Trace Ideal Properties of Perturbed Dirichlet Semigroups

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Introduction

We study semigroup differences of the form

$$e^{-t(H+\mu)} - e^{-t(H+\mu+\nu)},$$

where H is the generator of a regular Dirichlet form and μ and ν are suitable measures. In the first section the appropriate classes of measures are introduced. Moreover, we provide a list of examples which can be treated in the Dirichlet form framework.

In the second section the above semigroup is investigated in terms of Hilbert–Schmidt and trace class properties. It turns out that if the set on which ν "lives", the so–called *quasi–support*, has finite capacity then the above semigroup difference is Hilbert–Schmidt. A more restrictive condition on the quasi–support of ν is exhibited which implies that this difference is trace class.

1. Dirichlet forms plus measures

We consider trace ideal properties and their consequences in the framework of *regular Dirichlet forms*, see [10]. In [13] the more general setting of quasi–regular Dirichlet forms is described in detail; with minor changes, our results hold true in this setting. To begin we briefly recollect some basic material concerning the perturbation of Dirichlet forms by measures. Instead of reproducing all the definitions we illustrate the scope of this setting by some examples. In this respect, the monograph [6] is a good reference.

1.1. Examples. (1) On $L_2(\mathbb{R}^d)$ the form

$$\mathfrak{h}_0[u,v] := \int \nabla u \nabla \overline{v} dx, D(\mathfrak{h}_0) = W_0^{1,2}(\mathbb{R}^d)$$

is a regular Dirichlet form.

(2) More generally, forms of the type

$$\mathfrak{h}[u,v] = \sum_{i,j} \int a_{ij}(x)\partial_i u(x)\partial_j v(x)dx$$

with domain $W_0^{1,2}(\Omega)$ are regular Dirichlet forms for suitable a_{ij}; see [6], Section 1.2.

(3) In the last example also Neumann boundary conditions can be included, provided Ω has the extension property, see [1], p. 83f (in order that the form be regular, one needs that $C(\overline{\Omega})$ is dense in $W^{1,2}(\Omega)$).

(4) If $p : [0,\infty) \times E \times E \to \mathbb{R}$ satisfies the basic assumptions of stochastic spetral analysis, "BASSA", then the generator of the associated semigroup corresponds to a regular Dirichlet form on $L_2(E)$. Thus our setting incorporates all the examples listed in [7].

We consider measure perturbations of regular Dirichlet forms and refer the reader to [2, 3, 14, 19, 21] where he can find more details as well as more relevant literature. A recent investigation of spectral theoretic properties of $-\Delta + \mu$ can be found in [4]; see also the article of J. Brasche in the present volume.

Let us mention one point which makes measure perturbations interesting: They provide a convenient means of treating *potentials* and *Dirichlet boundary conditions* simultaneously.

We recall that a regular Dirichlet form $\mathfrak{h}$ with domain D defines a set–function, the *capacity*, by

$$\text{cap } (K) := \inf\{\mathfrak{h}[u,u] + \|u\|^2; u \in D, u \geq \chi_K\}$$

for compact K and regular extension to arbitrary sets. The right class for the positive part of our perturbations is

$$M_0 := \{\mu : \mathfrak{B} \to [0,\infty]; \mu \text{ countably additive}, \mu << \text{ cap}\},$$

the class of measures which do not carge sets of zero capacity. It is wellknown that

$$(\mathfrak{h} + \mu)[u,v] := \mathfrak{h}[u,v] + \int \tilde{u}\overline{\tilde{v}}d\mu$$

defines a closed form, where we write $\tilde{u}$ for the quasi–continuous representative of u.

1.2. Examples. (1) If $\mu = Vdx$ then $\mathfrak{h} + \mu = \mathfrak{h} + V$ is the usual form sum.

(2) (Cf. [3]) The measure $\infty_A(B) := \infty \cdot \text{cap}(A \cap B)$ defines an element of M_0. It is clear that

$$D(\mathfrak{h} + \infty_A) = \{u \in D; \tilde{u} = 0 \text{ q.e. on } A\} =: D_0(A^c).$$

(3) For the classical Dirichlet form $\mathfrak{h}_0$ (see Example 1.1 (1)) and $A = \overline{A}$ it is not hard to check that

$$D(\mathfrak{h}_0 + \infty_A) = W_0^{1,2}(A^c),$$

so that $\mathfrak{h}_0 + \infty_A$ is the form corresponding to the Dirichlet Laplacian on A^c. For more details on that we refer to [12, 9].

As negative parts of perturbations measures which satisfy a Kato condition turn out to be especially well suited. This is due to the fact that the corresponding semigroups still act on different L_p–spaces. We fix a regular Dirichlet form $\mathfrak{h}$ with domain D on some $L_2(X, m)$ and denote the corresponding operator by H. Let us now briefly recall the *extended Kato class*, introduced in [19]. We denote

$$\hat{S}_K := \{\mu \in M_0; \langle \mu; (H+E)^{-1} \cdot \rangle \in L_1(X, m)' = L_\infty(X, m)\},$$

$$c_E(\mu) := \|\langle \mu, (H+E)^{-1} \cdot \rangle\|_\infty,$$
$$c(\mu) := \inf_{E>0} c_E(\mu),$$

which generalizes the Kato class K_d of potentials. More precisely: $V \in K_d$ implies that $Vdx \in \hat{S}_K$ with respect to the classical Dirichlet form $\mathfrak{h}_0$, and $c(Vdx) = 0$. We refer to [2, 19] for details. In [19] it is in particular shown that each $\mu^- \in \hat{S}_K$ is form bunded with respect to $\mathfrak{h}$ with bound $c(\mu)$ so that $\mu = \mu^+ - \mu^-$, $\mu^+ \in M_0$, $\mu^- \in \hat{S}_K, c(\mu^-) < 1$ gives rise to a closed form $\mathfrak{h} + \mu$. Due to the fact that μ^+ may be infinite this form is not necessarily densely defined, but we can associate a self adjoint operator $H + \mu$ in the Hilbert space $\overline{D(\mathfrak{h} + \mu)}$ which is of the form $L_2(Y)$ for a subset Y of X. We extend the semigroup $e^{-t(H+\mu)}$ by 0 to all of $L_2(X)$ and recall from [19] that it also acts on the spaces $L_p(X)$.

A simple way of defining a suitable notion of support for measures in M_0 is the following: A quasi–closed set $\Sigma(\nu)$ is called *quasi–support* of ν if

$$\mathfrak{h} + \infty\nu = \mathfrak{h} + \infty_{\Sigma(\nu)}.$$

This definition from [17] is based on the characterization of closed lattice ideals in [15] and extends the notion of quasi–support which was given for the smaller class of smooth measures in [11]. It is easy to see that, for Radon measures, $\Sigma(\nu)$ is smaller than the usual support and that these sets need not be equal; the examples in [21], Section 9 can be interpreted in this sense.

2. Trace ideal properties of semigroup differences

Throughout this section we assume that $\mathfrak{h}$ is a regular Dirichlet form with the additional property that $e^{-tH} : L_1 \to L_\infty$ is bounded. We write $\|\cdot\|_{HS}$ for the Hilbert–Schmidt norm.

2.1. Theorem. *Let $\mu^+, \nu \in M_0$ and $\mu^- \in \hat{S}_K$ with $c(\mu^-) < 1/2$. If $\mathrm{cap}(\Sigma(\nu)) < \infty$ then*

$$\|e^{-t(H+\mu)} - e^{-t(H+\mu+\nu)}\|_{HS} \le 2\mathrm{cap}(\Sigma(\nu))^{1/2} \|e^{-t(H-2\mu^-)} : L_2 \to L_\infty\|$$

for all $t > 0$.

The proof goes essentially along the same lines as the proof of Theorem 1 in [16] and will be given in [5]. This Theorem can, in particular, be applied to $\mu = V \in L^1_{loc} - K_d$ and $\nu = \infty_\Gamma$ in the setting of "BASSA". As a consequence concerning large coupling limits we have (see [17], Corollary 2.4)

2.2. Corollary. *Let μ, ν be as in the Theorem. Then*

$$H + \mu + n \cdot \nu \xrightarrow{nrs} H + \mu + \infty_{\Sigma(\nu)},$$

which means convergence in the norm resolvent sense.

Our next aim is an estimate of the trace norm $\|\cdot\|_{tr}$ of the semigroup difference if ν has sufficiently small support. A recent application of such an inequality can be found in [18], where $H = -\Delta, \mu = V \in L^1_{loc} - K_d$ and $\nu = \infty_S$ is considered. To formulate the appropriate condition on $\Sigma(\nu)$ we have to recall that there exists a Markov process which is associated with $\mathfrak{h}$ in the sense that

$$e^{-tH} f(x) = \mathbb{E}^x[f \circ X_t].$$

By τ_B we denote the first hitting time of B,

$$\tau_B(\omega) = \inf\{t > 0 : X_t(\omega) \in B\}$$

for $B \subset E$. We have:

2.3. Theorem. *Let If $c_{\Sigma(\nu)} := \int \left(\mathbb{P}^x\left[\tau_{\Sigma(\nu)} \le t\right]\right)^{1/2} dx < \infty$ for some $t > 0$, where μ, ν are as in Theorem 2.1. Then*

$$\|e^{-2t(H+\mu)} - e^{-2t(H+\mu+\nu)}\|_{tr} \le 2c_B \cdot \|e^{-t(H-2\mu^-)} : L_1 \to L_2\|^2.$$

Again, this may be applied to obstacle scattering, setting $\nu := \infty_B$. The slightly simpler case $\mu = 0$ had been studied in [16]; under the more restrictive regime of "BASSA", the corresponding results for $\mu = V dx$, $\nu = \infty_\Gamma$ were obtained in [8]. Let us mention that the above Theorem can also be applied to scattering by non–closed obstacles. Of course, the Theorem implies trace class convergence

$$e^{-t(H+\mu+n\nu)} \to e^{-t(H+\mu+\infty_{\Sigma(\nu)})}.$$

In the case of the classical Dirichlet form on $\mathbb{R}^d$, the assumption on $\Sigma(\nu)$ occuring in the Theorem is satisfied for bounded sets. Related estimates for $c_{\Sigma(\nu)}$ can be found in Section 2 of [18].

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