Dynamical localization for continuum random surface models

Anne BOUTET de MONVEL and PETER STOLLMANN

Abstract

We prove Anderson localization and strong dynamical localization for random surface models in $\mathbb{R}^d$.[1]

1 Introduction, the model, and the results

Spectral and scattering theory for mathematical models of rough surfaces has attracted considerable interest in recent years, as witnessed in [2,3,7–13,16].

One of the reasons is that these models exhibit a metal-insulator transition. This transition is expected for typical random models in dimensions three and higher but, unfortunately, a proof is out of sight for operators with stationary disorder, e.g., Anderson type Schrödinger operators.

In contrast, models with decaying randomness offer the possibility to prove that their spectrum is pure point near fluctuation boundaries and purely absolutely continuous away from those spectral energies which are generated by the random perturbation.

For a general discussion of models with decaying randomness we refer to the recent paper [13] by Hundertmark and Kirsch and the literature quoted there.

In the present paper we concentrate on the study of surface models in the continuum case. While the presence of an absolutely continuous component is settled in [13], some questions concerning localization seem to be open, namely the question whether dynamical localization holds. We will prove that the latter is indeed the case, relying on [4, 17].

This appears to be new, even for discrete models to which the major part of the literature is devoted. Let us briefly comment on the basic difference between continuum and discrete models and start with the fundamental difficulty one meets when studying the localization phenomenon for surface models.

The defining property of these models is that the perturbation is supported on a small set: a surface S or a neighborhood of a surface. Consequently, a simple-minded application of the existing techniques to prove localization has to fail. This is the case for the Aizenman-Molchanov technique in the discrete as well as the multiscale technique in the continuum case.

[1] *Mathematics Subject Classification*: Primary 81Q10; Secondary 35J10, 47A10

The common way around this difficulty can roughly be explained as follows: the operator is decomposed into a surface term which lives on the Hilbert space $\ell^2(S)$ (this is possible in the discrete setting) and a bulk term. This auxiliary construction to handle the eigenvalue problem for the original operator depends on the spectral parameter. It leads to a surface operator which consists of the random perturbation (which now has full support) and a difference operator which is more complicated than the usual discrete Laplacian. In fact the off-diagonal elements of this operator are all non-zero. They reflect the interaction between different sites on the surface induced by the bulk. Anyway, this auxiliary random operator can be treated with the Aizenman-Molchanov method, e.g., giving rise to exponential decay estimates for the Green's functions with high probability at a fixed energy.

Since, as was remarked above, this auxiliary operator depends on the energy, it is not clear how to prove variable energy estimates needed for a proof of dynamical localization. While in [11], Remark 1.5 it is indicated that dynamical localization could be proven along the lines we sketched above there is no hint mentioning a way around the complications we just explained.

We now introduce the model we are going to study:

The model 1.1. Consider the following self-adjoint random operator in $L^2(\mathbb{R}^d)$,

$$H(\omega) = -\Delta + V_\omega,$$

where

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^m} q_k(\omega) f(x - (k, 0)),$$

and

(1) $0 < m < d$ and points in $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$ are written as pairs, if convenient;

(2) The single site potential $f \geq 0$, $f \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d \leq 3$ and $p > d/2$ if $d > 3$, and $f \geq \sigma > 0$ on some open set $U \neq \emptyset$ for some $\sigma > 0$.

(3) The q_k are i.i.d. random variables distributed with respect to a probability measure μ on $\mathbb{R}$, such that $\operatorname{supp} \mu = [q_{\min}, 0]$ with $q_{\min} < 0$.

We will sometimes need further assumptions on the single site distribution μ:

(4) μ is *Hölder continuous*, i.e. there are constants $C, \alpha > 0$ such that

$$\mu[a, b] \leq C(b-a)^\alpha \text{ for } q_{\min} \leq a \leq b \leq 0.$$

(5) *Disorder assumption*: there exist $C, \tau > 0$ such that

$$\mu[q_{\min}, q_{\min} + \varepsilon] \leq C \cdot \varepsilon^\tau \text{ for } \varepsilon > 0.$$

See Figure 1 for an illustration showing a typical realization of a random surface potential.

It is not hard to see that

$$\sigma(H(\omega)) = [E_0, \infty)$$

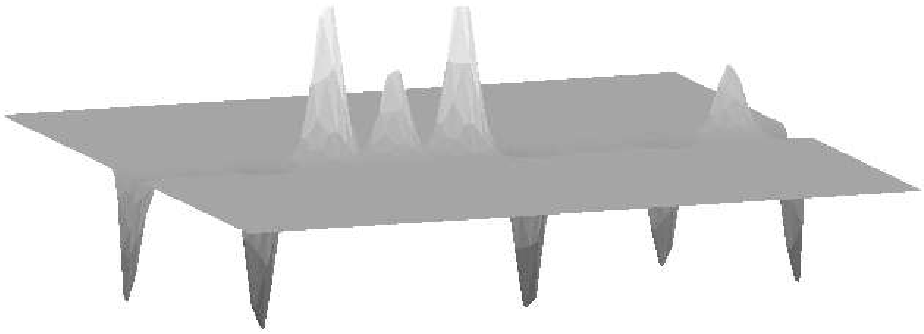

Figure 1: A typical random surface potential

where

$$E_0 = \inf \sigma(-\Delta + q_{\text{min}} \cdot f^{\text{per}}),$$

and

$$f^{\text{per}} = \sum_{k \in \mathbb{Z}^m} f(x - (k, 0))$$

denotes the periodic continuation of f along the surface.

Near the bottom of the spectrum E_0 one expects localization, i.e. suppression of transport as is typical for insulators. This is the content of our main result.

Theorem 1.2. *Let $H(\omega)$ be as in* 1.1(1)-(5) *with $\tau > d/2$.*

(a) *There exists an $\varepsilon > 0$ such that in $[E_0, E_0 + \varepsilon]$ the spectrum of $H(\omega)$ is pure point for almost every $\omega \in \Omega$, with exponentially decaying eigenfunctions.*

(b) *Assume that $p < 2(2\tau - m)$ for τ from* 1.1(5). *Then there exists an $\varepsilon > 0$ such that in $[E_0, E_0 + \varepsilon] = I$ we have strong dynamical localization in the sense that:*

$$\mathbb{E}\{\sup_{t>0} \| |X|^p e^{-itH(\omega)} P_I(H(\omega)) \chi_K \| \} < \infty$$

for every compact set $K \subset \mathbb{R}^d$.

The proof is given in Section 3 below after some necessary preparations. To stress the existence of a metal-insulator transition, let us cite Theorem 4.3 of [13]:

Theorem 1.3. *Let $H(\omega)$ be as in* 1.1(1)-(3). *Then we have, for every $\omega \in \Omega$:*

$$[0, \infty) \subset \sigma_{\text{ac}}(H(\omega)).$$

2 Wegner estimates and initial length scale estimates

In this section we prepare the ground for a multiscale analysis leading to strong dynamical localization. A new feature requires a modification of known principles:

disorder only influences the space near the surface

$$S = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^d$$

and is not seen in the rest of space. Thus, wiggling at the random coupling constants doesn't change the operator in the whole space, but only near S.

In most papers, notably those concerning discrete models, the following strategy is used: the full operator is decomposed into one part which lives on the surface S and one which lives away from the boundary. In this way, the eigenvalue problem for the original operator is reduced to an eigenvalue problem for an auxiliary operator living on the boundary. This decomposition works well in energy regions away from $[0, \infty)$ and in the resulting auxiliary problem the randomness concerns all of (the auxiliary space) S. The bad news is, that the auxiliary operators depend on the energy parameter. Thus it is hopeless to get variable energy multiscale estimates needed for dynamical localization.

Our way around this problem is to show that eigenfunctions corresponding to eigenvalues lying in the part $\sigma(H(\omega)) \setminus [0, \infty)$ live "near S". This means in turn that $\sigma(H(\omega)) \setminus [0, \infty)$ heavily depends on the disordered surface.

We will need to consider local Hamiltonians, i.e. the restriction to cubes of $H(\omega)$. More specifically, we have to analyze the dependence of the eigenvalues of $H_\Lambda(\omega)$ from the random parameter $\omega \in \Omega$. Here $\Lambda = \Lambda_L(x)$ is an open cube of sidelength L centered at $x \in \mathbb{R}^d$. Typically, we will be concerned with the case $L \in 3\mathbb{N} \setminus 2\mathbb{N}$ and $x \in (\frac{L}{3}\mathbb{Z})^d$ in the course of our multiscale analysis later.

As boundary conditions we choose the Dirichlet boundary conditions without expressing that explicitly in the notation.

As a basic input we need the following result from [14] which we infer for the convenience of the reader, put into a somewhat different but equivalent form.

Lemma 2.1. *Let $\Lambda \subset \mathbb{R}^d$ be open, $W \subset \Lambda$ be open and $P \subset W$ be such that* $\operatorname{dist}(P, W^c) := \vartheta > 0$. *Then there exists $C = C(\vartheta)$ such that the following holds: Let V_0 and V be uniformly locally in L^p with $p = 2$ if $d \leq 3$ and $p > d/2$ if $d > 3$. Denote the corresponding operators $-\Delta + V_0$ and $-\Delta + V$ in $L^2(\Lambda)$ by H_0 and H. Assume that $\{x \mid V_0(x) \neq V(x)\} \subset P$ and that Φ is an eigenfunction of H with eigenvalue $\mu \in \varrho(H_0)$. Then*

$$\|\Phi\| \leq [1 + C(\|(H_0 - \mu)^{-1}\| + \|(H_0 - \mu)^{-1}\nabla\|)] \, \|\chi_W \Phi\|.$$

Basically, this means that an eigenvalue μ induced by a perturbation $(V - V_0)$ supported in the region P comes with an eigenfunction Φ which is not too small in a neighborhood W of the perturbation region P. In fact, the estimate given in the lemma gives a lower bound on the portion $\|\chi_W \Phi\|$, which gets better the larger the distance of μ to the spectrum of H_0 is, and the larger (through $C(\vartheta)$) the neighborhood W of P is. For an illustration of the geometry appearing in the preceding Lemma, see Figure 2.

We go on to prove a Wegner estimate along the lines of [18, 14, 13] and single out one important step needed in the proof.

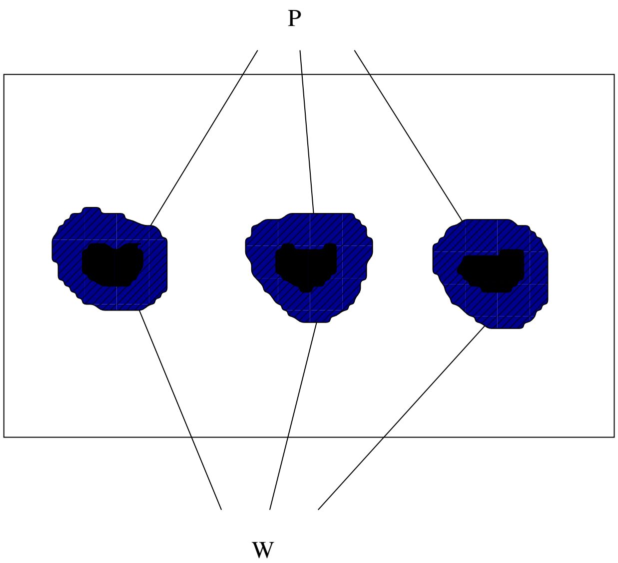

Figure 2: Geometry of Lemma 2.1

Proposition 2.2. *Let $H(\omega)$ be as in* 1.1(1)-(3). *Then there exist $\varrho > 0$ and $C > 0$ such that for every cube $\Lambda = \Lambda_L(x)$ where $L \in 2\mathbb{N} + 1$, $x \in \mathbb{Z}^d$, every $E \in [E_0, E_0 + \varrho]$, every $\omega \in \Omega$ and every eigenfunction Φ of $H_\Lambda(\omega)$ with eigenvalue E we get*

$$\|\Phi \chi_W\| \geq C \cdot \|\Phi\|,$$

where

$$W = \bigcup_{k \in \mathbb{Z}^m} (U + (k, 0)).$$

Note that U was defined in 1.1(2) to be an open set such that the single-site potential f has a lower bound $\sigma > 0$ on U. Thus W is the region in $\mathbb{R}^d$ near the surface, where the influence of the random potential is felt.

Proof. We want to apply Lemma 2.1 above. The right W has already been defined, where formally we consider, of course, $W \cap \Lambda$.

Taking $\varrho < -E_0$ from the beginning we only need to treat those cubes Λ for which $\Lambda \cap W \neq \emptyset$ since, otherwise $H_\Lambda(\omega) = -\Delta_\Lambda$ which has no eigenvalues below zero.

Let $P_0 \subset U$, $P_0 \neq \emptyset$ such that $\vartheta := \mathrm{dist}(P_0, U^{\mathrm{c}}) > 0$, which is possible since U was assumed to be open. Define $P = \bigcup_{k \in \mathbb{Z}^m} (P_0 + (k, 0))$. Then for every Λ as above we have that

$$\mathrm{dist}(P \cap \Lambda, W^{\mathrm{c}} \cap \Lambda) = \vartheta.$$

Next, define $f_0 = f \cdot \chi_{P_0^{\mathrm{c}}} < f$.

Therefore, for $f_0^{\mathrm{per}} := \sum_{k\in\mathbb{Z}^m} f_0(x-(k,0))$ we get that

$$E_0 = \inf\sigma(-\Delta + q_{\min}\cdot f^{\mathrm{per}}) < \tilde{E}_0 := \inf\sigma(-\Delta + q_{\min}\cdot f_0^{\mathrm{per}}).$$

Let $\varrho < \tilde{E}_0 - E_0$. We are going to apply Lemma 2.1 for $V = V_\omega \cdot \chi_\Lambda$, $V_0 = V_\omega \cdot \chi_{\Lambda\setminus P} = (\sum_{k\in\mathbb{Z}^m} q_k(\omega) f_0(x-(k,0)) \cdot \chi_\Lambda$. First note, that

$$\{x \mid V_0(x) \neq V(x)\} \subset P.$$

Moreover, since Dirichlet boundary conditions push the spectrum to the right and by basic monotonicity we get that

$$\inf\sigma(-\Delta + V_0) \geq \inf\sigma\big((-\Delta + q_{\min}\cdot f_0^{\mathrm{per}})_\Lambda\big) \geq \tilde{E}_0.$$

Thus, for every $\omega \in \Omega$ and $E \in [E_0, E_0+\varrho]$ we have that $E \in \varrho(H_0)$ and $\mathrm{dist}(E, \sigma(H_0)) \geq \tilde{E}_0 - E_0 - \varrho =: \delta > 0$. We are thus in position to apply Lemma 2.1, where the norms appearing $\|(H_0-E)^{-1}\|$, $\|(H_0-E)^{-1}\nabla\|$ are uniformly bounded in terms of δ. This ends the proof. □

We are now in position to prove a suitable form of the Wegner estimate.

Proposition 2.3. *Assume that $H(\omega)$ is as in 1.1(1)-(3) and that, moreover, the Hölder condition 1.1(4) is valid. Then, for every interval $J \subset I_\varrho := [E_0, E_0+\varrho]$, with ϱ taken from Proposition 2.2 above, we get*

$$\mathbb{P}\{\omega \mid \sigma(H_\Lambda(\omega)) \cap J \neq \emptyset\} \leq C_W |\Lambda| \cdot |J|^\alpha,$$

where C_W is independent of J and Λ, $\Lambda = \Lambda_L(x)$ with $L \in 2\mathbb{N}+1$, $x \in \mathbb{Z}^d$.

Proof. Since the event to be considered is void if $\Lambda \cap W = \emptyset$ we can assume $\Lambda \cap W \neq \emptyset$. For the course of this proof denote by

$$H(q) = \big(-\Delta + \sum_{k\in\mathbb{Z}^m,(k,0)\in\Lambda} q_k \cdot f(x-(k,0))\big)_\Lambda$$

with Dirichlet boundary conditions at $\partial\Lambda$ and by $E_n(q)$ the n-th eigenvalue of $H(q)$. We get that

$$\mathbb{P}\{\omega \mid \sigma(H_\lambda(\omega)) \cap J \neq \emptyset\} = \bigotimes_{(k,0)\in\Lambda} \mu(\{q \mid E_n(q) \in J \text{ for some } n \in \mathbb{N}\})$$

by definition of $\mathbb{P}$. We want to use the method from [18] to get an estimate on

$$\bigotimes_{(k,0)\in\Lambda} \mu(\{q \mid E_n(q) \in J\}),$$

since the number of $n \in \mathbb{N}$ for which $E_n(q) \leq 0$ can be roughly bounded by $|\Lambda|$. For the estimate we aim at, two properties of $E_n(q)$ as a function of $q \in \mathbb{R}^{\Lambda\cap(\mathbb{Z}^m\times\{0\})}$ have to be checked.

First, $E_n(\,\cdot\,)$ is monotonic. That follows readily from the min-max principle. Secondly, we have to check that, for some $\eta > 0$ and t small enough

$$E_n(q + t \cdot \mathbf{1}) - E_n(q) \geq t \cdot \eta, \tag{$*$}$$

where

$$\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^{\Lambda \cap (\mathbb{Z}^m \times \{0\})}$$

If this is achieved, we can use the Lemma from [18] as well as the proof of the Wegner estimate there and obtain the asserted bound.

Return to the proof of $(*)$.

Clearly, $E_n(q + t \cdot \mathbf{1})$ is continuous in t and admits left and right derivatives which by the celebrated Feynman-Hellmann Theorem, see [19], p. 151, can be calculated as follows:

$$E_n(q + t \cdot \mathbf{1}) = E_n\Big(\underbrace{-\Delta + \sum_{(k,0)\in\Lambda} q_k \cdot f(x - (k,0))}_{H_0} + t \cdot \underbrace{\sum_{(k,0)\in\Lambda} f(x - (k,0))}_{F}\Big),$$

so that

$$E_n(q + t \cdot \mathbf{1}) = E_n(H_0 + t \cdot F),$$

whence

$$\frac{d^+}{dt} E_n(H_0 + t \cdot F)\Big|_{t=s} = (F\Phi_s \mid \Phi_s),$$

where Φ_s is some specific eigenfunction of $H_0 + s \cdot F$ of norm one.

Since $H_0 + s \cdot F = H_\Lambda(\omega)$ for suitable ω, we can apply Proposition 2.2 and get

$$\begin{aligned}
\frac{d^+}{dt} E_n(H_0 + t \cdot F)\Big|_{t=s} &= \int_\Lambda f^{\mathrm{per}}(x)|\Phi_s(x)|^2 dx \\
&\geq \int_{\Lambda \cap W} f^{\mathrm{per}}(x)|\Phi_s(x)|^2 dx \\
&\geq \sigma \|\Phi_s \cdot \chi_W\|^2 \\
&\geq \sigma \cdot C.
\end{aligned}$$

Integrating this with respect to s from 0 to t implies the desired lower bound $(*)$ and thus ends the proof. $\square$

We go on to prove an estimate concerning the initial step in the multiscale induction.

Proposition 2.4. *Assume* 1.1(1)-(3) *and* 1.1(5). *Then, for any* $\xi \in (0, 2\tau - m)$ *there is a* $\beta > 0$ *and* $L_0 \in 2\mathbb{N} + 1$ *such that*

$$\mathbb{P}\{\mathrm{dist}(\sigma(H_\Lambda(\omega)), E_0) \leq L^{\beta-2}\} \leq L^{-\xi}$$

for $\Lambda = \Lambda_L(k)$, $k \in \mathbb{Z}^d$, $L \in 2\mathbb{N} + 1$, $L \geq L_0$.

Proof. We will essentially use the lower estimate $(*)$ from the preceding proof. Define

$$\Omega_{L,h} = \{\omega \in \Omega \mid q_k(x) \geq q_{\min} + h \text{ for all } k \in \mathbb{Z}^m, (k,0) \in \Lambda\}.$$

Then

$$\begin{aligned}\mathbb{P}(\Omega_{L,h}) &\geq 1 - |\Lambda \cap \mathbb{Z}^m \times \{0\}|\, \mu[q_{\min}, q_{\min} + h] \\ &\geq 1 - C\, |\Lambda \cap (\mathbb{Z}^m \times \{0\})| \cdot h^\tau.\end{aligned}$$

Choose $h = \eta^{-1} \cdot L^{\beta-2}$ with η from $(*)$. Then

$$\mathbb{P}(\Omega_{L,h}) \geq 1 - \eta^{-\tau} C \cdot L^{m-\tau(2-\beta)}.$$

Moreover, from $(*)$ we have that for $\omega \in \Omega_{L,h}$

$$E_1(H_\Lambda(\omega)) \geq E_0 + h \cdot \eta \geq E_0 + L^{\beta-2}.$$

Starting with $0 < \xi < 2\tau - m$ we find $\beta > 0$ such that $\xi < \tau(2-\beta) - m$ and for such a choice of ξ and β and $L \geq L_0$ large enough the assertion follows. □

3 Multiscale analysis and proof of the result

In this section we comment on how to obtain a proof of Theorem 1.2. It is a simple adaption of the multiscale method quite well-known in the random business. For a pedestrian version we refer to [17] and the references contained in that book.

The multiscale induction is used to provide exponential off-diagonal decay for the resolvent $(H_\Lambda(\omega) - E)^{-1}$ for $E \in I = [E_0, E_0 + \varepsilon]$ with high probability in terms of the sidelength of L. More precisely, consider open cubes $\Lambda = \Lambda_l(x)$, $x \in \mathbb{Z}^d$, $l \in 2\mathbb{N} + 1$; denote

$$\Lambda^{\text{int}} := \Lambda_{l/3}(x), \quad \Lambda^{\text{out}} := \Lambda_l(x) \setminus \Lambda_{l-2}(x)$$

and by $\chi^{\text{int}} = \chi^{\text{int}}_\Lambda$ and $\chi^{\text{out}} = \chi^{\text{out}}_\Lambda$ denote the respective characteristic functions. A cube Λ is called (γ, E)*–good for* $\omega \in \Omega$, if

$$\|\chi^{\text{out}} R_\Lambda(E) \chi^{\text{int}}\| \leq \exp(-\gamma \cdot l),$$

where

$$R_\Lambda(E) = R_\Lambda(\omega, E) = (H_\Lambda(\omega) - E)^{-1}$$

and $E \in \rho(H_\Lambda(\omega))$ is understood.

To prove localization for energies in an interval $I \subset \mathbb{R}$ consider, for $l \in 2\mathbb{N}+1$, $\gamma > 0$ and $\xi > 0$:

Estimate $G(I, l, \gamma, \xi)$:

$\forall x, y \in \mathbb{Z}^d$, $d(x,y) \geq l$ the following estimate holds:

$$\mathbb{P}\{\forall E \in I : \Lambda_l(x) \text{ or } \Lambda_l(y) \text{ is } (\gamma, E)\text{–good for } \omega\} \geq 1 - l^{-2\xi}.$$

Note that $d(x,y)$ denotes the distance with respect to the ℓ^∞–metric and the condition on the distance of x and y just ensures that the respective cubes are disjoint. Multiscale induction allows one to deduce $G(I, L, \gamma_L, \xi)$ from $G(I, l, \gamma_l, \xi)$ with $L \gg l$ and $\gamma_L \approx \gamma_l$. This will enable us to proceed by induction and prove the estimate $G(I, l_k, \gamma, \xi)$ for a sequence $l_k \nearrow \infty$ of length scales with γ, ξ independent of k.

Here it is important that we can start with l large enough and γ_l not too small, more precisely: $\gamma_l \geq l^{\beta-1}$ for some $\beta > 0$. (Note that for $\gamma_l = l^{-1}$ we would have $\exp(-\gamma_l \cdot l) = 1$ which doesn't lead to an interesting decay.)

The step from one length scale $l = l_k$ to the next one $L = l_{k+1}$ (this explains the name multi–scale analysis) makes use of the following basic idea: partition the cube $\Lambda = \Lambda_L$ into cubes of sidelength l. Each of the good small cubes will add to the exponential decay of R_Λ and since independence and $G(I, l, \gamma_l, \xi)$ guarantees a lot of good small boxes with high probability, we get exponential decay of R_Λ with high probability. The control of the bad cubes is established with the help of the Wegner–type estimate proven in 2.3 above. In our special situation all the cubes which do not touch the surface are automatically good which means that exponential decay off the surface comes for free. We get the following induction theorem:

Theorem 3.1. *Let $H(\omega)$ be as in 1.1, ρ as in Proposition 2.2 and $I_\rho = [E_0, E_0 + \rho]$. Furthermore, fix $\xi > 0$ and let $\alpha \in (1,2)$ be such that*

$$4d \frac{\alpha - 1}{2 - \alpha} \leq \xi.$$

Then there exists $\bar{l} = \bar{l}(\xi, \beta, \alpha)$ such that the following holds.
If $I \subset I_\rho$ and $G(I, l, \gamma, \xi)$ is satisfied for some $\gamma \geq l^{\beta-1}$ and some $l \geq \bar{l}$, then there exists a sequence $(l_k)_{k \in N}$ and a $\gamma_\infty > 0$ with the following properties:

(i) *For all $k \in \mathbb{N}$ the estimate $G(I, l_k, \gamma_\infty, \xi)$ is satisfied.*

(ii) $l_k^\alpha \leq l_{k+1} \leq l_k^\alpha + 6$.

This follows from the multiscale induction as presented in [17] which is a somewhat modified continuum version of the method of von Dreifus and Klein [5], see also [15].

Proof of Theorem 1.2. Observe that for $\xi \in (0, 2\tau - m)$ there is $\beta_1 > 0$ such that with $I_L = [E_0, E_0 + \frac{1}{2}L^{\beta_1 - 2}]$ and $E \in I_L$ we get

$$\mathbb{P}\{\text{dist}(\sigma(H_\Lambda(\omega)), E_0) \leq \frac{1}{2}L^{\beta_1 - 2}\} \leq L^{-\xi}$$

for L large enough. This implies $G(I_l, l, \gamma, \xi)$ with $\gamma > l^{\beta-1}$ for large enough l by Combes–Thomas estimates; see [1] and [17], Section 2.4. Thus we can apply Theorem 3.1 and get that $G(I, l_k, \gamma_\infty, \xi)$ is satisfied for $I = I_l$, l large enough.

Assertion (b) of Theorem 1.2 now follows from the main result in [4].

To deduce (a) we can apply the standard reasoning, proving that with probability 1 every polynomially bounded generalized eigenfunction of $H(\omega)$ is exponentially decaying. □

Concluding remarks

There are several directions in which our result could possibly be generalized. We mention here some which seem especially interesting to us and comment on the difficulties to overcome to this end.

(1) Periodic background:
By this we mean that $-\Delta$ is replaced by $H_0 = -\Delta + V_0$, where V_0 is a $\mathbb{Z}^d$–periodic function. Then $\sigma(H_0)$ has a band structure and it would be worthwhile to investigate localization near band edges of $\sigma(H_0 + V_\omega)$ created by the random surface potential V_ω. In view of the results of [14], this would seem to require a thorough analysis of the behaviour of band edges in the case where a periodic potential is perturbed by a potential which is periodic in a certain direction.

(2) Lifshitz asymptotics:
In the present paper we rely on the disorder assumption 1.1(5) in order to get the initial estimate for the multiscale induction. It would be desirable to dispense with this assumption and use a Lifshitz tail estimate instead. This seems to make a generalization of the Floquet–Bloch theory to partially periodic potentials necessary.

(3) Continuum quasi–periodic surface models:
The discrete case has been settled in [2] using KAM-theory. For a continuum version it doesn't seem clear whether one should try to take a multiscale approach instead.

(4) Bootstrap multiscale analysis:
In a recent preprint [6] a bootstrap multiscale analysis is presented which gives even subexponential decay in dynamical localization estimates. It is probably possible to use this approach also in the present setting.

Acknowledgments. This work was started, respectively finished during stays of the second author at the Université Paris 7 Denis Diderot in 2000 and 2001. He would like to express his sincere gratitude for the warm hospitality extended to him and special thanks to J.-J. Sansuc.

References

[1] J.M. Barbaroux, J.M. Combes, P.D. Hislop: Localization near band edges for random Schrödinger operators, *Helv. Phys. Acta* **70**(1–2), 16–43 (1997).

[2] A.Boutet de Monvel, A. Surkova: Localisation des états de surface pour une classe d'opérateurs de Schrödinger discrets à potentiels de surface quasi-périodiques, *Helv. Phys. Acta* **71**(5), 459–490 (1998).

[3] A. Boutet de Monvel, A. Surkova: Les états de surface d'un opérateur de Schrödinger discret à potentiel de surface périodique, to submit.

[4] D. Damanik, P. Stollmann: Multiscale analysis implies strong dynamical localization, *Geom. Funct. Anal.* **11** (2001), to appear.

[5] H. von Dreifus, A. Klein: A new proof of localization in the Anderson tight binding model, *Comm. Math. Phys.* **124**(2), 285–299 (1989).

[6] F. Germinet, A. Klein: Bootstrap Multiscale Analysis and Localization in Random Media, Preprint mp_arc 00-506 (2000).

[7] V. Grinshpun: Localization for random potentials supported on a subspace, *Lett. Math. Phys.* **34**(2), 103–117 (1995).

[8] V. Jakšić, Y. Last: Corrugated surfaces and a.c. spectrum, *Rev. Math. Phys.* **12**(11), 1465–1503 (2000).

[9] V. Jakšić, S. Molchanov: On the surface spectrum in dimension two, *Helv. Phys. Acta* **71**(6), 629–657 (1998).

[10] V. Jakšić, S. Molchanov: On the spectrum of the surface Maryland model, *Lett. Math. Phys.* **45**(3), 189–193 (1998).

[11] V. Jakšić, S. Molchanov: Localization of surface spectra, *Comm. Math. Phys.* **208**(1), 153–172 (1999).

[12] V. Jakšić, S. Molchanov, L. Pastur: On the propagation properties of surface waves, in "Wave propagation in complex media (Minneapolis, MN, 1994)," IMA Math. Appl., Vol. **96**, Springer, New York, 1998, pp. 143–154.

[13] D. Hundertmark, W. Kirsch: Spectral theory of sparse potentials, in "Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999)," Amer. Math. Soc., Providence, RI, 2000, pp. 213–238.

[14] W. Kirsch, P. Stollmann, G. Stolz: Localization for random perturbations of periodic Schrödinger operators, *Random Oper. Stochastic Equations* **6**(3), 241–268 (1998).

[15] W. Kirsch, P. Stollmann, G. Stolz: Anderson localization for random Schrödinger operators with long range Interactions, *Comm. Math. Phys.* **195**(3), 495–507 (1998).

[16] M. Schröder: On the spectrum of Schrödinger operators at the half space with a certain class of boundary conditions, *Z. Anal. Anwendungen* **7**(3), 233–239 (1988).

[17] P. Stollmann: *Caught by disorder – bound states in random media*, Progress in Math. Phys., Birkhäuser, Boston, 2001, to appear.

[18] P. Stollmann: Wegner estimates and localization for continuum Anderson models with some singular distributions, *Arch. Math.* **75**(4), 307–311 (2000).

[19] W. Thirring: *A Course in Mathematical Physics 3, Quantum Mechanics of Atoms and Molecules*, Springer Verlag, New York, Wien, 1981.

Anschriften der Autoren:

A. Boutet de Monvel
Université Paris-7
Mathématiques, case 7012
2 place Jussieu
75251 Paris Cedex 05, France

P. Stollmann
Fakultät für Mathematik
Technische Universität
09107 Chemnitz, Germany