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Anderson Localization for Random Schrödinger Operators with Long Range Interactions

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Abstract: We prove pure point spectrum with exponentially decaying eigenfunctions at all band edges for Schrödinger Operators with a periodic potential plus a random potential of the form $V_\omega(x) = \sum q_i(\omega) f(x-i)$, where f decays at infinity like $|x|^{-m}$ for $m > 4d$ resp. $m > 3d$ depending on the regularity of f. The random variables q_i are supposed to be independent and identically distributed. We assume that their distribution has a bounded density of compact support.

1. Introduction

At least since the groundbreaking work of Lifshitz in the early 60s it is widely accepted among physicists that random models of solid state physics should exhibit pure point spectrum near fluctuation boundaries. The latter are those parts of the spectrum which are determined by rather rare events. To present a more concrete picture consider a Schrödinger operator of the form $-\Delta + V_{per} + V_\omega$ describing a periodic solid with additional impurities given by the random perturbation. The spectrum will typically consist of a union of closed intervals, bands, whose edges correspond to the unlikely events that the random perturbation takes on its maximal respectively minimal value. Thus these band edges form the fluctuation boundaries.

Mathematical rigorous proofs of localization in multidimensional models have so far been restricted to Anderson type models with additional technical restrictions. They go back to the pioneering works for the discrete case ([12, 11, 5, 20]) with $V_{per} = 0$ which found considerable simplification in [6]. The first paper which treated the continuum case was [13] which was extended and simplified substantially in [3, 14, 17]. In the latter paper the case of a non-trivial periodic background potential was considered for the first time; however, as in the other articles mentioned so far, the results showed localization near the bottom of the spectrum, only.

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Localization near arbitrary band edges was investigated in a series of papers [8, 9, 1, 16, 21]: In [8] a discrete Schrödinger operator was considered, while [9] treats a divergence form model for acoustic waves and contains most of the technique necessary for the Schrödinger case (in [21] an extension of the results for anisotropic models is given). In [1] a fairly general situation is considered, including Schrödinger operators of the above type under some mild assumptions on the coefficients. In our recent paper [16] we also studied the latter case (allowing more general V_{per} and f) using among other ingredients a multi-scale analysis which is based on a variation of ideas from [3, 9].

To point out the progress achieved in the present note we recall that alloy type models have the form

$$V_\omega(x) = \sum q_i(\omega) f(x-i), \tag{1.1}$$

where we will assume that the q_i for $i \in \mathbb{Z}^d$ are independent identically distributed random variables which have a bounded density g. We suppose that supp(g) is an interval and denote by q_- and q_+ its infimum and supremum respectively. The function $f \geq 0$, $f \geq c > 0$ on an open set, is assumed to belong to $l^1(L^p)$ with $p = 2$ for $d \leq 3$, $p > \frac{d}{2}$ for $d \geq 4$.

Also let $H_0 = -\Delta + V_{per}$, where $V_{per} \in L^p_{loc}$ is a $\mathbb{Z}^d$-periodic potential with p as above. We will study the random Schrödinger operator $H_\omega = H_0 + V_\omega$.

Despite some discussion in [3, 14], localization had so far been settled for compactly supported f only, even in the case $V_{per} = 0$. Here we will prove localization for long range interaction f with

$$|f(x)| \leq C|x|^{-m} \quad \text{for } |x| \text{ large},$$

where m will be chosen suitably. We will follow the general strategy for proving localization which was used for discrete models in [6] and adapted in [9] to the continuous case. A few changes will be implemented into this strategy, which, as a by-product, lead to streamlined proofs of some of the results of [9] (see the comment at the end of Sect. 5). In addition, we will use some results from [16] respectively [3, 1]. Peter Hislop has announced results (joint work with Combes and Mourre, in preparation) on the problem of localization for long range f as well.

The difficulties one has to overcome are caused by the fact that $V_\omega(x)$ and $V_\omega(y)$ are statistically correlated even if the distance between x and y is large. For other results on localization for correlated potentials, see e.g. [7, 10].

A simple extension of the argument given in [16] shows that $\sigma(H_\omega) = \Sigma$ for almost every ω, where

$$\Sigma = \bigcup_{q \in [q_-, q_+]} \sigma(H_0 + q \cdot \sum_k f(\cdot - k)).$$

If $m > 4d$ we will prove pure point spectrum with exponentially decaying eigenfunctions near the bottom inf Σ of the spectrum. If we impose the conditions of [1] or [3] we may even have $m > 3d$ since their Wegner estimate is stronger. Note that [1, 3] in particular assume boundedness of f. Under slightly stronger assumptions on the density g we can prove localization near all band edges of the spectrum.

Let us formulate the main results of this paper.

Theorem 1.1. *If $m > 4d$ then for almost every ω the spectrum of H_ω is pure point in a neighborhood of* inf Σ *with exponentially decaying eigenfunctions.*

Theorem 1.2. *Assume that $m > 4d$ and that there exists $\tau > d/2$ such that g satisfies*

$$\int_{q_-}^{q_-+h} g(s)\, ds + \int_{q_+-h}^{q_+} g(s)\, ds \le h^{\tau} \quad \textit{for small } h. \tag{1.2}$$

Then for almost every ω the spectrum of H_ω is pure point in a neighborhood of $\partial\Sigma$ with exponentially decaying eigenfunctions.

In our proof the particular form of the Wegner bound turns out to be crucial. In the Wegner estimate we take from [14] and [16] the Wegner bound behaves quadratic in the volume term. The above results rely on this type of Wegner estimate. Barbaroux, Combes, Hislop and Mourre ([1, 3] and [4]) prove a Wegner estimate that is linear in the volume. In their case our proof below requires only $m > 3d$ instead of $m > 4d$.

One of us has found a simple proof of a Wegner estimate for Hölder continuous distribution of the coupling constant [22]. This implies that the above theorems remain valid under this more general assumption. For discrete models, a Wegner estimate was proven in [2] which allows for Hölder continuous measures and even certain measures with non-zero discrete parts.

We mention here without proof that the assumption on m can be weakened at the cost of a somewhat worse result on eigenfunction decay. In fact, by using the ideas described below in a somewhat different type of multiscale analysis (as used in [16] and based on methods from [20] and [3]) it can be shown that Theorems 1.1 and 1.2 hold for $m > 3d$ (respectively $m > 2d$ under the conditions of [1, 3]), but merely with eigenfunctions which decay more rapidly than any inverse polynomial. Of course, the optimal assumption, which we were not able to get, should be $m > d$.

The additional assumption (1.2) in Theorem 1.2 could be dropped if Lifshitz tail behavior of the integrated density of states would be known at internal band edges of H_ω (a property known at inf Σ and used in the proof of Theorem 1.1, compare in particular the proof of Proposition 4.2 in [16]). Klopp has recently shown in [18] that internal Lifshitz tails appear if and only if the integrated density of states at the corresponding band edge of the periodic operator H_0 is "non-degenerate" (see [18]).

2. Preliminaries and Strategy of Proof

In this paper we use the structure of the probability space Ω of the random variables which still has product structure although the random potential itself might be (and typically will be) deterministic in the sense of the technical definition of this term (see [15]). The strategy of our proof is to make probabilistic estimates for the box-hamiltonians $H_{\Lambda_l(x)}$ (see below) which are valid *uniformly* in q_i for those i with dist$(i, \Lambda_l(x)) \ge r_l$ for some "security distance" r_l.

To define this procedure more precisely note that our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given by $\Omega = \otimes_{i\in\mathbb{Z}^d} S$, $S \subset \mathbb{R}$, $\mathcal{F} = \otimes_{i\in\mathbb{Z}^d}\mathcal{B}(S)$, $\mathbb{P} = \otimes_{i\in\mathbb{Z}^d}\mathbb{P}_0$, where $\mathbb{P}_0$ is the distribution of q_0 and S its (compact) support. For any subset Λ of $\mathbb{R}^d$ we define the projection $\Pi_\Lambda : \Omega \longrightarrow \otimes_{i\in\Lambda\cap\mathbb{Z}^d} S$ by $(\Pi_\Lambda\omega)_i = \omega_i$ for $i \in \Lambda \cap \mathbb{Z}^d$.

Definition 2.1. *If $A \in \mathcal{F}$ is an event and $\Lambda \subset \mathbb{R}^d$ we define*

$$A^*_\Lambda = \{\omega \mid \exists\omega' \in A \quad \Pi_\Lambda\omega' = \Pi_\Lambda\omega\} = \Pi_\Lambda^{-1}(\Pi_\Lambda A). \tag{2.1}$$

Proposition 2.2. *If $\Lambda_1 \cap \Lambda_2 \cap \mathbb{Z}^d = \emptyset$, $\quad A, B \in \mathcal{F}$ then $A^*_{\Lambda_1}$ and $B^*_{\Lambda_2}$ are independent events.*

Proof. The event $A^*_{\Lambda_1}$ is a $(\Lambda_1 \cap \mathbb{Z}^d)$-cylinder set, while $B^*_{\Lambda_2}$ is a $(\Lambda_2 \cap \mathbb{Z}^d)$-cylinder. Thus they are independent if $\Lambda_1 \cap \Lambda_2 \cap \mathbb{Z}^d = \emptyset$. $\square$

In order to explain the key role of this elementary fact in extending known multiscale methods to the case of long range interactions we first introduce some notation and terminology.

We denote by $H_\Lambda = H_\Lambda(\omega)$ the operator $H_\omega = H_0 + V_\omega$ restricted to the cube Λ with periodic boundary conditions and by $R_\Lambda(z)$ its resolvent. By $\Lambda_l(x)$ we denote the cube of sidelength $l \in 2\mathbb{N} + 1$ around the point $x \in \mathbb{Z}^d$. We will drop the letter "x" whenever it is understood which center is meant or if x is the origin. For fixed x denote

$$\Lambda_l^{inn} := \Lambda_{l/3}, \quad \Lambda_l^{out} := \Lambda_l \setminus \Lambda_{l-2}.$$

Also let $\chi_l^{inn} := \chi_{\Lambda_l^{inn}}$ and $\chi_l^{out} := \chi_{\Lambda_l^{out}}$ be the corresponding indicator functions.

Definition 2.3. *A cube Λ_l is called (γ, E)-good for $\omega \in \Omega$, if*

$$\|\chi_l^{out} R_{\Lambda_l}(E) \chi_l^{inn}\| \le e^{-\gamma l},$$

where $E \in \mathbb{R} \setminus \sigma(H_{\Lambda_l}(\omega))$ is understood.

If the potential f is compactly supported then the events $\{\omega : \Lambda_l$ is not (γ, E)-good$\}$ and $\{\omega : \tilde{\Lambda}_l$ is not (γ, E)-good$\}$ are independent if the distance of the cubes Λ_l and $\tilde{\Lambda}_l$ is sufficiently large. This has been crucial in multiscale techniques as for example in [6] and [9], but does not hold in the long range case. Our basic idea to solve this problem is to do a multiscale analysis with "uniformly good" rather than "good" cubes:

Definition 2.4. *Let $G_l(\gamma, E) := \{\omega | \Lambda_l$ is (γ, E)-good$\}$. The cube Λ_l is called uniformly (γ, E)-good for a certain ω, if $\omega' \in G_l(\gamma, E)$ for all ω' with $\Pi_M \omega' = \Pi_M \omega$, where $M = \Lambda_{4l}$.*

Thus

$$\{\omega | \Lambda_l \text{ is not uniformly } (\gamma, E)\text{-good}\} = \{\omega | \Lambda_l \text{ is not } (\gamma, E)\text{-good}\}^*_{\Lambda_{4l}},$$

which implies independence of $\{\omega | \Lambda_l(x)$ is not uniformly (γ, E)-good$\}$ and $\{\omega | \Lambda_l(y)$ is not uniformly (γ, E)-good$\}$ if $d(x, y) > 4l$ by Proposition 2.2. Here $d(\cdot, \cdot)$ denotes the ∞-metric.

In order to make a multiscale analysis work with the stronger requirement of uniformly good cubes (see Sect. 4), we also need uniform versions of the two basic ingredients into the multiscale technique, the Wegner estimate and the initial scale estimate. These are provided in the next section, where only the Wegner estimate needs a little extra thought.

To understand how the concept of good cubes respectively uniformly good cubes will be used in the proof of localization in Sect. 5, we record the following *eigenfunction decay inequality* in part a) of the following Lemma. For this we recall that a function $\psi \in W^{1,2}_{loc}$ is called a *generalized eigenfunction* for $H = -\Delta + V$ ($V \in L^p_{loc}$, p as above) to $E \in \mathbb{R}$, if for every $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\langle \nabla\psi, \nabla\varphi \rangle + \langle V\psi, \varphi \rangle = E\langle \psi, \varphi \rangle.$$

We use the notation $\|V\|_{p,loc,unif} := \sup_x \left(\int_{\Lambda_1(x)} |V(y)|^p \, dy \right)^{1/p}$. Part b) is a *geometric resolvent inequality* used to compare R_Λ and $R_{\Lambda'}$ for different cubes in the multiscale analysis of Sect. 4.

Lemma 2.5. *Let $H = -\Delta + V$ be such that $\|V\|_{p,loc,unif} \leq M$. Then for every bounded set $U \subset \mathbb{R}$ there is a constant $C_{U,M}$ such that*

a) *every generalized eigenfunction ψ of H to $E \in U$ satisfies*

$$\|\chi_\Lambda^{inn}\psi\| \leq C_{U,M}\|\chi_\Lambda^{out}R_\Lambda(E)\chi_\Lambda^{inn}\|\,\|\chi_\Lambda^{out}\psi\|, \tag{2.2}$$

b) *if $\Lambda \subset \Lambda'$ are cubes with centers in $\mathbb{Z}^d$ and sidelengths in $2\mathbb{N}+1$, and if $A \subset \Lambda^{inn}$, $B \subset \Lambda' \setminus \Lambda$ and $E \in U$, then*

$$\|\chi_B R_{\Lambda'}(E)\chi_A\| \leq C_{U,M}\|\chi_B R_{\Lambda'}(E)\chi_\Lambda^{out}\|\,\|\chi_\Lambda^{out}R_\Lambda(E)\chi_A\|. \tag{2.3}$$

The proofs of these results involve the resolvent identity, the introduction of smooth cut-off functions and gradient estimates. Very similar results are provided in [9, Lemma 26 and Lemma 27]. An elementary discussion can also be found in [23], so we omit the proof.

3. Wegner Estimates and Initial Scale Estimate

Let us set $A = A(E,l,\varepsilon) = \{\omega\,|dist(E,\sigma(H_{\Lambda_l})) < \varepsilon\}$. The Wegner Lemma tells us that this event has small probability, more precisely

Proposition 3.1. *([14, 16]) Under the assumptions of Theorem 1.1 let $a \in \partial\Sigma$. Then there exists a neighborhood U of a and a constant $C > 0$ such that for $E \in U$,*

$$\mathbb{P}(A(E,l,\varepsilon)) \leq C\varepsilon\,|\Lambda_l|^2\,. \tag{3.1}$$

As usual $|M|$ denotes the Lebesgue-measure of the set M.

Combes-Hislop [3] and Barbaroux-Combes-Hislop [1] have a better Wegner estimate which gives $|\Lambda_l|$ instead of $|\Lambda_l|^2$ on the right-hand side of (3.1):

Proposition 3.2. *([3, 1]) Under the assumptions in [3] resp. [1] we have for E in a neighborhood U of $a \in \partial\Sigma$ that*

$$\mathbb{P}(A(E,l,\varepsilon)) \leq C\varepsilon\,|\Lambda_l|\,. \tag{3.2}$$

We will need a "uniform" Wegner-Lemma, i.e. an estimate of A^*_{l+r} (where $A = A(E,l,\varepsilon)$).

Proposition 3.3 (*uniform Wegner-estimate). For $E \in U$ as in Proposition 3.1 it holds that*

$$\mathbb{P}(A^*_{l+r}) \leq C(\varepsilon + r^{-(m-d)})\,|\Lambda_l|^2\,. \tag{3.3}$$

Moreover in the case of Proposition 3.2 we have

$$\mathbb{P}(A^*_{l+r}) \leq C(\varepsilon + r^{-(m-d)})\,|\Lambda_l|\,. \tag{3.4}$$

Note that the estimates (3.3) respectively (3.4) remain true if the event A is replaced by $\{\omega|\ \mathrm{dist}(M,\sigma(H_{\Lambda_l})) < \varepsilon\}$ and $M \subset U$ is such that diam $M < \varepsilon$.

Proof. Suppose $\omega \in A^*_{l+r}$ then there exists an $\omega' \in A$ such that

$$\Pi_{\Lambda_{l+r}}\omega' = \Pi_{\Lambda_{l+r}}\omega.$$

Consequently, for any $x \in \Lambda_l$ we have

$$\begin{aligned}|V_\omega(x) - V_{\omega'}(x)| &\leq \sum_{i \notin \Lambda_{l+r}} \left(|q_i(\omega)| + |q_i(\omega')|\right) f(x-i) \\ &\leq C \sum_{i \notin \Lambda_{l+r}} |x-i|^{-m} \;\leq\; C' r^{-(m-d)}.\end{aligned}$$

Thus dist$(E, \sigma(H_{\Lambda_l}(\omega'))) < \varepsilon$ implies dist$(E, \sigma(H_{\Lambda_l}(\omega))) < \varepsilon + c' r^{-(m-d)}$, and hence

$$\mathbb{P}(A^*_{l+r}) \leq \mathbb{P}(A(l, \varepsilon + c' r^{-(m-d)})).$$

The latter probability can be estimated by (3.1) und (3.2) respectively. $\square$

The induction in the multiscale method of [6] does not directly use a Wegner estimate but a consequence on the distance of the spectra of hamiltonians on disjoint boxes, see the estimate (4.4) in [6]. In the following we adapt this estimate to our situation.

Let I be an interval such that $\overline{I} \subset U$ for the neighborhood U found in Proposition 3.1 respectively Proposition 3.2. For a $\Lambda = \Lambda_l(x)$ define

$$\sigma_I(H_\Lambda) = \sigma(H_\Lambda) \cap \left[I + (-\frac{1}{2} l^{-(m-d)}, \frac{1}{2} l^{-(m-d)}) \right].$$

Note that $\sigma_I(H_\Lambda) \subset U$ for l sufficiently large.

Let $\Lambda_1 = \Lambda_{l_1}(x_1)$ and $\Lambda_2 = \Lambda_{l_2}(x_2)$ be cubes such that $\Lambda_{4l_1}(x_1) \cap \Lambda_{4l_2}(x_2) = \emptyset$. For fixed $\omega \in \Omega$ and $\omega_i := \Pi_{\Lambda_{4l_i}(x_i)}\omega$, $i = 1, 2$, we introduce the “uniform distance”

$$\tilde{d}(\sigma_I(H_{\Lambda_1}), \sigma_I(H_{\Lambda_2})) := \inf_{\substack{\tilde\omega_1 \in \Pi_{\mathbb{R}^d \setminus \Lambda_{4l_1}(x_1)}\Omega \\ \tilde\omega_2 \in \Pi_{\mathbb{R}^d \setminus \Lambda_{4l_2}(x_2)}\Omega}} \text{dist}\left(\sigma_I(H_{\Lambda_1}(\omega_1, \tilde\omega_1)),\ \sigma_I(H_{\Lambda_2}(\omega_2, \tilde\omega_2))\right).$$

We consider the event

$$A := \left\{\omega \middle|\ \tilde{d}(\sigma_I(H_{\Lambda_1}), \sigma_I(H_{\Lambda_2})) \leq \min\{l_1, l_2\}^{-(m-d)}\right\}.$$

Lemma 3.4. *There is a constant $C > 0$ independent of l_1, l_2, such that under the assumptions of Propositon 3.1,*

$$\mathbb{P}(A) \leq C \frac{\max\{l_1, l_2\}^d}{\min\{l_1, l_2\}^{m-3d}}, \tag{3.5}$$

and under the assumptions of Proposition 3.2,

$$\mathbb{P}(A) \leq C \frac{\max\{l_1, l_2\}^d}{\min\{l_1, l_2\}^{m-2d}}. \tag{3.6}$$

Proof. For subsets Λ of $\mathbb{R}^d$ we write $\mathbb{P}_\Lambda$ for the probability $\otimes_{i\in\Lambda\cap\mathbb{Z}^d}\mathbb{P}_0$ on $\Pi_\Lambda\Omega$ and $\mathbb{E}_\Lambda$ for its expectation.

We may assume $l_1 \le l_2$. The event A is a $(\Lambda_{4l_1} \cup \Lambda_{4l_2})$-cylinder and thus

$$\mathbb{P}(A) = \mathbb{P}_{\Lambda_{4l_1}\cup\Lambda_{4l_2}}(A) = \mathbb{E}_{\Lambda_{4l_2}}\mathbb{P}_{\Lambda_{4l_1}}(A). \tag{3.7}$$

Keeping ω_2 fixed for a moment, we pick an arbitrary $\tilde{\omega}_2^0 \in \Pi_{\mathbb{R}^d\setminus\Lambda_{4l_2}}\Omega$. From Weyl's asymptotic formula we conclude that $\sigma_I(H_{\Lambda_2}(\omega_2, \tilde{\omega}_2^0))$ has at most Cl_2^d elements (uniformly in ω_2). The decay assumption on f implies that for $x \in \Lambda_2$ and all $\tilde{\omega}_2 \in \Pi_{\mathbb{R}^d\setminus\Lambda_{4l_2}}\Omega$,

$$\left|V_{(\omega_2,\tilde{\omega}_2^0)}(x) - V_{(\omega_2,\tilde{\omega}_2)}(x)\right| \le Cl_2^{-(m-d)}.$$

Thus $\sigma_I(H_{\Lambda_2}(\omega_2, \tilde{\omega}_2)) \subset S_{\omega_2}$, where S_{ω_2} is independent of $\tilde{\omega}_2$ and a union of at most Cl_2^d intervals, each of length $\le Cl_2^{-(m-d)}$. We conclude

$$\begin{aligned}\mathbb{P}_{\Lambda_{4l_1}}(A) &\le \mathbb{P}_{\Lambda_{4l_1}}\left\{\omega_1 : \inf_{\tilde{\omega}_1\in\Pi_{\mathbb{R}^d\setminus\Lambda_{4l_1}}\Omega} \operatorname{dist}(\sigma_I(H_{\Lambda_1}(\omega_1,\tilde{\omega}_1)),\ S_{\omega_2}) \le l_1^{-(m-d)}\right\}\\ &= \mathbb{P}\left\{\omega : \operatorname{dist}(\sigma_I(H_{\Lambda_1}(\omega)),\ S_{\omega_2}) \le l_1^{-(m-d)}\right\}^*_{4l_1}. \end{aligned}\tag{3.8}$$

The set S_{ω_2} can be covered by at most Cl_2^d intervals of length $\le \frac{1}{2}l_2^{-(m-d)} \le \frac{1}{2}l_1^{-(m-d)}$. Thus an additivity argument and the uniform Wegner estimate (3.3) with $\varepsilon = l_1^{-(m-d)}$ (respectively the remark following Proposition 3.3) show that the r.h.s. of (3.8) can be further estimated by

$$\le Cl_2^d l_1^{-(m-d)} l_1^{2d} = C\frac{l_2^d}{l_1^{m-3d}}.$$

Since all constants are uniform in ω_2, (3.5) is now a direct consequence of (3.7). Using (3.4) instead of (3.3) we get (3.6). □

The other important tool for the multiscale analysis is an initial scale estimate, a result saying that with high probability the distance of the spectrum of finite box hamiltonians to $\partial\Sigma$ is not too small. If, as assumed here, $f \ge 0$, then proofs of this result (e.g. [6, 3, 9]) directly extend to the long range case by a monotonicity argument. It is more difficult to incorporate that we only want to assume that $f \ge c > 0$ on some open set (rather than $f \ge c\chi_{\Lambda_1(0)}$). Under this assumption a proof of the following result was given in Propositions 4.1 and 4.2 of [16]. It again extends to the long range case considered here by monotonicity. In fact, this shows that the following results hold for the uniform events from Definition 2.1.

Proposition 3.5. (a) *Under the assumptions of Theorem 1.1 let* $a = \inf\Sigma$. *Then for any* $\xi > 0$ *and* $\beta_0 \in (0, 2)$ *there is an* $l^* = l^*(\xi, \beta_0)$ *such that*

$$\mathbb{P}\left\{dist(\sigma(H_\Lambda(\omega)), a) \le l^{\beta_0-2}\right\}^*_{4l} \le l^{-\xi}$$

for $\Lambda = \Lambda_l(0)$ *and* $l \ge l^*$.

(b) *Under the assumptions of Theorem 1.2 let* $a \in \partial\Sigma$.
Then for any $\xi \in (0, 2\tau - d)$ *there is a* $\beta_0 > 0$ *and* $l^* = l^*(\tau, \xi)$ *such that*

$$\mathbb{P}\left\{\mathit{dist}(\sigma(H_\Lambda(\omega)) \cap (a,\infty),\ a) \leq l^{\beta_0 - 2}\right\}^*_{4l} \leq l^{-\xi} \quad \textit{if } a \textit{ is a lower band edge}$$

and

$$\mathbb{P}\left\{\mathit{dist}(\sigma(H_\Lambda(\omega)) \cap (-\infty, a),\ a) \leq l^{\beta_0 - 2}\right\}^*_{4l} \leq l^{-\xi} \quad \textit{if } a \textit{ is an upper band edge}$$

for $\Lambda = \Lambda_l(0)$ *and* $l \geq l^*$.

Note that $\sigma(H_\Lambda(\omega)) \subset \Sigma$ holds for every cube $\Lambda = \Lambda_l(0)$ (see e.g. the proof of $\sigma(H_\omega) = \Sigma$ a.s. in [16]) and thus part b) implies that $\mathbb{P}\{\mathrm{dist}(\sigma(H_\Lambda(\omega)), a) \leq l^{\beta_0 - 2}\}^*_{4l} \leq l^{-\xi}$ for every band edge a and sufficiently large l.

4. Multiscale Analysis

In this section we adapt the type of multiscale analysis which was used in [23] (which in turn is based upon [6] and [9]) to our situation. With the preparations from Sects. 2 and 3 at hand, the detailed proofs of the following results are very similar to the considerations in [6]. Thus we will only outline the main ideas and refer to [23] for a more detailed account.

In this and the next section we prove Theorems 1.1 and 1.2 under the assumption $m > 4d$. Simple modifications show that only $m > 3d$ is needed if the stronger form of the Wegner estimate is available.

For an interval $I \subset \mathbb{R}$, $l \in 2\mathbb{N} + 1$, $\gamma > 0$, and $\xi > 0$, we say that the estimate $G(I, l, \gamma, \xi)$ is satisfied if for all pairs $x, y \in \mathbb{Z}^d$ with $d(x, y) \geq 4l$ it holds that

$$\mathbb{P}\left\{\forall E \in I \text{ the cube } \Lambda_l(x) \text{ or } \Lambda_l(y) \text{ is uniformly } (\gamma, E)\text{-good for } \omega\right\} \geq 1 - l^{-2\xi}.$$

Note that by our general assumptions there is a bound $\|V_{per} + V_\omega\|_{p,loc,unif} \leq M$ uniformly in $\omega \in \Omega$.

Theorem 4.1. *Let* U *be the neighborhood of* $a \in \partial\Sigma$ *as given in Proposition 3.3. Fix* $\xi \in (0, \frac{m}{4} - d]$ *and* $\beta \in (0, 1)$. *Then there exist* $\alpha = \alpha(d, \xi) \in (1, 2)$, $l^* = l^*(d, \xi, \beta, m)$, $c_1 = c_1(d, C_{U,M})$ *(with* $C_{U,M}$ *from (2.2) respectively (2.3)),* $c_2 = c_2(d, m)$ *and a universal constant* c *such that the following implication holds:*

If I *is an open interval with* $\overline{I} \subset U$ *and for* $l \geq l^*$ *and* $\gamma \geq l^{\beta - 1}$ *the estimate* $G(I, l, \gamma, \xi)$ *is satisfied, then also* $G(I, L, \gamma_L, \xi)$ *is true, where*

(i)

$$l^\alpha \leq L \leq l^\alpha + 6,$$

(ii)

$$\gamma_L \geq \gamma(1 - cl^{1-\alpha}) - c_1 l^{-1} - c_2 \frac{\ln L}{L} \geq L^{\beta - 1}. \tag{4.1}$$

Moreover, for α *and* ξ *we have that* $(\alpha - 1)d < 2\xi$.

The *proof* of Theorem 4.1 starts by picking $\alpha = \alpha(d, \xi) \in (1, 2)$ such that $4d\left(\frac{\alpha-1}{2-\alpha}\right) \leq \xi$ and $L \in 3\mathbb{N} \setminus 6\mathbb{N}$ such that $l^\alpha \leq L \leq l^\alpha + 6$. With $\Gamma_x := x + \frac{l}{3}\mathbb{Z}^d$ we now define the event

$$\Omega_G(x) := \Big\{\omega \in \Omega : \forall E \in I \text{ there are no four cubes } \Lambda_l(b_i) \subset \Lambda_L(x) \\ \text{with } b_i \in \Gamma_x,\ i = 1, \ldots, 4 \text{ and } d(b_i, b_j) > 4l \text{ for } i \neq j \text{ such that} \\ \Lambda_l(b_i) \text{ is not uniformly } (\gamma, E)\text{-good for } i = 1, \ldots, 4\Big\}.$$

Using the remark on independence following Definition 2.4 and $4d(\alpha-1)/(2-\alpha) \leq \xi \leq m/4 - d$ it can be seen that

$$\mathbb{P}(\Omega_G(x)) \geq 1 - \frac{1}{3}L^{-2\xi} \tag{4.2}$$

for l sufficiently large. Simple geometrical considerations also show that for $\omega \in \Omega_G(x)$ and $E \in I$ there are disjoint cubes $\Lambda_{l_i}(b_i) \subset \Lambda_L(x)$, $i = 1, 2, 3$, such that

(i) $l_i \in \mathcal{L} := \{0, 5l, 10l, (10 + \frac{1}{3})l, 15l, (15 + \frac{1}{3})l, (15 + \frac{2}{3})l\}$,
(ii) $\sum_{i=1}^3 l_i \leq (15 + \frac{2}{3})l$,
(iii) if $b \in \Gamma_x$ and $\Lambda_l(b)$ is not uniformly (γ, E)-good, then $\Lambda_l(b) \subset \bigcup_i \Lambda_{l_i}(b_i)$,
(iv) $d(\Lambda_{l_i}(b_i), \Lambda_{l_j}(b_j)) \geq \frac{l}{3}$ for $i \neq j$.

Now fix $x, y \in \mathbb{Z}^d$ with $d(x, y) \geq 4L$ and define

$$\Omega_W := \Big\{\omega \in \Omega : \exists \Lambda_1 = \Lambda_{l_1}(z_1),\ \Lambda_2 = \Lambda_{l_2}(z_2) \text{ with} \\ \Lambda_1 \subset \Lambda_L(x),\ z_1 \in \Gamma_x,\ \Lambda_2 \subset \Lambda_L(y),\ z_2 \in \Gamma_y,\ l_i \in \mathcal{L} \cup \{L\} \\ \text{and } \tilde{d}(\sigma_I(H_{\Lambda_1}), \sigma_I(H_{\Lambda_2})) \leq \min\{l_1, l_2\}^{-(m-d)}\Big\}.$$

$\mathbb{P}(\Omega_W)$ can be estimated by counting the number of possible centers and sidelengths in question, and by using Lemma 3.4 for a fixed pair of boxes Λ_1 and Λ_2. Applying the definition of α, L and ξ one gets that

$$\mathbb{P}(\Omega_W) \leq \frac{1}{3}L^{-2\xi} \tag{4.3}$$

for l sufficiently large. Using (4.2) and (4.3) the theorem now follows from

Lemma 4.2. *If l is sufficiently large and $\omega \in \Omega_G(x) \cap \Omega_G(y) \cap \Omega_W^c$, then for arbitrary $E \in I$ there is $z \in \{x, y\}$ such that $\Gamma_L(z)$ is uniformly (γ_L, E)-good, where γ_L satisfies (4.1).*

Proof. The proof of this result is very similar to the arguments in Sect. 4 of [6] respectively their adaptation to the continuum in Sect. 6 of [9]. A detailed account of the proof of this result for the case of compactly supported single site potential f is also given in [23]. However, to adjust these arguments to the long range case, it has to be shown that the required estimate for $\|\chi_L^{out} R_{\Lambda_L}(E)\chi_L^{inn}\|$ with $\Lambda_L = \Lambda_L(z)$ holds uniformly for all $\omega' \in \Omega$ with $\Pi_{\mathbb{R}^d\setminus\Lambda_{4L}}\omega = \Pi_{\mathbb{R}^d\setminus\Lambda_{4L}}\omega'$. To this end one starts by using $\omega \in \Omega_W$ to show that for either $z = x$ or $z = y$ all cubes $\Lambda_{\tilde{l}}(u) \subset \Lambda_L(z)$ with $u \in \Gamma_z$ and $\tilde{l} \in \mathcal{L}$ are *uniformly non-resonant*, i.e. satisfy

$$\text{dist}(\sigma(H_{\Lambda_{\tilde{l}(u)}}), E) \geq \frac{1}{2}\tilde{l}^{-(m-d)}$$

uniformly for all ω' with $\Pi_{\mathbb{R}^d \setminus \Lambda_{4\tilde{l}(u)}}\omega = \Pi_{\mathbb{R}^d \setminus \Lambda_{4\tilde{l}(u)}}\omega'$.

For this z it can now be shown that $\Lambda_L(z)$ is uniformly (γ_L, E)-good. This uses $\omega \in \Omega_G(z)$ and an iteration procedure similar to the argument in [6], where the geometric resolvent inequality (2.3) is applied repeatedly. We omit further details and refer to either [6] or [23]. $\square$

We now combine Proposition 3.5 and Theorem 4.1 to get

Theorem 4.3. *Let* $a = \inf \Sigma$ *under the assumptions of Theorem 1.1 or* $a \in \partial\Sigma$ *under the assumptions of Theorem 1.2. Then there exists an open interval* I *containing* a, $\xi > 0$, $\gamma > 0$, $\alpha \in (1, 2)$, *and a sequence* $(l_k)_{k\in\mathbb{N}}$ *of length scales such that*

(i) $(\alpha - 1)d < 2\xi$,
(ii) $l_k^\alpha \leq l_{k+1} \leq l_k^\alpha + 6$ *for all* k,
(iii) $G(I, l_k, \gamma, \xi)$ *is satisfied for all* k.

Proof. Proposition 3.5 shows the existence of $\beta_0 > 0$ and $\xi > 0$ (which may be picked smaller than $m/4 - d$) such that for l sufficiently large and $E \in I := (a - \frac{1}{2}l^{\beta_0-2}, a + \frac{1}{2}l^{\beta_0-2})$ one has

$$\mathbb{P}\left\{\text{dist}(E, \sigma(H_{\Lambda_l})) \leq \frac{1}{2}l^{\beta_0-2}\right\}^*_{4l} \leq l^{-\xi}.$$

Here, $\sigma(H_{\Lambda_l})$ can get closer than $\frac{1}{2}l^{\beta_0-2}$ to E from only one side since from the fact that $\sigma(H_{\Lambda_l}) \subset \Sigma$ it follows that there is a gap (r, s) of $\sigma(H_{\Lambda_l})$ such that $E \in (r, s)$ and either dist$(r, E) \geq c$ or dist$(s, E) \geq c$ for a constant which does not depend on l and ω.

An improved version of the Combes-Thomas estimate introduced in [1] (see also [16, Lemma A.1]) can be used to conclude that

$$\|\chi_l^{out} R_{\Lambda_l}(E)\chi_l^{inn}\| \leq \frac{c'}{l^{\beta_0-2}} e^{-c'' l^{\frac{1}{2}(\beta_0-2)} \cdot l}$$

uniformly in the components of ω outside Λ_{4l} with probability $\mathbb{P} \geq 1 - l^{-\xi}$. Choosing $\beta \in (0, \beta_0/2)$ this implies that $G(I, l, \gamma_l, \xi)$ is satisfied for $l \geq l^*$ and $\gamma_l = l^{\beta-1}$, where independence was used.

We pick $l_1 = l$ and can now complete the proof of Theorem 4.3 by an inductive application of Theorem 4.1. If l_1 was picked sufficiently large, it can be checked by using (4.1) that there is a $\gamma > 0$ with $\gamma_{l_k} \geq \gamma$ for all k. $\square$

5. Proof of Localization

In this section we complete the proof of localization by adapting the line of reasoning from [23], where an improved continuum version of the arguments from Sect. 3 of [6] is presented.

For an operator $H = -\Delta + V$, where $V \in L^p_{loc,unif}$, p as above, it is true that for almost every E with respect to a spectral measure for H there exists a polynomially bounded generalized eigenfunction for H, e.g. [19]. Therefore the proof of Theorems 1.1 and 1.2 is completed once we have shown

Proposition 5.1. *Let I be an interval as provided by Theorem 4.3. Then it holds with probability one that every polynomially bounded generalized eigenfunction ψ for H_ω to an $E \in I$ is exponentially decaying, in fact*

$$\limsup_{x\to\infty} \frac{\log|\psi(x)|}{d(x,0)} \leq -3\gamma, \tag{5.1}$$

where $\gamma > 0$ is the decay rate found in Theorem 4.3.

(Note that the γ given in Theorem 4.3 describes the decay of the "averaged Green's function" $\|\chi_l^{out} R_l(E)\chi_l^{inn}\|$ between the cube $\Lambda_{l/3}$ and the boundary of Λ_l, i.e. over a distance $l/3$ in ∞-metric. Thus the factor 3 on the r.h.s. of (5.1) shows that ψ decays at the same rate as the Green's function.)

Proof. With small modifications we follow the lines of the proof of Lemma 3.1 in [6]. For $x_0 \in \mathbb{Z}^d$ let

$$A_{k+1}(x_0) = \Lambda_{8bl_{k+1}}(x_0) \setminus \Lambda_{8l_k}(x_0)$$

for $b > 1$ to be chosen later, and consider the event

$$E_k(x_0) := \Big\{\text{There is some } E \in I \text{ and } x \in A_{k+1}(x_0) \cap \Gamma_k \text{ such that} \\ \Lambda_{l_k}(x_0) \text{ and } \Lambda_{l_k}(x) \text{ are not } (\gamma, E)\text{-good}\Big\},$$

where $\Gamma_k = x_0 + \frac{l_k}{3}\mathbb{Z}^d$. Since $G(I, \xi, l_k, \gamma)$ holds and $A_{k+1}(x_0) \cap \Gamma_k$ has $\leq C_b(l_{k+1}/l_k)^d$ elements, we can estimate

$$\mathbb{P}(E_k(x_0)) \leq c_b l_k^{-2\xi} l_k^{d(\alpha-1)}.$$

This is summable over k by Theorem 4.3 and thus by Borel-Cantelli and stationarity we get that $\mathbb{P}(\Omega_0) = 1$ for

$$\Omega_0 := \left\{\omega : \forall x \in \mathbb{Z}^d\ \exists k_x \in \mathbb{N} \text{ such that } \omega \notin E_k(x) \text{ for } k \geq k_x\right\}.$$

If $\omega \in \Omega_0$ and $\psi \neq 0$ is a polynomially bounded generalized eigenfunction for H to $E \in I$, then it is shown as in [6] that $\Lambda_{l_k}(x)$ is (γ, E)-good for all $k \geq k_0(\omega)$ and $x \in A_{k+1}(x_0) \cap \Gamma_k$ ($\|\chi_{x_0}\psi\|$ replaces $|\psi(x_0)|$ from the discrete case and (2.2) is used).

Continuing the argument as in [6] and with $b > \frac{1+\rho}{1-\rho}$ for some $\rho \in (0,1)$ it can be shown that for $y \in \tilde{A}_{k+1}(x_0)$ (defined as in [6] with factors 2 replaced by 8) and some $n \geq \frac{3}{l_k}\rho d(y, x_0)$ it holds that

$$\|\chi_y\psi\| \leq \|\chi_{y_0,l_k}^{inn}\psi\| \leq (C_\psi(3^d-1)e^{-\gamma l_k})^j \|\chi_{l_k,y_j}^{out}\psi\|, \quad j = 1, \ldots, n.$$

Here $y_0, \ldots, y_n$ is a sequence in $\Gamma_k \cap A_{k+1}(x_0)$ with $y \in \Lambda_{l_k/3}(y_0)$ and $d(y_j, y_{j+1}) \leq l_k/3$.

The lower estimate for n and the polynomial bound for ψ imply that for every $\tilde{\gamma} < 3\gamma$ we have

$$\|\chi_y\psi\| \leq C(\psi, d, \rho, \tilde{\gamma})e^{-\tilde{\gamma}\rho d(y,x_0)}.$$

Since $\rho \in (0,1)$ and $\tilde{\gamma} < 3\gamma$ were arbitrary, (5.1) follows from a subsolution estimate $|\psi(y)| \leq C_{I,M}\|\chi_y\psi\|$ (e.g. [19]). $\square$

We point out that some of the changes in the above argument compared to [6] and [9] can also be used to streamline the proof of Theorem 6 in [9] as was already observed in [23]:

Note that our result in Theorem 4.3 is actually somewhat stronger than the corresponding results in [6] respectively [9], even in the case of compactly supported f. The difference is that in the definition of γ-good cubes we work with $\Lambda_l^{inn} = \Lambda_{l/3}$ as the inner cube, while [6] and [9] use unit cubes around the center in the same context. This has the effect that in the estimate for $\mathbb{P}(E_k(x_0))$ above we can work with the counting factor $(l_{k+1}/l_k)^d$, where the proof of [6, Lemma 3.1] needs l_{k+1}^d. A consequence of this fact is that we can now directly complete the proof of exponential decay with $\xi > 0$ as provided in Theorem 4.3, while the reasoning in [6] needs $\xi > d$ in this context. In the proof of their Theorem 6 on band edge localization in [9] (which in the case of compactly supported f corresponds to our Theorem 1.2), Figotin and Klein provide an alternative argument to show that, roughly speaking, a result as Theorem 4.3 with $\xi > 0$ implies that the same result holds for some $\xi > d$. This requires an additional multiscale analysis, which can be dropped by using our more direct argument.

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