DELONE DYNAMICAL SYSTEMS AND ASSOCIATED RANDOM OPERATORS

DANIEL LENZ AND PETER STOLLMANN

ABSTRACT. We carry out a careful study of basic topological and ergodic features of Delone dynamical systems. We then investigate the associated topological groupoids and in particular their representations on certain direct integrals with non constant fibres. Via noncommutative integration theory these representations give rise to von Neumann algebras of random operators. Features of these algebras and operators are discussed. Restricting our attention to a certain subalgebra of tight binding operators, we then discuss a Shubin trace formula.

INTRODUCTION

The study of disorder is one of the most important issues today. In mathematical models of solid state physics order or disorder are expressed in terms of the hamiltonian that drives the system. The latter operator appears in the Schrödinger equation and its properties are used to describe the electronic properties of the solid under consideration.

Both order and disorder can be cast in the framework of parametrized operators. Let us be a little more precise on that point. We start at one extreme: a perfectly orderered solid, an ideal crystal. Since the atomic positions belong to a lattice Γ in this case, the hamiltonian exhibits a corresponding translational symmetry. This symmetry is encoded in the quotient X/Γ of the underlying space $X \in \{\mathbb{R}^d, \mathbb{Z}^d\}$ which leads to a family of operators parametrized by X/Γ. As elementary as this observation is, it leads to a number of important consequences. One of these consequences concerns the nature of the spectrum and the dynamics of the hamiltonian: periodic operators (at least under some additional assumptions) have purely absolutely continuous spectrum and only extended states. The other extreme case concerns models that are statistically independent at distant parts of space. Important and well studied are models of Anderson type for which the parameter space is typically of the form $I^{\mathbb{Z}^d}$, I a compact interval in $\mathbb{R}$. For these models a completely different spectral and dynamical picture is expected with energy regions filled by dense pure point spectrum with localized eigenfunctions and absence of mobility. The key notion is localization, a phenomenon that is rigorously proved in certain cases, see [7, 28, 36]. The models we want to study are situated in between ordered (e.g. periodic) and heavily disordered (e.g. Anderson models) and are used to describe quasicrystals. The discovery of quasicrystals (see the celebrated 1984 paper [32]) has led to an impressive research activity. However, mathematically rigorous results concerning the spectral theory

1991 *Mathematics Subject Classification.* 47B80; 82B44; 46L60; 52C23.
Key words and phrases. Random operators; disordered systems; quasicrystals; aperiodic tilings.

and dynamics of quasicrystalline Schrödinger operators are somewhat rare except for one dimensional papers.

In the present paper we report on some basic issues concerning the treatment of the multidimensional case. Namely, we describe how Delone dynamical systems of finite type can be used as the relevant parameter spaces. It turns out that there are aspects common with the one dimensional situation: local finiteness and a hierarchical order. There is one important difference as well: a more complicated geometry and the absence of a lattice structure. This leads to complications concerning the parametrized family of hamiltonians that appears. The latter are defined on different Hilbert spaces. The first point we mentioned is reflected in the strong ergodic properties that are the same as in the one dimensional case. The second point leads to the lack of spectral consequences that follow from these ergodic properties.

At this point let us end this general preview and refer the reader to the main text for more details. Instead we go on to relate our results to what can be found in the literature:

Various features of Delone sets and tilings have been considered in the literature. See, e.g., [6, 17, 19, 20, 21, 22, 31, 34, 35] and the literature cited there. However, as will be seen in Section 1 below there are certain misunderstandings concerning the problem of defining a suitable topology on the set of Delone sets.

On the other hand, a thorough study of "almost random operators" in an algebraic setting focusing on K-theory can be found in [3, 4, 5]. This has been taken up by Kellendonk who introduced certain C^*-algebras associated to tilings and studied their K-theory [15, 16], see [1, 17, 29] as well. In fact, we have been inspired by these works although they do not cover all the results mentioned here. Our focus is rather on general features of random operators associated to tilings. These features include (almost sure) constancy of the spectrum and its various parts, absence of discrete spectrum and validity of a Shubin trace formula.

The important method here is Connes noncommutative integration theory, which we use to associate a von Neumann algebra to Delone dynamical systems. We should stress that this does not mean to associate in some general fashion a von Neumann algebra to a dynamical system. Rather we use the specific situation at hand as well as the guidance provided by the physical background.

Acknowledgement. It is our pleasure to acknowledge fruitful correspondence with J. Lagarias, P. Pleasants and B. Solomyak. Moreover, both authors are grateful for financial support from the German Science Foundation in the priority programs "Interacting stochastic systems of high complexity" and "Quasicrystals".

1. Generalities on Delone sets

The aim of this section is to recall standard concepts from the theory of Delone sets and to introduce a suitable topology on the closed sets in euclidian space.

We will be concerned with Delone sets in $\mathbb{R}^d$. A subset ω of $\mathbb{R}^d$ is called a *Delone set* if there exist $0 < r, R < \infty$ such that $r \leq \|x - y\|$ whenever $x, y \in \omega$ with $x \neq y$, and $B_R(x) \cap \omega \neq \emptyset$ for all $x \in \mathbb{R}^d$. Here, the Euclidean norm on $\mathbb{R}^d$ is denoted by $\|\cdot\|$

and $B_s(x)$ denotes the (closed) ball in $\mathbb{R}^d$ around x with radius s. The set ω is then also called an (r,R)-set. We will be particularly interested in the restrictions of Delone sets to bounded sets. In order to treat these restrictions, we introduce the following definition.

Definition 1.1. (a) A pair (Λ, Q) consisting of a bounded subset Q of $\mathbb{R}^d$ and $\Lambda \subset Q$ finite is called a pattern. The set Q is called the support of the pattern.
(b) A pattern (Λ, Q) is called a ball pattern if $Q = B_s(x)$ with $x \in \Lambda$ for suitable $x \in \mathbb{R}^d$ and $s \in (0,\infty)$.

The pattern (Λ_1, Q_1) is contained in the pattern (Λ_2, Q_2) written as $(\Lambda_1, Q_1) \subset (\Lambda_2, Q_2)$ if $Q_1 \subset Q_2$ and $\Lambda_1 = Q_1 \cap \Lambda_2$. Diameter, volume etc. of a pattern are defined to be the diameter, volume etc of its support. For patterns $X_1 = (\Lambda_1, Q_1)$ and $X_2 = (\Lambda_2, Q_2)$, we define $\sharp_{X_1} X_2$, *the number of occurences of* X_1 *in* X_2, to be the number of elements in $\{t \in \mathbb{R}^d : \Lambda_1 + t \subset \Lambda_2, Q_1 + t \subset Q_2\}$.

For further investigation we will have to identify patterns which are equal up to translation. Thus, on the set of patterns we introduce an equivalence relation by setting $(\Lambda_1, Q_1) \sim (\Lambda_2, Q_2)$ if and only if there exists a $t \in \mathbb{R}^d$ with $\Lambda_1 = \Lambda_2 + t$ and $Q_1 = Q_2 + t$. In this latter case we write $(\Lambda_1, Q_1) = (\Lambda_2, Q_2) + t$. The class of a pattern (Λ, Q) is denoted by $[(\Lambda, Q)]$. The notions of diameter, volume, occurence etc. can easily be carried over from patterns to pattern classes.

Every Delone set ω gives rise to a set of pattern classes, $\mathcal{P}(\omega)$ viz $\mathcal{P}(\omega) = \{Q \wedge \omega : Q \subset \mathbb{R}^d$ bounded and measurable$\}$, and to a set of ball pattern classes $\mathcal{P}_B(\omega)) = \{B_s(x) \wedge \omega : x \in \omega, s > 0\}$. Here we set $Q \wedge \omega = [(\omega \cap Q, Q)]$.

For $s \in (0,\infty)$, we denote by $\mathcal{P}_B^s(\omega)$ the set of ball patterns with radius s; note the relation with s-patches as considered in [19]. A Delone set is said to be of *finite type* if for every radius s the set $\mathcal{P}_B^s(\omega)$ is finite. We refer the reader to [19] for a detailed discussion of Delone sets of finite type.

Next we introduce a suitable topology on the set of closed subsets of $\mathbb{R}^d$. Although it is basically known how this can be done, we will take some care. Actually, it turns out that certain statements in [20, 35] concerning this issue are, while morally true, not completely correct ;-)

Denote by $\mathcal{F}(\mathbb{R}^d)$ the set of closed subsets of $\mathbb{R}^d$ and recall that there is a natural action T of $\mathbb{R}^d$ on $\mathcal{F}(\mathbb{R}^d)$ given by $T_t G = G + t$. We aim at a topology on $\mathcal{F}(\mathbb{R}^d)$ that fulfills two requirements: the action T should be continuous and two sets that are close to each other with respect to the topology are supposed to be such that their finite parts have small *Hausdorff distance*. The latter can be defined by

$$d_H(K_1, K_2) := \inf(\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \wedge K_2 \subset U_\varepsilon(K_1)\} \cup \{1\}),$$

where K_1, K_2 are compact subsets of a metric space (X,d) and $U_\varepsilon(K)$ denotes the open ε-neighborhood around K. The extra 1 is to deal with the empty set that is included in $\mathcal{K}(X) := \{K \subset X : K \text{ compact}\}$. It is wellknown that $(\mathcal{K}(X), d_H)$ is complete if (X,d) is complete and compact if (X,d) is compact. Quite often, the Hausdorff distance is defined between nonvoid compact sets only. The way we defined it, $\emptyset$ is added as an isolated point. For our purposes later on it will be important to have the empty set at our disposal and it will no longer be an isolated point.

A natural first attempt to define a suitable topology goes as follows. Abbreviate $B_r(0) =: B_r$ and $\mathcal{K}(B_r) := \mathcal{K}_r$ which is a compact metric space by what we just mentioned. We call the initial topology on $\mathcal{F}(\mathbb{R}^d)$, induced by the restriction mappings

$$J_R : \mathcal{F}(\mathbb{R}^d) \to \mathcal{K}_R, F \mapsto F \cap B_R, R > 0$$

the *topology of local Hausdorff convergence*. Of course this topology satisfies the second requirement listed above. However a serious problem connected with that topology comes from the fact, that F and G might be closer in Hausdorff distance than $F \cap B_k$ and $G \cap B_k$ as two relatively close portions of F and G might just lie inside respectively outside the ball B_R. Put differently, the restriction $F \mapsto F \cap B_r$ is by no way a contraction from $\mathcal{K}_R$ to $\mathcal{K}_r$. In particular, T does not act continuously. Consider, e.g., $F := \mathbb{Z} \in \mathcal{F}(\mathbb{R})$; for any $t \in (0, 1)$ we get that

$$d_H((T_t F) \cap B_k, F \cap B_k) = \max\{|t|, 1 - |t|\}.$$

Consequently, for any $k \geq 3$ and $t \in (0, 1)$, $T_t F \notin V_k(F)$. To circumvent this lamentable fact we introduce

$$d_k(F, G) := \inf(\{\varepsilon > 0 : F \cap B_k \subset U_\varepsilon(G) \wedge G \cap B_k \subset U_\varepsilon(F)\} \cup \{1\}),$$

a measure for the distance of F and G that is monotone in the cutoff parameter k, i.e., $d_k(F, G) \leq d_{k+1}(F, G)$. Moreover, $d_k(F, G) \leq d_H(F \cap B_k, G \cap B_k)$. Unfortunately, the d_k do not satisfy the triangle inequality. To see this, consider, in $\mathbb{R}$, the case $k = 1$ and the sets $F = \{1 - \varepsilon\}, G = \{1 + \varepsilon\}, H = \emptyset$, leading to $d_k(F, G) = 2\varepsilon$, $d_k(G, H) = 0$ but $d_k(F, H) = 1$ (if ε is small enough).

We use them to define a topology coarser than the topology of local Hausdorff convergence via the neighborhood basis

$$U_{\varepsilon,k}(F) := \{G \in \mathcal{F}(\mathbb{R}^d) : d_k(F, G) \leq \varepsilon\}, \varepsilon > 0, k \in \mathbb{N};$$

we call the corresponding topology τ_{nat} the *natural topology*. By the definition of d_k it is clear that $d_k(T_t F, F) \leq |t|$ for any $F \in \mathcal{F}(\mathbb{R}^d), t \in \mathbb{R}^d$ so that translations are continuous. Fortunately this topology has nice compactness properties as seen in:

Theorem 1.2. *$\mathcal{F}(\mathbb{R}^d)$ endowed with the natural topology τ_{nat} is compact.*

Proof. Let $(F_\iota)_{\iota \in I}$ be a net in $\mathcal{F}(\mathbb{R}^d)$. We use that $\mathcal{K}_{k+\frac{1}{2}}$ is a compact metric space for every k. Therefore, $(F_\iota \cap B_{k+\frac{1}{2}})_{\iota \in I}$ has an accumulation point $C_{k+\frac{1}{2}}$ in $\mathcal{K}_{k+\frac{1}{2}}$ for every k and we find a subsequence converging in the Hausdorff metric. By a standard diagonal argument we can arrange a common subsequence, call it $(F_m)_{m\in\mathbb{N}}$, such that $(F_m \cap B_{k+\frac{1}{2}})_{m\in\mathbb{N}}$ converges to $C_{k+\frac{1}{2}}$ for every $k \in \mathbb{N}$;

$$C := \bigcup_{k\in\mathbb{N}} C_{k+\frac{1}{2}}$$

is a good candidate for the limit of the (F_m) in the natural topology. Let us first note that

$$C \cap B_k = C_{k+\frac{1}{2}} \cap B_k \qquad (*).$$

(This is clear but it is here that the $\cdots + \frac{1}{2}$ in the definitions above gets important. It is not true in general that $C \cap B_{k+\frac{1}{2}} = C_{k+\frac{1}{2}} \cap B_{k+\frac{1}{2}}$. We will meet this kind of effect again later when discussing the lack of compactness for the topology of local Hausdorff convergence in the paragraph following the present proof.) From $(*)$ it follows that C is closed. We have to show that

$$d_k(F_m, C) \to 0 \text{ as } m \to \infty$$

for any k; so fix k. If

$$1 > \delta_m > d_H(F_m \cap B_{k+\frac{1}{2}}, C_{k+\frac{1}{2}})$$

we get that

$$C \cap B_k \subset C_{k+\frac{1}{2}} \subset U_{\delta_m}(F_m \cap B_{k+\frac{1}{2}})$$

by definition of the Hausdorff metric. Conversely,

$$F_m \cap B_k \subset F_m \cap B_{k+\frac{1}{2}} \subset U_{\delta_m}(C_{k+\frac{1}{2}}) \subset U_{\delta_m}(C).$$

Put together, we get

$$d_k(F_m, C) \leq \delta_m$$

for $\delta_m > d_H(F_m \cap B_{k+\frac{1}{2}}, C_{k+\frac{1}{2}})$, i.e.,

$$d_k(F_m, C) \leq d_H(F_m \cap B_{k+\frac{1}{2}}, C_{k+\frac{1}{2}})$$

and the latter tends to 0 as $m \to \infty$. Thus we have proved that every net in $\mathcal{F}(\mathbb{R}^d)$ has a converging subsequence. □

Note that, interestingly, no additional properties are needed for compactness. Of course, this result immediately gives compactness of certain subsets of the set of all Delone sets, e.g. compactness of the union over R of the (r, R)-sets for any fixed value of r.

One might think that the topology of local Hausdorff leads to a compact space as well, since in this topology, $\mathcal{F}(\mathbb{R}^d)$ is considered as a subset of the product of the $\mathcal{K}_R$, which is compact by Tychonovs theorem. However, it is not closed, as seen for $F_n := \{1 + \frac{1}{n}\}$; of course, $R = 1$ is the crucial value. This latter sequence also shows directly that the topology of local Hausdorff convergence is not compact and not too natural either ;-)

We will also use the natural topology to define a topology on tiling spaces in a quite general setting.

Let us note in passing that the metric ρ proposed in [35] as well as the metric in [20] do not satisfy the triangle inequality and that these metrics are restricted to Delone sets. This is again due to the phenomenon alluded to above, namely that restricting sets does not make the Hausdorff distance smaller. See, however, [22], in which a metric on the set of Delone sets is constructed. A discussion in a more general framework can be found in [30], where the author constructs a topology on the set of closed discrete subsets of a locally compact σ-compact space. In the case of $\mathbb{R}^d$ this topology coincides with the restriction of the above given natural topology.

To settle the issue of metrizability of the natural topology we next mention an alternative approach. The method we are going to outline now has most definitely been pointed out to us by someone else. Unfortunately, we were not able to find out by whom.

We use the stereographic projection to identify points $x \in \mathbb{R}^d \cup \{\infty\}$ in the one-point-compactification of $\mathbb{R}^d$ with the corresponding points $\tilde{x} \in \mathbb{S}^d$. Clearly, the latter denotes the d-dimensional unit sphere $\mathbb{S}^d = \{\xi \in \mathbb{R}^{d+1} : \|\xi\| = 1\}$. Now $\mathbb{S}^d$ carries the euclidean metric ρ. Since the unit sphere is compact and complete, we can associate a complete metric ρ_H on $\mathcal{K}(\mathbb{S}^d)$ by what we said above.

For $F \in \mathcal{F}(\mathbb{R}^d)$ write $\tilde{F}$ for the corresponding subset of $\mathbb{S}^d$ and define

$$\rho(F,G) := \rho_H(\widetilde{F \cup \{\infty\}}, \widetilde{G \cup \{\infty\}}) \text{ for } F, G \in \mathcal{F}(\mathbb{R}^d).$$

Although this constitutes a slight abuse of notation it makes sense since $\widetilde{F \cup \{\infty\}}, \widetilde{G \cup \{\infty\}}$ are compact in $\mathbb{S}^d$ provided F, G are closed in $\mathbb{R}^d$.

We have the following result:

Proposition 1.3. *The metric ρ above induces the natural topology on $\mathcal{F}(\mathbb{R}^d)$.*

Proof. An explicit calculation of the stereographic projection shows that

$$\rho(\widetilde{x}, \widetilde{\infty}) \le \frac{42}{\|x\|} \tag{1}$$

for $x \in \mathbb{R}^d$ as well as

$$\rho(\widetilde{x}, \widetilde{y}) \le 2\|x - y\|. \tag{2}$$

We want to show that the identity $id : (\mathcal{F}(\mathbb{R}^d), \tau_{nat}) \to (\mathcal{F}(\mathbb{R}^d), \rho)$ is continuous. Since $(\mathcal{F}(\mathbb{R}^d), \tau_{nat})$ is a compact space this implies that $id : (\mathcal{F}(\mathbb{R}^d), \tau_{nat}) \to (\mathcal{F}(\mathbb{R}^d), \rho)$ is in fact a homeomorphism. To prove the desired continuity, fix $F \in \mathcal{F}(\mathbb{R}^d)$ and $\varepsilon > 0$. We have to find a basic neighborhood $U_{\delta,k}$ that is contained in the ε- ball around F (with respect to ρ, of course). To this end choose $k \in \mathbb{N}$ such that $42/k < \varepsilon$ and $\delta = \varepsilon/2$.

This is a good choice; in fact, let $G \in U_{\delta,k}$. Combining $F \cap B_k \subset U_\delta(G)$ with (2) we get that

$$\widetilde{F \cap B_k} \subset U^\rho_\varepsilon(\widetilde{G}), \tag{3}$$

where the superscript ρ indicates the underlying metric space. By (1) and the choice of k we know that

$$F \cap (\widetilde{B^c_k \cup \{\infty\}}) \subset U^\rho_\varepsilon(\widetilde{G \cup \{\infty\}}). \tag{4}$$

Relations (3) and (4) together yield that

$$\widetilde{F \cup \{\infty\}} \subset U^\rho_\varepsilon(\widetilde{G \cup \{\infty\}}).$$

The corresponding relation with F and G interchanged follows in the same way so that

$$\rho_H(\widetilde{F \cup \{\infty\}}, (\widetilde{G \cup \{\infty\}}) \le \varepsilon.$$

This proves the asserted continuity. $\square$

Next, we define Delone dynamical systems, following [21] and single out some important properties:

Definition 1.4. (a) Let Ω be a set of Delone sets. The pair (Ω, T) is called a *Delone dynamical system* (DDS) if Ω is invariant under the shift T and closed in the natural topology.
(b) A DDS (Ω, T) is said to be of *finite type* if $\cup_{\omega \in \Omega} P^s_B(\omega)$ is finite for every $s > 0$.
(c) Let $0 < r, R < \infty$ be given. A DDS (Ω, T) is said to be an (r, R)-*system* if every $\omega \in \Omega$ is an (r, R)-set.

(d) The set $\mathcal{P}(\Omega)$ of *pattern classes associated to a DDS* Ω is defined by $\mathcal{P}(\Omega) = \cup_{\omega\in\Omega}\mathcal{P}(\omega)$.

Remark 1.5. (a) Whenever (Ω, T) is a Delone dynamical system, there exists an $R > 0$ with $B_R(x) \cap \omega \neq \emptyset$ for every $\omega \in \Omega$ and every $x \in \mathbb{R}^d$. This follows easily as Ω is closed and invariant under the action of T.
(b) Every DDSF is an (r, R)-system for suitable $0 < r, R < \infty$.
(c) Let ω be an (r, R)-set and let Ω_ω be the closure of $\{T_t\omega : t \in \mathbb{R}^d\}$ in $\mathcal{F}(\mathbb{R}^d)$ with respect to the natural topology. Then (Ω_ω, T) is an (r, R)-system.

For a DDSF, there is a simple way to describe convergence in the natural topology. This is shown in the following lemma. We omit the straightforward proof.

Lemma 1.6. *If (Ω, T) is a DDSF then a sequence (ω_n) converges to ω in the natural topology if and only if there exists a sequence (t_n) converging to 0 such that for every $L > 0$ there is an $n_0 \in \mathbb{N}$ with $(\omega_n + t_n) \cap B_L = \omega \cap B_L$ for $n \geq n_0$.*

We will need standard notions from the theory of dynamical systems. Namely, a DDS is called *minimal* if every orbit is dense. It is called *uniquely ergodic* if there exists exactly one T-invariant probability measure. It is called *aperiodic* if T_t does not have a fixed point for any $t \neq 0$.

For $s > 0$ and $Q \in \mathbb{R}^d$, we denote by $\partial_s Q$ the set of points in $\mathbb{R}^d$ whose distance to the boundary of Q is less than s. A sequence (Q_n) of bounded subsets of $\mathbb{R}^d$ is called a van Hove sequence if $|Q_n|^{-1}|\partial_s Q_n| \longrightarrow 0, n \longrightarrow 0$ for every $s > 0$.

Theorem 1.7. *Let (Ω, T) be a DDSF. Then (Ω, T) is uniquely ergodic if and only if, for every pattern class P the frequency $\lim_{n\to\infty} |Q_n|^{-1}\sharp_P(\omega \wedge Q_n)$ exists uniformly in $\omega \in \Omega$ for every van Hove sequence Q_n.*

Remark 1.8. After a first version of the present paper was on the web, B. Solomyak kindly informed us of the work [22] by Lee, Moody and Solomyak. As [22], Theorem 2.7 the reader can find a result analogous to the preceding Theorem.

Proof. For the "if" part we can refer to [34], Theorem 3.3. There, it is additionally assumed that the tiling dynamical system has the local isomorphism property. The latter is only used, however, to guarantee that frequencies are strictly positive. We will describe how to pass from Delone dynamical systems to tilings in detail later.

For the "only if" part we construct a continuous function on Ω that essentially counts occurences of a fixed pattern. Here are the details:

Fix $0 < r < R$ such that Ω is an (r, R)-system. We fix a pattern $P = (\Lambda_P, Q_P)$ with $0 \in \Lambda_P$ and diameter R_P. Moreover, we choose an auxiliary function $g \in C_c(\mathbb{R}^d)$, $g \geq 0$ with support contained in $B_{\frac{r}{42}}$ and normalized to $\int g(x)dx = 1$. With the help of g we define

$$f_P : \Omega \to \mathbb{R}, f_P(\omega) = g(t) \text{ iff } Q_P \wedge (\omega + t) = P$$

(where such a t is uniquely determined in case $g(t) \neq 0$) and 0 if no translate of P appears in ω. We consider all the points of ω at which a copy of P is centered, namely

$$S_\omega := \{s_n \in \mathbb{R}^d : Q_P \wedge (\omega + s_n) = P\}$$

and remark that the distinct points of S_ω have distance at least r since Ω is an (r,R)-system. Using the set S_ω, we get

$$f_P(\omega+t)=\sum_n g(s_n-t)$$

and, therefore,

$$\begin{aligned}\int_Q f_P(\omega+t)dt &= \sum_n \int_Q g(s_n-t)dt\\ &= \sum_{B_{\frac{r}{42}(s_n)}\cap Q\neq\emptyset} \int_Q g(s_n-t)dt\end{aligned}$$

Thus, if we let

$$Q^- := \{y\in Q : y+B_{R_P+r}\subset Q\}, Q^+ := U_r(Q),$$

we get that

$$\sharp_P Q\wedge\omega = \#\{n : s_n+Q_P\subset Q\}$$

along with the inequalities

$$\begin{aligned}\int_{Q^-} f_P(\omega+t)dt &\leq \#\{n : B_{\frac{r}{42}}(s_n)\cap Q^-\neq\emptyset\}\\ &\leq \sharp_P Q\wedge\omega = \#\{n : s_n+Q_P\subset Q\}\\ &\leq \#\{n : B_{\frac{r}{42}}(s_n)\subset Q^+\}\\ &\leq \int_{Q^+} f_P(\omega+t)dt.\end{aligned}$$

Now, take a van Hove sequence (Q_n). Then, since (Ω,T) is uniquely ergodic,

$$\lim_{n\to\infty}\frac{1}{|Q_n|}\int_{Q_n^-} f_P(\omega+t)dt = \lim_{n\to\infty}\frac{1}{|Q_n|}\int_{Q_n} f_P(\omega+t)dt = \int_\Omega f_P(\omega)d\mu(\omega),$$

where μ is the unique normalized invariant measure. Similarly,

$$\lim_{n\to\infty}\frac{1}{|Q_n|}\int_{Q_n^+} f_P(\omega+t)dt = \int_\Omega f_P(\omega)d\mu(\omega).$$

Using the inequality above, we now find that

$$\frac{1}{|Q_n|}\int_{Q_n^-} f_P(\omega+t)dt \leq \frac{1}{|Q_n|}\sharp_P Q_n \leq \frac{1}{|Q_n|}\int_{Q_n^+} f_P(\omega+t)dt$$

and consequently that

$$\lim_{n\to\infty}\frac{1}{|Q_n|}\sharp_P Q_n = \int_\Omega f_P(\omega)d\mu(\omega)$$

uniformly in ω. $\square$

In order to take advantage of the analysis from the theory of tilings, let us now relate Delone dynamical systems and tiling dynamical systems. To do so, we start with a slight generalization of what can, e.g., be found in [34]. For the readers convenience we will repeat all the necessary definitions. We follow mainly [34] (see [31] as well) although we rephrase some notions slightly.

A *tile* is just a set T that agrees with the closure of its interior. A *tiling* S is a countable family $S = (T_n)$ of tiles with disjoint interiors covering the whole space $\mathbb{R}^d$. It is sometimes very useful to supply tiles with an additional mark or decoration. To do so, consider a finite set $\mathbb{A}$ called the alphabet or the set of *decorations*. An $\mathbb{A}$-*decorated tiling* S consists of a countable family $S = ((T_n, a_n))_{n\in\mathbb{N}}$ such that (T_n) is a tiling and the decorations of two tiles agree only if the tiles are translates of each other. Following [34] we also call a_n the *type* of the tile T_n.

In order to describe a convenient topology for tilings let us first ignore decorations. Then, the natural topology provides a suitable topology. In fact, let us call

$$\Sigma(S) := \cup_n \partial T_n$$

the shape of the tiling $S = (T_n)$ and Σ the shape map acting from the set $\mathcal{T}$ of all tilings to $\mathcal{F}(\mathbb{R}^d)$. We call the initial topology with respect to the shape map the *natural topology* on $\mathcal{T}$. On the space of $\mathbb{A}$-decorated tilings $\mathcal{T}_{\mathbb{A}}$ we define the shape map as

$$\Sigma_{\mathbb{A}} : \mathcal{T}_{\mathbb{A}} \to \mathcal{F}(\mathbb{R}^d)^{\mathbb{A}}, ((T_n, a_n)) \mapsto (\cup\{\partial T_n : a_n = a\})_{a\in\mathbb{A}}.$$

Note that in the case of nonconnected tiles the shape map might loose some information. This won't bother us in what follows, since we are dealing with convex polygonal tiles.

Definition 1.9. (a) Let $X \subset \mathcal{T}_{\mathbb{A}}$ be a set of $\mathbb{A}$-decorated tilings. The pair (X, T) is called an $\mathbb{A}$-*tiling dynamical system*, $\mathbb{A}$-TDS, if X is invariant under the shift $T = (T_t)_{t\in\mathbb{R}^d}$ and closed in the natural topology.

(b) An $\mathbb{A}$-TDS (X, T) is said to be of *finite type*, if there is a finite set of tiles P such that every tile from one of the tilings in X is the translate of one of the tiles in P.

Next we describe the Voronoi construction. It enables us to pass from Delone dynamical to decorated tiling dynamical systems. See the discussion in [19], Section 2 as well. There, the reader will also find an account of the *Delone tesselation*, a possibility to pass from Delone sets to tilings that goes back to Delone (Delaunay), [10].

Let $\omega \subset \mathbb{R}^d$ be a Delone set; for $x \in \omega$ define

$$\begin{aligned} T(x,\omega) &:= \{y \in \mathbb{R}^n : d(x,y) \le d(y,\omega)\} \\ &= \{y \in \mathbb{R}^n : \|x-y\| \le \|z-y\| \quad (z \in \omega)\}. \end{aligned}$$

Clearly, $T(x,\omega)$ is the convex hull of finitely many points, a polygon, called the Voronoi cell of ω around x. If ω is an (r, R)-set, it follows that

$$B_{\frac{r}{2}}(x) \subset T(x,\omega) \subset B_{2R}(x).$$

Moreover,

$$S_\omega := (T(x,\omega))_{x\in\omega}$$

defines a tiling of $\mathbb{R}^d$. In this way, we get a TDS $X_\Omega = \{S_\omega : \omega \in \Omega\}$. It is clear that

$$V : \Omega \to X_\Omega, \omega \mapsto S_\omega$$

is continuous and respects translations.

Unfortunately, V doesn't need to be injective. In fact, you will easily find different periodic sets (e.g. $\mathbb{Z}$ and $(\frac{1}{4} + 2\mathbb{Z}) \cup (\frac{3}{4} + 2\mathbb{Z})$) that lead to the same Voronoi tiling.

However, decorations can help to recover the original Delone set in the DDSF case. In fact, starting from a DDSF Ω define

$$\mathbb{A} := \{(\omega \cap B_{2R}(x) - x, T(x,\omega) - x) =: a(x,\omega) : \omega \in \Omega, x \in \omega\},$$

which is a finite set if Ω is of finite type. Define

$$V : \Omega \to X_\Omega^{\mathbb{A}}, V_\omega := ((T(x,\omega), a(x,\omega))_{x\in\omega}.$$

It is not hard to see that we can reconstruct Ω from $X_\Omega^{\mathbb{A}}$ and that V is an isomorphism of dynamical systems.

Therefore, we can use analogs for Delone dynamical systems of the results from [34] on ergodic properties of tiling dynamical systems.

For a quite different approach to the topology on the set of Delone sets we refer to [6], where Delone sets are identified with the sum of delta measures sitting at the points of the Delone set. Then one has the w*-topology on the set of measures at ones disposal, providing good compactness properties. The approach presented here has the advantage that a topology is induced on the set of closed sets. That can be used to define a topology on (decorated) tilings via the shape map.

2. Groupoids and non commutative integration theory

In this section we introduce groupoids and basic notions from Connes non-commutative integration theory.

We will be concerned with several locally compact topological spaces. Given such a space Z, we denote the set of continuous functions on Z with compact support by $C_c(Z)$. The support of a function in $C_c(Z)$ is denoted by supp(f). The topology gives rise to the Borel-σ-algebra. The measurable nonnegative functions with respect to this σ-algebra will be denoted by $\mathcal{F}^+(Z)$. The measures on Z will be denoted by $\mathcal{M}(Z)$.

A set $\mathcal{G}$ together with a partially defined associative multiplication $\cdot : \mathcal{G}^2 \subset \mathcal{G}\times\mathcal{G} \longrightarrow \mathcal{G}$, and an inversion $-1 : \mathcal{G} \longrightarrow \mathcal{G}$ is called a groupoid if the following holds:

- $(g^{-1})^{-1} = g$ for all $g \in \mathcal{G}$,
- If $g_1 \cdot g_2$ and $g_2 \cdot g_3$ exist, then $g_1 \cdot g_2 \cdot g_3$ exists as well,
- $g^{-1} \cdot g$ exists always and $g^{-1} \cdot g \cdot h = h$, whenever $g \cdot h$ exists,
- $h \cdot h^{-1}$ exists always and $g \cdot h \cdot h^{-1} = g$, whenever $g \cdot h$ exists.

A groupoid is called topological groupoid if it carries a topology making inversion and multiplication continuous. Here, of course, $\mathcal{G} \times \mathcal{G}$ carries the product topology and $\mathcal{G}^2 \subset \mathcal{G}\times\mathcal{G}$ is equipped with the induced topology.

A given groupoid $\mathcal{G}$ gives rise to some standard objects: The subset $\mathcal{G}^0 = \{g \cdot g^{-1} : g \in \mathcal{G}\}$ is called the set of *units*. For $g \in \mathcal{G}$ we define its *range* $r(g)$ by $r(g) = g\cdot g^{-1}$ and its *source* by $s(g) = g^{-1}\cdot g$. Moreover, we set $\mathcal{G}^\omega = r^{-1}(\{\omega\})$ for any unit $\omega \in \mathcal{G}^0$. One easily checks that $g \cdot h$ exists if and only if $r(h) = s(g)$.

By a standard construction we can assign a groupoid $\mathcal{G}(\Omega,T)$ to a Delone dynamical system. As a set $\mathcal{G}(\Omega, T)$ is just $\Omega \times \mathbb{R}^n$. The multiplication is given by $(\omega, x)(\omega - x, y) = (\omega, x + y)$ and the inversion is given by $(\omega, x)^{-1} = (\omega - x, -x)$. The groupoid operations can be visualized by considering an element (ω, x) as an arrow $\omega - x \xrightarrow{x}$

ω. Multiplication then corresponds to concatenation of arrows; inversion corresponds to reversing arrows.

Apparently this groupoid $\mathcal{G}(\Omega, T)$ is a toplogical groupoid when Ω is equipped with the topology of the previous section and $\mathbb{R}^n$ carries the usual topology.

The groupoid $\mathcal{G}(\Omega, T)$ acts naturally on a certain topological space $\mathcal{X}$. This space and the action of $\mathcal{G}$ on it are of crucial importance in the sequel. The space $\mathcal{X}$ is given by

$$\mathcal{X} = \{(\omega, x) \in \mathcal{G} : x \in \omega\} \subset \mathcal{G}(\Omega, T).$$

In particular, it inherits a topology form $\mathcal{G}(\Omega, T)$. Two features of the topology are given in the following proposition.

Proposition 2.1. (a) *$\mathcal{X} \subset \mathcal{G}$ is closed.*

(b) *Let (Ω, T) be an (r, R)-system and $\omega \in \Omega$ and $x \in \omega$ be arbitrary. Then there exist a neighbourhood U of $\omega \in \Omega$ and a continuous function $h : U \longrightarrow B_{\frac{r}{2}}(x)$ with $\omega' \cap B_{\frac{r}{2}}(x) = \{h(\omega')\}$ for every $\omega' \in U$.*

Proof. (a) Let $((\omega_\iota, x_\iota)_{\iota \in I})$ be a net in $\mathcal{X}$ converging to $(\omega, x) \in \mathcal{G}$. Thus, $\omega_\iota \longrightarrow \omega$ and $x_\iota \longrightarrow x$ and it remains to show $x \in \omega$. Assume the contrary, i.e. $x \notin \omega$. As ω is closed, there exists a $\delta > 0$ with $B_\delta(x) \cap \omega = \emptyset$. Thus, $\omega_\iota \longrightarrow \omega$ implies $\omega_\iota \cap B_{\frac{\delta}{2}}(x) = \emptyset$ for all i large. But this is a contradiction to $x_\iota \longrightarrow x$.

(b) By the definition of the topology, there exists a neighbourhood $\widetilde{U}$ of ω with $\omega' \cap B_{\frac{r}{2}}(x) \neq \emptyset$ for every $\omega' \in \widetilde{U}$. As (Ω, T) is an (r, R)-system, $\omega' \cap B_{\frac{r}{2}}(x)$ consists of only one element. Denoting this element by $h(\omega')$, we get a function $h : \widetilde{U} \longrightarrow B_{\frac{r}{2}}(x)$. Continuity of h is now a direct consequence of the definition of the topology on Ω. $\square$

Remark 2.2. Note that the proof of part (a) does not use that Ω is a set of Delone sets. Thus, the corresponding statement remains valid whenever Ω is a subset of $\mathcal{F}(\mathbb{R}^d)$ which is closed in the natural topology.

Corollary 2.3. $C_c(\mathcal{X}) = \{f|_{\mathcal{X}} : f \in C_c(\mathcal{G})\}$.

Proof. As $\mathcal{X}$ is closed in $\mathcal{G}$ be the foregoing proposition, this follows by standard arguments involving Uryson's Lemma. $\square$

A key feature of $\mathcal{X}$ is its bundle structure. More precisely, we have a continuous map $p : \mathcal{X} \longrightarrow \Omega$, $p((\omega, x)) = \omega$ making $\mathcal{X}$ into a bundle over Ω with fibres $\mathcal{X}^\omega = p^{-1}(\omega) = \{(\omega, p) : p \in \omega)\} \simeq \omega \subset \mathbb{R}^n$.

Now, we can discuss the action of $\mathcal{G}$ on $\mathcal{X}$. Every $g = (\omega, x)$ gives rise to a map $J(g) : \mathcal{X}^{s(g)} \longrightarrow \mathcal{X}^{r(g)}$, $J(g)(\omega - x, p) = (\omega, p + x)$. A simple calculation shows that $J(g_1 g_2) = J(g_1)J(g_2)$ and $J(g^{-1}) = J(g)^{-1}$, whenever $s(g_1) = r(g_2)$. Thus, $\mathcal{X}$ is an $\mathcal{G}$-space (see [24]). These $\mathcal{G}$-spaces are important objects in Connes non-commutative integration theory. They give rise to random variables. More precisely we have the following definition.

Definition 2.4. Let (Ω, T) be an (r, R)-system.

(a) A choice of measures $\beta : \Omega \to \mathcal{M}(\mathcal{X})$ is called a positive random variable with values in $\mathcal{X}$ if the map $\omega \mapsto \beta^\omega(f)$ is measurable for every $f \in \mathcal{F}^+(\mathcal{X})$, β^ω is supported on $\mathcal{X}^\omega$, i.e., $\beta^\omega(\mathcal{X} - \mathcal{X}^\omega) = 0$, $\omega \in \Omega$, and β satisfies the following invariance condition

$$\int_{\mathcal{X}^{s(g)}} f(J(g)p) d\beta^{s(g)}(p) = \int_{\mathcal{X}^{r(g)}} f(q) d\beta^{r(g)}(q)$$

for all $g \in \mathcal{G}$ and $f \in \mathcal{F}^+(\mathcal{X}^{r(g)})$.

(b) A map $\Omega \times C_c(\mathcal{X}) \longrightarrow \mathbb{C}$ is called a complex random variable if there exist an $n \in \mathbb{N}$, positive random variables β_i, $i = 1, \ldots, n$ and $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$ with $\beta^\omega(f) = \sum_{i=1}^k \beta_i^\omega(f)$.

We are now heading towards introducing and studying a special random variable. This variable is quite important. It will give rise to the ℓ^2-spaces on which the Hamiltonians act. Later we will see that these Hamiltonians also induce random variables. We will need some information on the continuous functions on $\mathcal{X}$.

Proposition 2.5. *Let Ω be an (r, R)-system.*

(a) *Let $g \in C_c(\mathcal{X})$ be given. Then, the function $\alpha(g) : \Omega \longrightarrow \mathbb{R}$, $\alpha^\omega(g) = \sum_{p \in \omega} g(\omega, p)$ belongs to $C_c(\mathcal{X})$.*

(b) *For $g \in \mathcal{F}^+(\mathcal{X})$ the function $\alpha(g) : \Omega \longrightarrow \mathbb{R}$, $\alpha^\omega(g) = \sum_{p \in \omega} g(\omega, p)$ is measurable.*

Proof. (a) We fix $\omega_0 \in \Omega$ arbitrary and show continuity of $\alpha(g)$ at ω_0. As g has compact support, there exists $s > 0$, with $g(\omega, p) = 0$ whenever $p \notin B_s$. Apparently, $\omega_0 \cap B_{s+r}$ is finite. Thus, there exists $k \in \mathbb{N}$ and pairwise different $x_1, \ldots, x_k \in \mathbb{R}^d$ with $\{x_1, \ldots, x_k\} = \omega_0 \cap B_{s+r}$. By Proposition 2.1, there exist $\widetilde{U} \subset \Omega$ with $\omega_0 \in \widetilde{U}$ and $h_i : \widetilde{U} \longrightarrow B_{\frac{r}{2}}(x_i)$ continuous with $\{h_i(\omega)\} = B_{\frac{r}{2}}(x_i) \cap \omega$ for every $\omega \in \widetilde{U}$, $i = 1, \ldots, k$. As (Ω, T) is an (r, R)-system, the $B_{\frac{r}{2}}(x_i)$ are pairwise disjoint. By definition of the topology on Ω, we can find $U \subset \widetilde{U}$ such that

$$\omega \cap B_s \subset \{h_i(\omega) : i = 1, \ldots, k\} \text{ for all } \omega \in U .$$

(Note that the ball appearing on the left hand side of this inclusion has radius strictly less than $r + s$.) Thus, for $\omega \in U$, the following holds

$$\alpha(g)(\omega) = \sum_{p \in \omega \cap B(0,s)} g(\omega, p) = \sum_{i=1}^k g(\omega, h_i(\omega))$$

and the desired continuity follows. It is not hard to see that $\alpha(g)$ has compact support.

(b) This follows from (a) by standard monotone class arguments. □

As apparently $\alpha^\omega(h) = \alpha^{\omega+x}(h(\cdot - x))$, we immediately have the following corollary from part (b) of the Proposition.

Corollary 2.6. *The map $\alpha : \Omega \longrightarrow \mathcal{M}(\mathcal{X})$, $\alpha^\omega(f) = \sum_{p \in \omega} f(p)$ is a random variable with values in $\mathcal{X}$.*

Now, let μ be a measure on Ω. By (b) of Proposition 2.5, we see that $(\mu \circ \alpha)(g) = \int_\Omega \alpha^\omega(g)\, d\mu(\omega)$ exists for every $g \in C_c(\mathcal{X})$. The following is the key lemma on integration of random variables.

Lemma 2.7. *Let μ be T-invariant (i.e. $\mu(f) = \mu(f(\cdot - t))$ for every $t \in \mathbb{R}^n$).*

(a) *Let β be a nonnegative random variable. Then $\int_\Omega \beta^\omega(F(\omega, \cdot))\, d\mu(\omega)$ does not depend on $F \in \mathcal{F}^+(\mathcal{X})$ provided F satisfies $\int F((\omega + t, x + t)\, dt = 1$ for every $(\omega, x) \in \mathcal{X}$.*

(b) *Let β be an arbitrary random variable. $\int_\Omega \beta^\omega(F(\omega, \cdot))\, d\mu(\omega)$ does not depend on $F \in \mathcal{F}^+(\mathcal{X}) \cap C_c(\mathcal{X})$ provided F satisfies $\int F((\omega + t, x + t)\, dt = 1$ for every $(\omega, x) \in \mathcal{X}$.*

Proof. (a) This follows from [9] (see [24] for a discussion as well).

(b) This is an immediate consequence of (a). □

It is instructive to consider a special instance of the lemma. Namely, consider $f \in C_c(\mathbb{R}^n)$. Apparently f gives rise to a function $F_f \in C_c(\mathcal{X})$ given by $F_f((\omega, x)) = f(x)$. Now, let μ be an invariant measure on Ω and β a random variable. Then, the invariance properties of μ and β show that the functional $I : C_c(\mathbb{R}^n) \longrightarrow \mathbb{R}$, $I(f) = (\mu \circ \alpha)(F_f)$ is translation invariant and positive. By uniqueness of the Haarmeasure on $\mathbb{R}^n$, we infer that there exists a constant $\lambda(\beta)$ with $\mu \circ \beta(F_f) = \lambda(\beta) \int_{\mathbb{R}^n} f(t)\, dt$. This shows, in particular, that the integral $\int_\Omega \beta^\omega(F_f(\omega, \cdot))\, d\mu(\omega)$ does not depend on $f \geq 0$ provided f satisfies $\int_{\mathbb{R}^n} f(t)\, dt = 1$.

Let an invariant measure μ on Ω and the random variable α as above be given. We can then introduce the space $L^2(\mathcal{X}, \mu \circ \alpha)$. The bundle structure of $\mathcal{X}$ and of α suggest, that this space can be considered as a direct integral. This means we aim at giving sense to the equation

$$L^2(\mathcal{X}, \mu \circ \alpha) = \int_\Omega^\oplus \ell^2(\mathcal{X}^\omega, \alpha^\omega)\, d\mu(\omega). \tag{5}$$

As the fibres in this direct integral are not constant, we need to be careful about the notion of measurability. More precisely, we need to introduce a set $\mathcal{V}$ of functions f on Ω with $f(\omega) \in \ell^2(\mathcal{X}^\omega, \alpha^\omega)$, $\omega \in \Omega$, satsifying the following properties

- (V) $\mathcal{V}$ is a vectorspace under the the obvious operations (i.e. $(f + g)(\omega) = f(\omega) + g(\omega)$ and $(\lambda f)(\omega) = \lambda f(\omega)$).
- (M) $\omega \mapsto \langle f(\omega), g(\omega)\rangle_\omega$ is measurable for arbitrary $f, g \in \mathcal{V}$. Here, $\langle \cdot, \cdot \rangle_\omega$ is the inner product on $\ell^2(\mathcal{X}^\omega, \alpha^\omega)$.
- (S) If f is a function on Ω with $f(\omega) \in \ell^2(\mathcal{X}^\omega, \alpha^\omega)$, $\omega \in \Omega$ and $\omega \mapsto \langle f(\omega), g(\omega)\rangle_\omega$ measurable for every $g \in \mathcal{V}$, then f belongs to $\mathcal{V}$ as well.
- (D) There exists a countable set $D \subset \mathcal{V}$ such that the set $\{d(\omega) : d \in D\}$ is total in $\ell^2(\mathcal{X}^\omega, \alpha^\omega)$ for every $\omega \in \Omega$.

Such a set will be called a measurable structure on the family $(\ell^2(\mathcal{X}^\omega, \alpha^\omega))_{\omega \in \Omega}$. Note, that the condition (M) says that the functions in $\mathcal{V}$ have a certain measurability property. Condition (S) is a maximality assumption.

Given the special structure of $\mathcal{X}$, we can actually identify functions f on Ω with values in $\ell^2(\mathcal{X}^\omega, \alpha^\omega)$ with functions on $\mathcal{X}$. This will be done tacitely in the sequel. There are at least three good canditates for measurable structures. They are given as follows:

- The set $\mathcal{V}_1$ consists of all $f : \mathcal{X} \longrightarrow \mathbb{C}$ which are measurable and satisfy $f(\omega, \cdot) \in \ell^2(\mathcal{X}^\omega, \alpha^\omega)$ for every $\omega \in \Omega$
- The set $\mathcal{V}_2$ consists of all $f : \mathcal{X} \longrightarrow \mathbb{C}$ such that $f(\omega, \cdot) \in \ell^2(\mathcal{X}^\omega, \alpha^\omega)$, for all $\omega \in \Omega$, and $\omega \mapsto \langle f(\omega, \cdot), F(\omega, \cdot)\rangle_\omega$ is measurable for all $F \in C_c(\mathcal{X})$.
- Finally, the set $\mathcal{V}_3$ is given by all $f : \mathcal{X} \longrightarrow \mathbb{C}$ such that $f(\omega, \cdot) \in \ell^2(\mathcal{X}^\omega, \alpha^\omega)$, for all $\omega \in \Omega$, and $\omega \mapsto \langle f(\omega, \cdot), F(\omega, \cdot)\rangle_\omega$ is measurable for all $F \in C_c(\mathcal{G})$.

It is not too hard to check that these are all measurable structures. In fact, they are even equal. This is shown next.

Proposition 2.8. $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}_3$.

Proof. The equality of $\mathcal{V}_2$ and $\mathcal{V}_3$ is immediate from Corallary 2.3.

$\mathcal{V}_1 \subset \mathcal{V}_2$: Let $f \in \mathcal{V}_1$ be given. Without loss of generality we can assume that f is the characteristic function χ_M of a measurable set $M \subset \mathcal{X}$ with $M \cap \omega \subset B(0, s)$ for all $\omega \in \Omega$ and a certain $s > 0$ not depending on ω. As $\mathcal{X}$ is both locally compact and σ-compact, its Borel-σ-algebra is generated by compact sets. Thus, it suffices to consider χ_K with $K \subset \mathcal{X}$ compact. By standard arguments, it then suffices to consider $f \in C_c(\mathcal{X})$. For such f, measurability follows from Proposition 2.5.

$\mathcal{V}_2 \subset \mathcal{V}_1$: Let $f \in \mathcal{V}_2$ be given. We have to show that $f : \mathcal{X} \longrightarrow \mathbb{C}$ is measurable. By σ-compactness of $\mathcal{X}$, it suffices, to find, for every $(\omega_0, x_0) \in \mathcal{X}$, an open set $U \subset \mathcal{X}$ with $(\omega_0, x_0) \in U$ such that $f|_U$ is measurable. To provide such an U, we associate to (ω_0, x_0) an open set $U_1 \subset \Omega$ containing ω_0 as well as $h : U_1 \longrightarrow B_{\frac{r}{2}}(x_0)$ according to Proposition 2.1. By Uryson's lemma, we can find $U_2 \subset U_1$ open containing ω_0 and $g \in C_c(\Omega)$ with support contained in U_1 and $g \equiv 1$ on U_2. Moreover, let $s : \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous with $s(0) = 1$ and support contained in $B_{\frac{r}{4}}$. Then, $F : \mathcal{X} \longrightarrow \mathbb{R}$, with $F(\omega, x) = g(\omega)s(x - h(\omega))$ whenever $\omega \in U$ and $F(\omega, x) = 0$ otherwise, is continuous with compact support. It is immediate that

$$F(\omega, x) = \begin{cases} 1 & : \quad \omega \in U_2 \text{ and } x = h(\omega), \\ 0 & : \quad \omega \in U_2, \text{ and } x \neq h(\omega). \end{cases}$$

On $U \equiv (U_2 \times B_{\frac{r}{4}}(x_0)) \cap \mathcal{X}$, we then have

$$f(\omega, x) = f(\omega, h(\omega)) = \langle F(\omega, \cdot), f(\omega, \cdot)\rangle_\omega$$

and we infer measurability of $f|_U$ as $f \in \mathcal{V}_2$.

It remains to show that $\mathcal{V}_1 = \mathcal{V}_2$ is a measurable structure, i.e. satisfies the conditions $(V), (S), (M)$ and (D). Now, (V) is clear and (M) is a simple consequence of Corollary 2.6.

Moreover, obviously, $C_c(\mathcal{X})$ belongs to $\mathcal{V}_1$ and (S) follows as $\mathcal{V}_2 \subset \mathcal{V}_1$. To show (D), choose for each $q \in \mathbb{Q}^d$ a function $f_q \in C_c(\mathbb{R}^d)$ with support contained in $B_{\frac{r}{2}}(q)$ and $f_q \equiv 1$ on $B_{\frac{r}{4}}(q)$. Then, $\{f_q|_\mathcal{X} : q \in \mathbb{Q}^d\}$ has the desired properties. $\square$

Having discussed the appropriate notion of measurability, we can now give sense to equation (5). This is the content of the following lemma.

Lemma 2.9. *The map* $U : L^2(\mathcal{X}, \mu \circ \alpha) \longrightarrow \int_\Omega^\oplus \ell^2(\mathcal{X}^\omega, \alpha^\omega)\, d\mu(\omega)$ *with* $U(f)(\omega)(x) = f((\omega, x))$ *is unitary.*

Proof. By the foregoing proposition, $U(f)$ belongs indeed to $\mathcal{V}_1$. Direct calculations invoking Fubini's Theorem show that U is isometric. Thus, U indeed maps into $\int_\Omega^\oplus \ell^2(\mathcal{X}^\omega, \alpha^\omega)\, d\mu(\omega)$ and is injective. To show that U is surjective, is suffices to show that its image is dense. This can be done as follows: Let D be a dense set of bounded functions in $L^2(\Omega, \mu)$ and f_q, $q \in \mathbb{Q}^d$, as in the proof of the foregoing proposition. Then $\{h : \mathcal{X} \longrightarrow \mathbb{C} : h(\omega, x) = g(\omega) f_q(x)$ for suitable $g \in D$ and $q \in \mathbb{Q}^d\}$ has dense image under U. □

This lemma shows that $L^2(\mathcal{X}, \mu \circ \alpha)$ can be identified with $\int_\Omega^\oplus \ell^2(\mathcal{X}^\omega, \alpha^\omega)\, d\mu(\omega)$ in a canonical way.

Remark 2.10. In the above considerations, we have introduced $\mathcal{X}$ as a tautological bundle over Ω and then constructed an action of $\mathcal{G}$ on $\mathcal{X}$ as well as a family α of measures on $\mathcal{X}$. An alternative point of view is given as follows: A slight rearrangement of the arguments in the proof of Proposition 2.5 shows that

$$\alpha^\omega : C_c(\mathbb{R}^d) \longrightarrow \mathbb{C}, \;\; \alpha^\omega(f) = \sum_{p \in \omega} f(p)$$

is continuous in ω and satisfies an invariance condition. Thus, $\omega \mapsto \alpha^\omega$ is a transverse function on the groupoid $\mathcal{G}$ in the sense of Connes non-commutative-integration theory. The space $\mathcal{X}$ is then nothing but the "support" of α.

3. The von Neuman algebra of random operators

In this section we discuss the von Neuman algebra associated to a uniquely ergodic dynamical system. Details and proofs will be given in [26].

Let (Ω, T) be an (r, R)-system and let μ be an invariant measure on Ω. As there exists a canonical isomorphism between $L^2(\mathcal{X}, \mu \circ \alpha)$ and $\int_\Omega^\oplus \ell^2(\mathcal{X}^\omega, \alpha^\omega)\, d\mu(\omega)$, a special role is played by operators on $L^2(\mathcal{X}, \mu \circ \alpha)$ which respect this fibre structure. More precisely, we consider families $(A_\omega)_{\omega \in \Omega}$ of bounded operators $A_\omega : \ell^2(\omega, \alpha^\omega) \longrightarrow \ell^2(\omega, \alpha^\omega)$. Such a family is called *measurable* if $\omega \mapsto \langle f(\omega), (A_\omega g)(\omega)\rangle_\omega$ is measurable for every $f \in \mathcal{V}_1$. It is called *bounded* if the norms of the A_ω are uniformly bounded. It is called *covariant* if it satisfies the covariance condition

$$H_{\omega+t} = U_t H_\omega U_t^*, \; \omega \in \Omega, t \in \mathbb{R}^d, \tag{6}$$

where $U_t : \ell^2(\omega) \longrightarrow \ell^2(\omega + t)$ is the unitary operator induced by translation. Now, we can define

$$\mathcal{N}(\Omega, T, \mu) := \{A = (A_\omega)_{\omega \in \Omega} | A \text{ covariant, measurable and bounded}\}/\sim, \tag{7}$$

where $\sim$ means that we identify families which agree μ almost everywhere.

Remark 3.1. It is possible to define $\mathcal{N}(\Omega, T, \mu)$ by requiring seemingly weaker conditions. Namely, one can consider families (H_ω) which are essentially bounded and which satisfy the covariance condition almost everywhere. However, by standard procedures (see [9, 23]), it is possible to show that each of these families agrees almost everywhere with a family satisfying the stronger conditions discussed above.

As is clear from the definition, the elements of $\mathcal{N}(\Omega, T, \mu)$ are classes of families of operators. However, we will not distinguish too pedantically between classes and their representatives in the sequel.

Apparently, $\mathcal{N}(\Omega, T, \mu)$ is an involutive algebra under the obvious operations. There is an immediate representation $\pi : \mathcal{N}(\Omega, T, \mu) \longrightarrow B(L^2(\mathcal{X}, \mu \circ \alpha))$ given by $\pi(A) f((\omega, x)) = (A_\omega f_\omega)((\omega, x))$. Obviously, π is injective.

Lemma 3.2. *$\pi(\mathcal{N}(\Omega, T, \mu))$ is a von Neuman algebra.*

The elements of $\mathcal{N}(\Omega, T, \mu)$ and $\pi(\mathcal{N}(\Omega, T, \mu))$ are called random operators.

Lemma 3.3. *Let μ be ergodic and $(A_\omega) \in \mathcal{N}(\Omega, T, \mu)$ be selfadjoint. Then there exists $\Sigma, \Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp}, \Sigma_{ess} \subset \mathbb{R}$ and a subset $\widetilde{\Omega}$ of Ω of full measure such that $\Sigma = \sigma(A_\omega)$ and $\sigma_\bullet(A_\omega) = \Sigma_\bullet$ for $\bullet = ac, sc, pp, ess$ and $\sigma_{disc}(A_\omega) = \emptyset$ for every $\omega \in \widetilde{\Omega}$.*

Each random operator gives rise to a random variable.

Proposition 3.4. *Let $(A_\omega) \in \mathcal{N}(\Omega, T, \mu)$ be given. Then the map $\beta_A : \Omega \longrightarrow \mathcal{M}(\mathcal{X})$, $\beta_A^\omega(f) = \operatorname{tr}(A_\omega M_f(\omega))$ is a complex random variable.*

Now, choose a nonnegative $f \in C_c(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f(x)dx = 1$. Combining the previous proposition with Lemma 2.7, we infer that the map

$$\tau : \mathcal{N}(\Omega, T, \mu) \longrightarrow \mathbb{C}, \quad \tau(A) = \int_\Omega \operatorname{tr}(A_\omega M_f)\, d\mu(\omega)$$

does not depend on the choice of f. Important feature of τ are given in the following lemma.

Lemma 3.5. *The map $\tau : \mathcal{N}(\Omega, T, \mu) \longrightarrow \mathbb{C}$ is continuous, faithful, nonegative on $\mathcal{N}(\Omega, T, \mu)^+$ and satisfies $\tau(AB) = \tau(BA)$.*

Having defined τ, we can now associate a canonial measure ρ_A to every selfadjoint $A \in \mathcal{N}(\Omega, T, \mu)$.

Definition 3.6. For $A \in \mathcal{N}(\Omega, T, \mu)$ selfadjoint, and $B \subset \mathbb{R}$ Borel measurable, we set let $\rho_A(B) \equiv \tau(\chi_B(A))$, where χ_B is the characteristic function of B.

Lemma 3.7. *Let $A \in \mathcal{N}(\Omega, T, \mu)$ selfadjoint be given. Then ρ_A is a spectral measure for A. In particular, the support of ρ_A agrees with the almost sure spectrum Σ of A and the equality $\rho_A(F) = \tau(F(A))$ holds for every bounded measurable F on $\mathbb{R}$.*

Theorem 3.8. *Let (Ω, T) be a uniquely ergodic, aperiodic DDSF. Let μ be the unique invariant probability measure. Then $\mathcal{N}(\Omega, T, \mu)$ is a factor of type II_1, and*

$$\tau(1) = D := lim_{R \to \infty} \frac{\#(\omega \cap B_R(0))}{|B_R(0)|}$$

is the density *of ω.*

4. Tight binding operators

In order to describe the properties of disordered models quantum mechanically it is common to use a tight binding approach. E.g., a random model is often described by an operator on $\ell^2(\mathbb{Z}^d)$ consisting of the Laplacian that stands for nearest neighbor interactions plus a random potential perturbation. We search for an analogous description of quasicrystals, introducing the following notion that still leaves a lot of flexibility. In comparison with the random or almost random case it is again the fact that the space varies that makes the fundamental difference.

Related constructions have been introduced by Kellendonk [15, 16] and later been discussed by Kelledonk/Putnam [17] and Bellissard/Hermann/Zarrouati [6] (see [1, 29] as well). All these works are concerned with K-theory. The relevant C^*-algebras of tight binding operators are then discussed within the framework of discrete goupoids. These groupoids are transversals of $\mathcal{G}(\Omega, T)$ [17, 6] (see [4] for discussion of transversals and tight binding operators as well). Our discussion below does not use transversals and in fact not even groupoids. We rather directly introduce a C^*-algebra of tight binding operators. For further details and proofs we refer the reader to [25, 26].

Definition 4.1. Let Ω be a DDSF. A family $A = (A_\omega)$, $A_\omega \in \mathcal{B}(\ell^2(\omega))$ is said to be an *operator (family) of finite range* if there exists $s > 0$ such that

- $(A_\omega \delta_x | \delta_y) = 0$ if $x, y \in \omega$ and $|x - y| \geq s$.
- $(A_{\omega+t}\delta_{x+t}|\delta_{y+t}) = (A_{\tilde\omega}\delta_x|\delta_y)$ if $\omega \cap B_s(x+t) = \tilde\omega \cap B_s(x) + t$ and $x, y \in \tilde\omega$.

This merely says that the matrix elements $A_\omega(x, y) = (A_\omega\delta_x|\delta_y)$ of A_ω only depend on a sufficiently large patch around x and vanish if the distance between x and y is too large. Since there are only finitely many nonequivalent patches, an operator of finite range is bounded in the sense that

$$\|A\| = \sup_{\omega\in\Omega} \|A_\omega\| < \infty.$$

Moreover it is clear that every such A is covariant and consequently $A \in \mathcal{N}(\Omega, T, \mu)$ for every invariant measure μ. The completion of the space of all finite range operators with respect to the above norm is a C^*–algebra that we denote by $\mathcal{A}(\Omega, T)$. The representations $\pi_\omega : A \mapsto A_\omega$ can be uniquely extended to representations of $\mathcal{A}(\Omega, T)$ and are again denoted by $\pi_\omega : \mathcal{A}(\Omega, T) \to \mathcal{B}(\ell^2(\omega))$. We have the following result:

Theorem 4.2. *The following conditions on Ω are equivalent:*

(i) (Ω, T) *is minimal.*
(ii) *For any selfadjoint $A \in \mathcal{A}(\Omega, T)$ the spectrum $\sigma(A_\omega)$ is independent of $\omega \in \Omega$.*
(iii) π_ω *is faithful for every $\omega \in \Omega$.*

Next we relate the "abstract integrated density of states" ρ_A to the integrated density of states as considered in random or almost random models and defined by a volume limit over finite parts of the operator.

Note that for selfadjoint $A \in \mathcal{A}(\Omega, T)$ and bounded $Q \subset \mathbb{R}^d$ the restriction $A_\omega|_Q$ defined on $\ell^2(Q \cap \omega)$ has finite rank. Therefore, the spectral counting function

$$n(A_\omega, Q)(E) := \#\{ \text{ eigenvalues of } A_\omega|_Q \text{ below } E\}$$

is finite and $\frac{1}{|Q|}n(A_\omega, Q)$ is the distribution function of the measure $\rho(A_\omega, Q)$, defined by

$$\langle \rho(A_\omega, Q), \varphi \rangle := \frac{1}{|Q|}\mathrm{tr}(\varphi(A_\omega|_Q)) \text{ for } \varphi \in C_b(\mathbb{R}).$$

One of the fundamentals of random operator theory is the existence of the infinite volume limit

$$N(E) = \lim_{Q \nearrow \mathbb{R}^d} \frac{1}{|Q|} n(A_\omega, Q)(E)$$

for every $\omega \in \Omega$. This amounts to the convergence in distribution of the measures $\rho(A_\omega, Q)$ just defined. As a first result on weak convergence we get:

Theorem 4.3. *Let (Ω, T) be a uniquely ergodic DDSF and $A \in \mathcal{A}(\Omega, T)$ selfadjoint. Then, for any van Hoove sequence Q_n, $\rho(A_\omega, Q_n) \to \rho_A$ weakly as $n \to \infty$.*

Remark 4.4. This result is analogous to corresponding results for random or almost periodic operators as e.g. [33, 3, 4]. It generalizes results in Kellendonk's [15] on tilings associated to primitive substitutions. Its proof uses ideas of the cited works of Bellissard (see [15] as well) and of Hof [12].

For strictly ergodic, aperiodoc DDSF, we actually have a much stronger result. Namely, we can show pointwise and even uniform convergence of the corresponding distribution functions. Of course, uniform convergence follows from vage convergene if the limit is continuous. Thus, let us emphasize that in the context of DDSF continuity of the distribution function of ρ is wrong in general, see [18]. Still uniform convergence holds. Let us mention that this fits well within the general philosophy that everything behaves very uniformly within the reign of quasicrystals. All of this will be discussed in [26].

REFERENCES

[1] J. E. ANDERSON and I. F. PUTNAM, Topological invariants for substitution tilings and their associated C^*-algebras, *Ergodic Theory Dynam. Systems* **18** (1998), no. 3, 509–537

[2] J. AVRON and B. SIMON, Almost periodic Schrödinger Operators, II: The integrated density of states, *Duke Math. J.*, 50, 1982, pp. 369-391

[3] J. BELLISSARD and R. LIMA and D. TESTARD, Almost Periodic Schrödinger Operators. In: Mathematics + physics. Vol. 1, World Sci. Publishing, Singapore, 1995, pp. 1-64

[4] J. BELLISSARD, K–theory of C^*–algebras in solid state physics. In: Statistical mechanics and field theory: mathematical aspects (Groningen, 1985), Lecture Notes in Physics, Vol. 257, Berlin, 1986, pp. 99-156

[5] J. BELLISSARD, Gap labelling theorems for Schrödinger operators. In: From Number Theory to Physics, (M. Walsdschmidt, P. Moussa, J.M. Luck, C. Itzykson eds.) Springer, Berlin, (1992), pp. 539–630

[6] J. BELLISSARD, D. J. L. HERMANN, and M. ZARROUATI, Hulls of Aperiodic Solids and Gap Labelling Theorem, In: Directions in mathematical quasicrystals, CRM Monogr. Ser., 13, Amer. Math. Soc., Provicence, RI, 2000, pp. 207–258

[7] R. CARMONA and J. LACROIX, *Spectral theory of random Schrödinger operators,* Birkhäuser, Boston, 1990

[8] L. A. COBURN, R. D. MOYER and I. M. SINGER, C^*-algebras of almost periodic pseudo-differential operators, *Acta Math.*, 130, (1973), pp. 279–307

[9] A. CONNES, Sur la théorie non commutative de l'intégration. LNM, vol. 725, Springer, Berlin, 1979

[10] B. DELAUNAY [B.N. DELONE], Sur la sphére vide, *Izvestia Akad Nauk SSSR Otdel. Mat. Sov. Nauk.* 7, 1934, pp. 793-800

[11] C.P.M. GEERSE and A. HOF, Lattice gas models on self-similar aperiodic tilings, *Rev. Math. Phys.*, 3, 1991, pp. 163-221

[12] A. HOF, Some remarks on discrete aperiodic Schrödinger operators, *J. Statist. Phys.*, 72, (1993) pp. 1353–1374

[13] A. HOF, A remark on Schrödinger operators on aperiodic tilings, *J. Statist. Phys.*, 81, (1996) pp. 851–855

[14] C. JANOT, *Quasicrystals: A Primer*, Oxford University Press, Oxford, 1992

[15] J. KELLENDONK, Noncommutative geometry of tilings and gap labelling, *Rev. Math. Phys.*, 7, 1995, pp. 1133-1180

[16] J. KELLENDONK, The local structure of tilings and their integer grouip of coinvariants, *Comm. Math. Phys.*, 187, 1997, pp.115–157

[17] J. KELLENDONK and I. F. PUTNAM, Tilings; C^*-algebras, and K-theory. In: Directions in mathematical quasicrystals, CRM Monogr. Ser., 13, Amer. Math. Soc., Provicence, RI, 2000, pp. 177-206

[18] S. KLASSERT, D. LENZ and P. STOLLMANN, Discontinuities of the integrated density of states for random operators on Delone sets, `eprint: arXiv math-ph/0208027`, to appear in *Commun. Math. Phys.*

[19] J. C. LAGARIAS, Geometric Models for Quasicrystals I. Delone Sets of Finite Type, *Disc. and Comp. Geom.*, to appear

[20] J. C. LAGARIAS, Geometric Models for Quasicrystals II. Local Rules Under Isometries, *Disc. and Comp. Geom.*, to appear

[21] J. C. LAGARIAS and P.A.B. PLEASANTS, Repetitive Delone sets and Quasicrystals, *Ergodic Theory Dynam. Systems*, to appear

[22] J.-Y. LEE, R.V. MOODY and B. SOLOMYAK, Pure Point Dynamical and Diffraction Spectra, preprint 2001, available from
`http://www.math.washington.edu/ solomyak/personal.html`

[23] D. LENZ, Random Operators and Crossed Products, *Math. Phys. Anal. Geom.*, 2, 1999, pp. 197-220

[24] D. LENZ, N. PEYERIMHOF and I. VESELIC, Von Neumann algebras, Groupoids and the integrated density of states, `eprint arXiv math-ph/0203026`

[25] D. LENZ and P. STOLLMANN, Delone dynamical systems, groupoid von Neuman algebras and Hamiltonians for quasicrystals, *C. R. Acad. Sci. Paris, Ser. I*, **334**, 2002, 1131-1136

[26] D. LENZ and P. STOLLMANN, An ergodic theorem for Delone dynamical systems and existence of the density of states, in preparation

[27] D. LENZ and P. STOLLMANN, Algebras of operators associated to Delone dynamical systems, `eprint arXiv math-ph/0210031`, to appear in *Math. Phys. Anal. Geom.*

[28] L. PASTUR and A. FIGOTIN, *Spectra of random and almost periodic operators*, Springer-Verlag, Berlin, 1992

[29] I. F. PUTNAM, *The ordered K-theory of C^*-algebras associated with substitution tilings*, Comm. Math. Phys. **214** (2000), 593–605

[30] M. SCHLOTTMANN, *Generalized model sets and dynamical systems*. In: Directions in Mathematical Quasicrystals, eds M. Baake and R.V. Moody, CRM Monograph series, AMS, Providence RI (2000), 143-159

[31] M. SENECHAL, *Quasicrystals and Geometry*, Cambridge University Press, Cambridge, 1995

[32] D. SHECHTMAN, I. BLECH, D. GRATIAS and J.W. CAHN: Metallic phase with long-range orientational order and no translation symmetry, *Phys. Rev. Lett.*, 53, 1984, pp. 1951-1953

[33] M. SHUBIN, *The Spectral Theory and the Index of Elliptic Operators with Almost Periodic Coefficients*, Russian Math. Surveys, vol 34, 1979

[34] B. SOLOMYAK, Dynamics of self-similar tilings, *Ergodic Theory Dynam. Systems*, 17, 1997, pp. 695-738

[35] B. SOLOMYAK, Spectrum of a dynamical system arising from Delone sets. In: Quasicrystals and Discrete Geometry, ed. J. Patera, Fields Institute Monographs, vol. 10, AMS, Providence, RI 1998, pp. 265-275

[36] P. STOLLMANN, *Caught by disorder: bound states in random media*, Progress in Math. Phys., vol 20, Birkhäuser, Boston 2001

FAKULTÄT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT CHEMNITZ, D-09107 CHEMNITZ, GERMANY; E-MAIL: D.Lenz@mathematik.tu-chemnitz.de

FAKULTÄT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT CHEMNITZ, D-09107 CHEMNITZ, GERMANY; E-MAIL: P.Stollmann@mathematik.tu-chemnitz.de