{\nu-1}\|a_{0,\nu}\|_{A_0} \le 2\hat{K}(\eta_{\nu-1},a) \quad , \quad \nu = 1,2,...$$

Since $\hat{K}(t,a) \to 0$ as $t \to 0$, we have that

$$\|a_{1,\nu}\|_{A_1} \to 0 \qquad \text{as} \qquad \nu \to \infty \, . \tag{7.4}$$

Let

$$u_0 = a_{0,0} \ , \ u_1 = a_{0,1} - a_{0,0} \ , \ \dots \ , \ u_\nu = a_{0,\nu} - a_{0,\nu-1} \ , \dots$$

We put further

$$L_0 = (1/2, 1] \quad , \quad L_\nu = (\eta_\nu, \eta_{\nu-1}] \quad , \quad \nu = 1,2,...$$

and define

$$w(t) = \begin{cases} \frac{1}{\log 2} u_0 & \text{if } t \in L_0 \ , \\ \frac{1}{2^{\nu-1}\log 2} u_\nu & \text{if } t \in L_\nu \ , \ \nu = 1,2,... \end{cases}$$

By (7.4), we have that $a = \int_0^1 w(t)\, dt/t$. Moreover, for $t \in L_0$, we get

$$\hat{J}(t, w(t)) \lesssim \hat{J}(1, u_0) \leq \|a - a_{1,0}\|_{A_1} + \|a_{0,0}\|_{A_0} \lesssim \hat{K}(1, a)\,.$$

So,

$$\begin{aligned}\int_{L_0} [\psi(t)\hat{J}(t, w(t))]^q \frac{dt}{t} &\lesssim (1 + \log 2)^q \hat{K}(1, a)^q \log 2 \\ &\lesssim \hat{K}(1/2, a)^q \log 2 \leq \int_{L_0} \hat{K}(t, a)^q \frac{dt}{t}\,.\end{aligned}$$

For $\nu = 1, 2, ...$ and $t \in L_\nu$, we obtain

$$\begin{aligned}\hat{J}(t, w(t)) &\leq \frac{1}{2^{\nu-1}\log 2} \hat{J}(\eta_{\nu-1}, u_\nu) \\ &\leq \frac{1}{2^{\nu-1}\log 2} [\, \|a_{1,\nu-1}\|_{A_1} + \|a_{1,\nu}\|_{A_1} + \eta_{\nu-1}(\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0})] \\ &\leq \frac{4}{2^{\nu-1}\log 2} \cdot \begin{cases} \hat{K}(\eta_{\nu-2}, a) & \text{if } \nu \geq 2 \\ \hat{K}(1, a) & \text{if } \nu = 1 \end{cases}\,.\end{aligned}$$

For $\nu = 1$ this gives

$$\begin{aligned}\int_{L_1} [\psi(t)\hat{J}(t, w(t))]^q \frac{dt}{t} &\lesssim (1 + 2\log 2)^q \hat{K}(1, a)^q \log 2 \\ &\lesssim \hat{K}(1/4, a)^q \log 2 \leq \int_{L_1} \hat{K}(t, a)^q \frac{dt}{t}\,.\end{aligned}$$

And for $\nu \geq 2$ we obtain

$$\int_{L_\nu} (\psi(t)\hat{J}(t, w(t)))^q \frac{dt}{t} \leq \frac{4^q(1 + 2^\nu)^q}{(2^{\nu-1}\log 2)^q} \hat{K}(\eta_{\nu-2}, a) 2^{\nu-1} \log 2 \lesssim \int_{L_{\nu-2}} \hat{K}(t, a)^q \frac{dt}{t}\,.$$

This yields the embedding $(A_1, A_0)_{0,q;K} \hookrightarrow (A_1, A_0)_{\psi,q;J}$ and finishes the proof. □

Remark 7.7 *In the special case $q = 1$ and A_0 equal to its Gagliardo completion in A_1, the equivalence theorem for the $(1, 1; K)$-method is proved in [32, page 22] as a consequence of the "strong fundamental lemma".*

Remark 7.8 *Using the J-description of $(1, q; K)$-spaces, it is easy to adapt the arguments in the proof of Theorem 3.5 to derive the following. For $1 \leq$*

$q < \infty$ and any sequences of Banach spaces (A_n), (B_n) with $A_n \hookrightarrow B_n$ for each $n \in \mathbb{N}$ and $\sup\{\|I_n\|_{A_n,B_n} : n \in \mathbb{N}\} < \infty$, we have

$$(\ell_q(A_n), \ell_q(B_n))_{1,q;K} = \ell_q((A_n, B_n)_{1,q;K}) .$$

Next we establish a norm estimate for interpolated operators. We state it in terms of the function $\varphi(t,s) = t(1 + (\log s/t)_+)$ introduced in (4.5).

Theorem 7.9 *Let $\bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1)$ be couples of Banach spaces with $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$. Suppose that $T \in \mathcal{L}(\bar{A}, \bar{B})$ and let $M_j = \|T\|_{A_j,B_j}$ for $j = 0, 1$. Then for $1 \le q < \infty$ we have*

$$\|T\|_{(A_0,A_1)_{\rho,q;J},(B_0,B_1)_{\rho,q;J}} \le \varphi(M_1, M_0) .$$

Proof. Assume first $M_0/M_1 \ge 1$. Take $a \in (A_0, A_1)_{\rho,q;J}$ and let

$$a = \int_1^\infty u(t)\,\frac{dt}{t} = \int_{\frac{M_0}{M_1}}^\infty u(M_1 t/M_0)\,\frac{dt}{t} .$$

Since $J(t, Tu(s)) \le M_0 J(M_1 t/M_0, u(s))$, we obtain

$$\begin{aligned}
\|Ta\|_{\rho,q;J} &\le M_0\Big(\int_{\frac{M_0}{M_1}}^\infty \Big[\frac{1+\log t}{t} J(M_1 t/M_0, u(M_1 t/M_0))\Big]^q \frac{dt}{t}\Big)^{1/q} \\
&\le M_1 \sup_{\frac{M_0}{M_1}<t<\infty} \left\{\frac{1+\log\frac{M_1 t}{M_0} + \log\frac{M_0}{M_1}}{1+\log\frac{M_1 t}{M_0}}\right\}\Big(\int_1^\infty [\rho(s)J(s,u(s))]^q \frac{ds}{s}\Big)^{1/q} \\
&\le \varphi(M_1, M_0)\Big(\int_1^\infty [\rho(s)J(s,u(s))]^q \frac{ds}{s}\Big)^{1/q} .
\end{aligned}$$

The remaining case $M_0/M_1 \le 1$ is a direct consequence of the inequality

$$J(t, Tu(s)) \le \max\{M_0\|u(s)\|_{A_0}, tM_1\|u(s)\|_{A_1}\} \le M_1 J(t, u(s)) . \qquad \square$$

Note that the estimate in Theorem 7.9 is the same as in Theorem 4.9 but exchanging the roles of M_0 and M_1. There is a similar "duality relationship" between the two methods that concerns entropy numbers of interpolated operators.

Theorem 7.10 *Let A, B_0, B_1 be Banach spaces with $B_0 \hookrightarrow B_1$ and let $T \in \mathcal{L}(A, B_0)$. Then for $1 \le q < \infty$ we have*

$$e_{n_0+n_1-1}(T : A \to (B_0, B_1)_{1,q;K}) \lesssim \varphi(e_{n_1}(T : A \to B_1), e_{n_0}(T : A \to B_0)) .$$

Proof. It suffices to check the inequality for $q = 1$ because $(B_0, B_1)_{1,1;K} \hookrightarrow (B_0, B_1)_{1,q;K}$. We claim that

$$\|b\|_{1,1;K} \leq \varphi(\|b\|_{B_1}, \|b\|_{B_0}) \quad \text{for} \quad b \in B_0 \,. \tag{7.5}$$

Indeed, for any $M > 1$ we have

$$\|b\|_{1,1;K} = \int_1^\infty \frac{K(t,b)}{t} \frac{dt}{t} \leq \int_1^M \|b\|_{B_1} \frac{dt}{t} + \int_M^\infty \frac{\|b\|_{B_0}}{t} \frac{dt}{t} = \|b\|_{B_1} \log M + \frac{\|b\|_{B_0}}{M} \,.$$

Then, (7.5) follows, if we choose $M = 1$ for $\|b\|_{B_0} \leq \|b\|_{B_1}$ and $M = \|b\|_{B_0}/\|b\|_{B_1}$ for $\|b\|_{B_0} > \|b\|_{B_1}$.

Take $\varepsilon_j > e_{n_j}(T : A \to B_j)$, $j = 0, 1$. According to definition of entropy numbers, we can find $b_1^j, ..., b_{s_j}^j$ with $s_j \leq 2^{n_j - 1}$ such that

$$T(U_A) \subset \bigcup_{k=1}^{s_j} (b_k^j + \varepsilon_j U_{B_j}) \quad , \quad j = 0, 1.$$

Whence,

$$T(U_A) \subset \bigcup_{\substack{1 \leq i \leq s_0 \\ 1 \leq j \leq s_1}} (b_i^0 + \varepsilon_0 U_{B_0}) \cap (b_j^1 + \varepsilon_1 U_{B_1}) \,.$$

If $(b_i^0 + \varepsilon_0 U_{B_0}) \cap (b_j^1 + \varepsilon_1 U_{B_1}) \neq \emptyset$, choose $c_{i,j}$ in this intersection. For any $a \in U_A$, we can find $c_{i,j}$ such that $\|Ta - c_{i,j}\|_{B_k} \leq 2\varepsilon_k$ for $k = 0, 1$. Then, by (7.5), we get

$$\|Ta - c_{i,j}\|_{(B_0,B_1)_{1,q;K}} \leq 2\varphi(\varepsilon_1, \varepsilon_0)$$

Since $s_0 s_1 \leq 2^{n_0 + n_1 - 1 - 1}$, the result follows by letting $\varepsilon_k \to \mathrm{e}_{n_k}(T : A \to B_k)$, $k = 0, 1$. $\square$

We have proved in Theorem 6.4 that compactness of $T : A_0 \to B_0$ implies that the interpolated operator by the $(0, q; J)$-method is compact. This fails for the $(1, q; K)$-method as we show next with an example.

Counterexample 7.11 Let $1 \leq q < \infty$, consider the couples of ordered spaces $\ell_q \hookrightarrow \ell_q(3^{-n})$, $\ell_q \hookrightarrow \ell_q(2^{-n})$, let o_n be the sequence having all coordinates equal to 0 except for the n-th coordinate which is 1 and let D be the diagonal operator defined by $D(x_n) = ((2/3)^n x_n)$. It is clear that $D : \ell_q \to \ell_q$ is compact and $D : \ell_q(3^{-n}) \to \ell_q(2^{-n})$ is bounded. However, D acting between the $(1, q; K)$-spaces is not compact. Indeed, by Lemma 7.2 and Remark 7.3, we have

$$(\ell_q, \ell_q(3^{-n}))_{1,q;K} = \ell_q(n^{1/q} 3^{-n}) \quad \text{and} \quad (\ell_q, \ell_q(2^{-n}))_{1,q;K} = \ell_q(n^{1/q} 2^{-n}) \,,$$

and the sequence $(x_n) = (3^n n^{-1/q} o_n)$ is bounded in $\ell_q(n^{1/q}3^{-n})$ but $(Dx_n) = (2^n n^{-1/q} o_n)$ does not have any convergent subsequence in $\ell_q(n^{1/q}2^{-n})$.

As we have already pointed out, it is proved in [11, Theorem 3.5], that $(A_0, A_1)_{1,q;K} = (A_0, A_1, A_1, A_0)_{(\alpha,1-\alpha),q;K}$. Whence, the known results on compactness of interpolated operators by the methods associated to the unit square (see [17, Theorem 6.3] or [24, Corollary 4.4]) yield that if $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $T : A_j \to B_j$ is compact for $j = 0, 1$ then $T : (A_0, A_1)_{1,q;K} \to (B_0, B_1)_{1,q;K}$ is also compact.

Next we show that the sole assumption of compactness in the restriction $T : A_1 \to B_1$ suffices for the interpolated operator to be compact. For this aim, we need two auxiliary results.

Lemma 7.12 *Let (F_n) be a sequence of Banach spaces and let $1 \le q < \infty$. Then*

$$(\ell_\infty(F_n), \ell_\infty(2^{-n}F_n))_{1,q;K} \hookrightarrow \ell_q(2^{-n}F_n) .$$

Proof. Assume that $(x_n) = (a_n) + (b_n)$ with $a = (a_n) \in \ell_\infty(F_n)$ and $b = (b_n) \in \ell_\infty(2^{-n}F_n)$. Then

$$\|x_m\|_{F_m} \le \|a_m\|_{F_m} + \|b_m\|_{F_m} \le \|a\|_{\ell_\infty(F_n)} + 2^m \|b\|_{\ell_\infty(2^{-n}F_n)} .$$

Hence,

$$\begin{aligned} \|(x_n)\|_{\ell_q(2^{-n}F_n)} &= \Big(\sum_{m=1}^{\infty} (2^{-m} \|x_m\|_{F_m})^q \Big)^{1/q} \\ &\le \Big(\sum_{m=1}^{\infty} (2^{-m} K(2^m, (x_n)))^q \Big)^{1/q} = \|(x_n)\|_{1,q} . \qquad \square \end{aligned}$$

The following result is easily checked.

Lemma 7.13 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$, let A_1^0 be the closure of A_0 in A_1 and let $1 \le q < \infty$. Then we have*

(i) $a \in A_1^0$ if and only if $t^{-1}K(t, a) \to 0$ as $t \to \infty$,
(ii) $(A_0, A_1)_{1,q;K} = (A_0, A_1^0)_{1,q;K}$.

Now we are ready for the compactness result.

Theorem 7.14 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be couples of Banach spaces with $A_0 \hookrightarrow A_1$, $B_0 \hookrightarrow B_1$, let $1 \le q < \infty$ and $T \in \mathcal{L}(\bar{A}, \bar{B})$. If $T : A_1 \to B_1$ is compact, then*

$$T : (A_0, A_1)_{1,q;K} \longrightarrow (B_0, B_1)_{1,q;K}$$

is compact as well.

Proof. For $m \in \mathbb{N}$ we denote by F_m the Banach space B_1 endowed with the norm $K(2^m, \cdot)$ and we let P_k be the operator which associates to each vector-valued sequence (x_n), the sequence $P_k(x_n) = (x_1, ..., x_k, 0, 0, ...)$. Let j be the operator defined by $j\,b = (b, b, b, ...)$ for $b \in B_1$.

By Lemma 7.13, we know that if we replace A_1 by A_1^0 (resp. B_1 by B_1^0) the interpolation space does not change. Moreover, it is not hard to check that $T : A_1^0 \to B_1^0$ is compact as well. Let $\tilde{T} = j \circ T$.
Factorization

$$(A_0, A_1^0)_{1,q;K} \hookrightarrow A_1^0 \xrightarrow{T} B_1^0 \xrightarrow{j} \ell_\infty(2^{-n}F_n) \xrightarrow{P_k} (\ell_\infty(F_n), \ell_\infty(2^{-n}F_n))_{1,q;K}$$

implies that $P_k\tilde{T} : (A_0, A_1^0)_{1,q;K} \to (\ell_\infty(F_n), \ell_\infty(2^{-n}F_n))_{1,q;K}$ is compact for any $k \in \mathbb{N}$. We are going to prove that

$$\|P_k\tilde{T} - \tilde{T}\|_{(A_0,A_1^0)_{1,q;K},(\ell_\infty(F_n),\ell_\infty(2^{-n}F_n))_{1,q;K}} \longrightarrow 0 \quad \text{as} \quad k \to \infty\,, \tag{7.6}$$

which will show that

$$\tilde{T} : (A_0, A_1^0)_{1,q;K} \longrightarrow (\ell_\infty(F_n), \ell_\infty(2^{-n}F_n))_{1,q;K} \qquad \text{is compact.} \tag{7.7}$$

By Theorem 7.9, in order to prove (7.6) it suffices to show that

$$\|P_k\tilde{T} - \tilde{T}\|_{A_1^0,\ell_\infty(2^{-n}F_n)} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{7.8}$$

Factorization

$$A_1^0 \xrightarrow{T} B_1^0 \xrightarrow{j} c_0(2^{-n}F_n) \hookrightarrow \ell_\infty(2^{-n}F_n)$$

yields that (7.8) follows if we prove

$$\|P_k\tilde{T} - \tilde{T}\|_{A_1^0,c_0(2^{-n}F_n)} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{7.9}$$

By compactness of $\tilde{T} : A_1^0 \to c_0(2^{-n}F_n)$, given any $\varepsilon > 0$, there exists a finite subset $\{d_1, ..., d_r\} \subset A_1^0$ such that

$$\tilde{T}(U_{A_1^0}) \subset \bigcup_{k=1}^{r} \left(\tilde{T}d_k + \frac{\varepsilon}{3}U_{c_0(2^{-n}F_n)}\right).$$

Find $N \in \mathbb{N}$ such that if $m \geq N$

$$\|(I - P_m)\tilde{T}d_k\|_{c_0(2^{-n}F_n)} \leq \frac{\varepsilon}{3} \quad \text{for} \quad k = 1, ..., r\,.$$

For any $m \geq N$ and $a \in U_{A_1^0}$, if we choose k such that

$$\tilde{T}a \in \tilde{T}d_k + \frac{\varepsilon}{3}U_{c_0(2^{-n}F_n)}\,,$$

and we write $\|\cdot\| = \|\cdot\|_{c_0(2^{-n}F_n)}$ then we obtain

$$
\begin{aligned}
&\|\tilde{T}a - P_m\tilde{T}a\| \\
&\quad \le \|\tilde{T}a - \tilde{T}d_k\| + \|\tilde{T}d_k - P_m\tilde{T}d_k\| + \|P_m\tilde{T}d_k - P_m\tilde{T}a\| \\
&\quad \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.
\end{aligned}
$$

This establishes (7.9) and so (7.7).

By (7.7) and Lemma 7.12, we derive that $\tilde{T} : (A_0, A_1^0)_{1,q;K} \to \ell_q(2^{-n}F_n)$ is also compact. Finally, since $j : (B_0, B_1^0)_{1,q;K} \to \ell_q(2^{-n}F_n)$ is a metric injection when we consider the discrete K-norm in $(B_0, B_1^0)_{1,q;K}$, it follows from the factorization

$$(A_0, A_1^0)_{1,q;K} \xrightarrow{T} (B_0, B_1^0)_{1,q;K} \xrightarrow{j} \ell_q(2^{-n}F_n)$$

that the operator $T : (A_0, A_1^0)_{1,q;K} \longrightarrow (B_0, B_1^0)_{1,q;K}$ is compact. □

8 Duality

We have seen in Section 7 a number of results concerning $(1, q; K)$-spaces which are "dual" of results for the $(0, q; J)$-spaces established in Sections 4 and 6. It is time now for showing that these methods are indeed dual one of the other. For this aim, we shall make use of some ideas introduced in [13] and [14] to investigate duality for methods associated to polygons. Since $(A_0, A_1)_{0,1;J} = A_0$ and $(A_0, A_1)_{1,\infty;K} = A_1$, we only pay attention to the case $1 < q < \infty$.

Subsequently, we suppose that $A_0 \hookrightarrow A_1$ with A_0 dense in A_1. So the dual space A_1^* of A_1 can be identified with a subspace A_1' of A_0^*. Using the discrete representations of $(A_0, A_1)_{0,q;J}$ and $(A_0, A_1)_{\rho,q;J} = (A_0, A_1)_{1,q;K}$ it is easy to see that A_0 is also dense in these two spaces. Let $(A_0, A_1)_{0,q;J}'$ and $(A_0, A_1)_{1,q;K}'$ be the dual spaces of $(A_0, A_1)_{0,q;J}$ and $(A_0, A_1)_{1,q;K}$ realized as subspaces of A_0^*, respectively. We also put $A_0' = A_0^*$.

Theorem 8.1 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and A_0 dense in A_1. Assume that $1 < q < \infty$ and let $1/q + 1/q' = 1$. Then we have*

$$(A_0, A_1)_{0,q;J}' = (A_1', A_0')_{1,q';K}$$

and the dual norm of $\|\cdot\|_{0,q}$ coincides with $\|\cdot\|_{1,q'}$.

Proof. For $n \in \mathbb{N}$, let G_n be A_0 normed by $J(2^n, \cdot; A_0, A_1)$. Put $W = \ell_q(G_n)$, let

$$M = \left\{ (w_n) \in W : \sum_{n=1}^{\infty} w_n = 0 \text{ (convergence in } A_1) \right\}$$

and

$$M^{\perp} = \{ \tilde{f} \in W^* : \tilde{f}(w_n) = 0 \text{ for each } (w_n) \in M \} .$$

Since $(A_0, A_1)_{0,q;J}$ endowed with the discrete J-norm $\| \cdot \|_{0,q}$ coincides with the quotient space W/M, we obtain

$$(A_0, A_1)^*_{0,q;J} = (W/M)^* = M^{\perp} .$$

In order to identify this space, let F_n be A_0' normed by

$$2^{-n} K(2^n, \cdot, A_1', A_0') = K(2^{-n}, \cdot, A_0', A_1') .$$

Using that J- and K-functional are in duality (see [5, Theorem 2.7.1]), for each $n \in \mathbb{N}$ we have $G_n' = F_n$ with equality of norms. It follows that $W^* = \ell_{q'}(F_n)$. Therefore, for any $\tilde{f} \in W^*$ there is a sequence (f_n) with $f_n \in F_n$ such that

$$\tilde{f}(w_n) = \sum_{n=1}^{\infty} f_n(w_n) \quad \text{and} \quad \|\tilde{f}\|_{W^*} = \left(\sum_{n=1}^{\infty} \|f_n\|_{F_n}^{q'} \right)^{1/q'} .$$

Moreover, if $\tilde{f} \in M^{\perp}$ then $f_n = f_m$ for all $n, m \in \mathbb{N}$. Indeed, if there is $a \in A_0$ such that $f_n(a) \neq f_m(a)$, then writing $w_k = a$ if $k = n$, $w_k = -a$ if $k = m$ and $w_k = 0$ for $k \neq n, m$, we obtain a sequence $w = (w_k) \in M$ such that $\tilde{f}(w) = f_n(a) - f_m(a) \neq 0$.

Conversely, we claim that if $f \in A_0'$ and $(f, f, \dots) \in W^*$ then the functional $\tilde{f}$ defined by the constant sequence $(f, f, \dots)$ belongs to $M^{\perp}$. Indeed, take any $(w_n) \in M$. We should show that $\tilde{f}(w_n) = \sum_{n=1}^{\infty} f(w_n) = 0$. Since

$$f \in (A_1', A_0')_{1,q';K} = (A_1', A_0')_{\rho,q';J} ,$$

there is a representation $f = \sum_{j=1}^{\infty} g_j$ (convergence in A_0') with $g_j \in A_1'$ such that

$$\|f - \sum_{j=1}^{m} g_j\|_{(A_1', A_0')_{1,q';K}} \to 0 \quad \text{as} \quad m \to \infty.$$

Let $c = \|(w_n)\|_W$ and take any $\varepsilon > 0$. We can find $k \in \mathbb{N}$ such that

$$\|(f - \sum_{j=1}^{k} g_j \, , \, f - \sum_{j=1}^{k} g_j \, , \, \dots)\|_{W^*} = \|f - \sum_{j=1}^{k} g_j\|_{(A_1', A_0')_{1,q';K}} < \frac{\varepsilon}{2c} .$$

Put $g = \sum_{j=1}^{k} g_j$. Then, noting that $g \in A_1'$ and that $\sum_{r=1}^{\infty} w_r = 0$ in A_1, we can choose $N \in \mathbb{N}$ such that for any $m \geq N$ we have $|g(\sum_{r=1}^{m} w_r)| < \varepsilon/2$. Hence, for any $m \geq N$, we obtain

$$
\begin{aligned}
|\sum_{r=1}^{m} f(w_r)| &= |\sum_{r=1}^{m} f(w_r) - g(\sum_{r=1}^{m} w_r) + g(\sum_{r=1}^{m} w_r)| \\
&\le \|(f-g\,,\, f-g\,,\,\dots)\|_{W^*}\|(w_n)\|_W + |g(\sum_{r=1}^{m} w_r)| \\
&\le \frac{\varepsilon}{2c}\cdot c + \frac{\varepsilon}{2} = \varepsilon .
\end{aligned}
$$

Consequently, the space $(A_0, A_1)'_{0,q;J}$ consists of all $f \in A'_0$ such that

$$
\Big(\sum_{n=1}^{\infty}[2^{-n}K(2^n, f; A'_1, A'_0)]^{q'}\Big)^{1/q'} < \infty
$$

and the dual norm of $\|\cdot\|_{0,q}$ is $\|\cdot\|_{1,q'}$. □

We finish the paper with the duality theorem for $(1, q; K)$-spaces.

Theorem 8.2 *Let A_0, A_1 be Banach spaces with $A_0 \hookrightarrow A_1$ and A_0 dense in A_1. Assume that $1 < q < \infty$ and let $1/q + 1/q' = 1$. Then we have*

$$
(A_0, A_1)'_{1,q;K} = (A'_1, A'_0)_{0,q';J}
$$

and the dual norm of $\|\cdot\|_{1,q}$ is $\|\cdot\|_{0,q'}$.

Proof. Let $F_n = A_1$, endowed with the norm $K(2^n, \cdot; A_0, A_1)$ and $G_n = A'_1$ normed by

$$
2^{-n}J(2^n, \cdot; A'_1, A'_0) = J(2^{-n}, \cdot; A'_0, A'_1)\,.
$$

By [5, Theorem 2.7.1] for any $n \in \mathbb{N}$ we have the relation $F'_n = G_n$ with equality of norms.

The space $(A_0, A_1)_{1,q;K}$ normed by the discrete K-norm $\|\cdot\|_{1,q}$ is isometric to the diagonal D of the space $X = \ell_q(2^{-n}F_n)$. It follows that

$$
(A_0, A_1)^*_{1,q;K} = D^* = X^*/D^{\perp}\,.
$$

Consequently, functionals f belonging to $(A_0, A_1)'_{1,q;K}$ are those given by a sequence $(f_n) \in X^* = \ell_{q'}(2^n G_n)$ by means of the formula $f(a) = \sum\limits_{n=1}^{\infty} f_n(a)$, $a \in A_0$, and the norm of f is

$$
\inf\Big\{\Big(\sum_{n=1}^{\infty} J(2^n, f_n; A'_1, A'_0)^{q'}\Big)^{1/q'} : f = \sum_{n=1}^{\infty} f_n\Big\} = \|f\|_{0,q'}\,. \qquad \square
$$

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