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Discontinuities of the Integrated Density of States for Random Operators on Delone Sets*

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Abstract: Despite all the analogies with "usual random" models, tight binding operators for quasicrystals exhibit a feature that clearly distinguishes them from the former: the integrated density of states may be discontinuous. This phenomenon is identified as a local effect, due to the occurrence of eigenfunctions with bounded support.

1. Introduction

In the present article we study the occurrence of discontinuities of the integrated density of states (IDS). For the special case of a tight binding model associated with the Penrose tiling the occurrence of this effect has been known for quite some time as witnessed for example in [ATF, FATK, KF, KS]. We present two results. The first aims at showing that the occurrence of jumps in the IDS cannot be excluded by global assumptions concerning, e.g., ergodic or combinatorial properties. To this end we present a theorem saying that starting from some model of aperiodic order (phrased in the language of Delone sets) one can construct a model that is "basically the same" and gives rise to a tight binding operator for which the IDS is discontinuous. Here "basically the same" is cast in the notion of "mutually locally derivable" for Delone dynamical systems. We discuss this notion analogous to the respective notion for tilings found in [BSJ]. In the construction we use that Laplacians on certain graphs have finitely supported eigenfunctions. It now becomes clear that it is the more complex structure of graphs in higher dimension that makes such a phenomenon possible. On lattices (and, consequently, in one-dimensional systems) such finitely supported eigenfunctions cannot occur.

Our second theorem says that this is the only possibility to create a jump of the IDS, at least when starting from a reasonable Delone dynamical system. It is a consequence of a rather strong ergodic theorem in [LS3]. This theorem states that the IDS is in fact the uniform limit of eigenvalue counting distributions. It is therefore substantially stronger

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than the weak convergence results typically proven in connection with the IDS see for example [BLT, H, K]. The fact that such a strong convergence holds true is special for models of aperiodic order and not met in usual random systems.

2. Notation and Results

In this section we introduce some notation and present our results. We will use the same setting as the one found [LS2]. For completeness reasons we include the necessary definitions.

Let $d \geq 1$ be a fixed integer and all Delone sets, patterns etc. will be subsets of $\mathbb{R}^d$. The Euclidean norm on $\mathbb{R}^d$ will be denoted by $\|\cdot\|$. For $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^d$, we let $B(p,r)$ be the closed ball in $\mathbb{R}^d$ centered at p with radius r.

A subset ω of $\mathbb{R}^d$ is called a Delone set if there exist $r(\omega)$ and $R(\omega) > 0$ such that $2r(\omega) \leq \|x-y\|$ whenever $x, y \in \omega$ with $x \neq y$, and $B(x, R(\omega)) \cap \omega \neq \emptyset$ for all $x \in \mathbb{R}^d$.

We are dealing in this paper with local structures of Delone sets, therefore the restrictions of ω to bounded subsets of $\mathbb{R}^d$ are of particular interest. In order to treat these restrictions, we introduce the following definition.

Definition 2.1. (a) *A pair (Λ, Q) consisting of a bounded subset Q of $\mathbb{R}^d$ and $\Lambda \subset Q$ finite is called a* **pattern**. *The set Q is called the* **support of the pattern**.
(b) *A pattern (Λ, Q) is called a* **ball pattern** *if $Q = B(x,r)$ with $x \in \Lambda$ for some $x \in \mathbb{R}^d$ and $r \in (0,\infty)$.*

The diameter and the volume of a pattern are defined to be the diameter and the volume of its support respectively.

We will have to identify patterns that are equal up to translation. More precisely, on the set of patterns we introduce an equivalence relation by setting $(\Lambda_1, Q_1) \sim (\Lambda_2, Q_2)$ if and only if there exists a $t \in \mathbb{R}^d$ with $\Lambda_1 = \Lambda_2 + t$ and $Q_1 = Q_2 + t$. The class of a pattern (Λ, Q) is then denoted by $[(\Lambda, Q)]$. Obviously the notions of diameter, volume occurrence etc. can easily be carried over from patterns to pattern classes.

Every Delone set ω gives rise to a set of pattern classes, $\mathcal{P}(\omega) = \{[Q \wedge \omega] : Q \subset \mathbb{R}^d$ bounded and measurable$\}$, and to a set of ball pattern classes $\mathcal{P}_B(\omega) = \{[B(p,r) \wedge \omega] : p \in \omega, r \in \mathbb{R}^+\}$. Here we set $Q \wedge \omega = (\omega \cap Q, Q)$. We define the radius $s = s(P)$ of an arbitrary ball pattern P to be the radius of the underlying ball. For $s \in (0,\infty)$, we denote by $\mathcal{P}_B^s(\omega)$ the set of ball pattern classes with radius s. A Delone set is said to be of *finite type* or of *finite local complexity* if for every radius $s > 0$ the set $\mathcal{P}_B^s(\omega)$ is finite.

The Hausdorff metric on the set of compact subsets of $\mathbb{R}^d$ induces the so called *natural topology* on the set of closed subsets of $\mathbb{R}^d$. It is described in detail in [LS2] and shares some nice properties: firstly, the set of all closed subsets of $\mathbb{R}^d$ is compact in the natural topology. Secondly, and this is of prime importance in view of the dynamical system we are to consider, the natural action T of $\mathbb{R}^d$ on the closed sets in $\mathbb{R}^d$ given by $T_t G = G + t$ is continuous.

Furthermore, a *Delone dynamical system (DDS)* consists of a set Ω of Delone sets, which is invariant under the shift T and closed in the natural topology. A DDS is said to be of finite type (DDSF) if $\cup_{\omega\in\Omega}\mathcal{P}_B^s(\omega)$ is finite for every s and the set $\mathcal{P}(\Omega)$ of patterns classes associated to a DDS Ω is defined by $\mathcal{P}(\Omega) = \cup_{\omega\in\Omega}\mathcal{P}(\omega)$. Due to the compactness of the set of all closed sets in the natural topology a DDS Ω is compact. We refrain from a precise discussion of the topology but we give the following lemma from [LS2].

Lemma 2.2. *If (Ω, T) is a DDSF then a sequence (ω_k) converges to ω in the natural topology if and only if there exists a sequence (t_k) converging to 0 such that for every $L > 0$ there is an $k_0 \in \mathbb{N}$ with $(\omega_k + t_k) \cap B(0, L) = \omega \cap B(0, L)$ for $k \geq k_0$.*

Roughly speaking, ω is close to $\tilde{\omega}$ if ω equals $\tilde{\omega}$ on a large ball up to a small translation.

We now recall some standard notions from the theory of dynamical systems and some available equivalent "combinatorial" characterizations. A dynamical system (Ω, T) is called *minimal* if the orbit $\{T_t\omega : t \in \mathbb{R}^d\}$ of any ω is dense in Ω. For a DDS this is equivalent to the property that $\mathcal{P}(\Omega) = \mathcal{P}(\omega)$ for any ω. This latter property is called a *local isomorphism property* in the tiling framework; see [Sol1]. A sequence (Q_k) of subsets in $\mathbb{R}^d$ is called a van Hove sequence if the sequence $|\partial^R Q_k||Q_k|^{-1}$ tends to zero for every $R \in (0, \infty)$. Here, $\partial^R Q$ denotes the set of those $x \in \mathbb{R}^d$ whose distance to the boundary of Q is less than R. Furthermore, a dynamical system (Ω, T) is called *uniquely ergodic* if it admits only one T-invariant measure (up to normalization). For a Delone dynamical system, this is equivalent to the fact that for every nonempty pattern class P the frequency

$$\nu(P) \equiv \lim_{k\to\infty} |Q_k|^{-1} \sharp_P(Q_k \wedge \omega),$$

exists uniformly in $\omega \in \Omega$ for every van Hove sequence (Q_k). Here $\sharp_P Q$ denotes the number of occurrences of P in Q. We call a dynamical system (Ω, T) *strictly ergodic* if it is minimal and uniquely ergodic. Note that in this case the frequency $\nu(P)$ is positive for every $P \in \mathcal{P}(\Omega)$.

Definition 2.3. *Let (Ω, T) be a DDSF. A family (A_ω) of bounded operators $A_\omega : \ell^2(\omega) \longrightarrow \ell^2(\omega)$ is called a random operator of finite range on (Ω, T) if there exists a constant r_A with*

- *$A_\omega(x, y) = 0$ whenever $\|x - y\| \geq r_A$.*
- *$A_\omega(x, y)$ only depends on the pattern class of $(B(x, r_A) \cup B(y, r_A)) \wedge \omega$.*

Usually, random operators are defined with respect to a measure. In our situation, however, it seems natural to define them without a given measure, as the setting of Delone sets is a purely topological one. Moreover, in the case of uniquely ergodic DDS, which is our main concern, a measure arises naturally as discussed above.

Note that the above defined operators provide a framework including Laplace type operators defined on $\ell^2(\mathbb{Z}^d)$. Let us mention that for $\tilde{\omega}, \omega \in \Omega$ the operators act on different Hilbert spaces $\ell^2(\omega)$ and $\ell^2(\tilde{\omega})$ unless ω and $\tilde{\omega}$ differ only by translation. Thus, to deal with operators on $\ell^2(\omega)$ is more complicated than in the lattice case.

The aim of this article is to discuss the phenomenon of discontinuities of the integrated density of states of random operators (A_ω) on a DDSF (Ω, T). This might rather come as a surprise in view of what is known for random models as well as one dimensional quasicrystals. It turns out that this phenomenon occurs if and only if there exist locally supported eigenfunctions of (A_ω). One can find examples of locally supported eigenfunctions on the Penrose tiling in [KS] and [ATF]. An eigenfunction f is said to be locally supported if $\operatorname{supp} f \subset K$, with K a compact set. The phenomenon of locally supported eigenfunctions is by no means pathological. Rather from any given DDSF (Ω, T) we can construct an, in some sense, local equivalent DDSF (Ω^b, T) such that a random operator of finite range (A^b_ω) defined on (Ω^b, T) has locally supported eigenfunctions. More precisely (Ω, T) and (Ω^b, T) are *mutually locally derivable* (MLD). The equivalence concept of mutual local derivability for tilings was discussed in detail in [BSJ]. This will all be discussed below. Our first result reads as follows.

Theorem 1. *Let (Ω, T) be a DDSF. Then there exists a DDSF (Ω^b, T) and a random operator of finite range (A^b_w) on (Ω^b, T) such that (Ω, T) and (Ω^b, T) are mutually locally derivable and (A^b_w) has locally supported eigenfunctions with the same eigenvalue for every $\omega \in \Omega^b$. Moreover, (A^b_w) can be chosen to be the nearest neighbor Laplacian of a suitable graph.*

Remark 2.4. The theorem also holds in the tiling setting. Here a single tile of the original tiling will be replaced by tiles of a new tiling which is MLD to the originally given one (see below for further discussion).

Note that for a selfadjoint random operator A and bounded $Q \subset \mathbb{R}^d$ the restriction $A_\omega|_Q$ defined on $\ell^2(Q \cap \omega)$ has finite rank. Therefore, the spectral counting function

$$n(A_\omega, Q)(E) := \#\{ \text{ eigenvalues of } A_\omega|_Q \text{ not exceeding } E\}$$

is finite and $\frac{1}{|Q|} n(A_\omega, Q)$ is the distribution function of the measure $\rho_Q^{A_\omega}$, defined by

$$\langle \rho_Q^{A_\omega}, \varphi \rangle := \frac{1}{|Q|} \mathrm{tr}(\varphi(A_\omega|_Q)) \text{ for } \varphi \in C_b(\mathbb{R}).$$

For a uniquely ergodic DDSF the measures $\rho_{Q_k}^{A_\omega}$ converge in distribution to a measure ρ^A which is independent of $\omega \in \Omega$ and called the integrated density of states (IDS) for any van Hove sequence Q_k as $k \to \infty$. This is described in [LS2, LS4]. There one can also find an interpretation of the IDS as a certain trace on a von Neumann algebra. Now, we can state our main theorem.

Theorem 2. *Let (Ω, T) be a strictly ergodic DDSF. Let A be a selfajoint random operator of finite range. Then E is a point of discontinuity of ρ^A if and only if there exists a locally supported eigenfunction of A_ω to E for one (all) $\omega \in \Omega$.*

Remarks 2.5. (1) It rather straightforward to see that locally supported eigenfunctions lead to a discontinuity of the IDS. The more interesting part of the equivalence is that discontinuities only happen in that way.

(2) As pointed out already the theorem gives rise to a complete characterization of the phenomenon of locally supported eigenfunctions in quasicrystal settings (i.e. DDSF and tiling settings).

(3) Let us emphasize that the integrated density of states is continuous in the case of almost periodic and random operators on lattices. Due to the more complex geometry this does not follow in the quasicrystal framework.

3. Preliminaries

In this section we will study the equivalence concept of MLD. Further we are going to construct a map that maps any given DDSF to a DDSF which is MLD to the original one and which admits a random operator with locally supported eigenfunctions. We record as well some tools that we will use later on to prove our results.

As mentioned already the equivalence concept of MLD for patterns on tilings was discussed in [BSJ]. We give the obvious definition for Delone dynamical systems of finite type.

Definition 3.1. *Let (Ω, T) and (Ω^b, T) both be DDSF. A map $D : \Omega \to \Omega^b$ is called a* **local derivation map** *if there exists a radius $r_D > 0$ such that $D(\omega) \cap \{x\} = (t + D(\omega)) \cap \{x\}$ holds whenever $\omega \cap B(x, r_D) = (t + \omega) \cap B(x, r_D)$. In this case (Ω^b, T) is called* **locally derivable** *from (Ω, T). Two DDSF (Ω, T) and (Ω^b, T) are* **mutually locally derivable** *if (Ω, T) is locally derivable from (Ω^b, T) and vice versa (with a possibly different radius r'_D).*

Note that the map D is local in the sense that $D(\omega) \cap B(x, s)$ only depends on $\omega \cap B(x, s + 2r_D)$.

Proposition 3.2. *Let (Ω, T) be a DDSF, $D : \Omega \to \Omega^b$, $\omega \mapsto D(\omega)$, a local derivation map. Then D is continuous with respect to the natural topology.*

Proof. This is immediate as D is local and the topology is local in the sense of Lemma 2.2. □

We now want to insert a well scaled local structure into a given Delone set wherever a certain pattern occurs. Let ω be a Delone set and P be a ball pattern class with $P \in \mathcal{P}_B(\omega)$. Then, we define ω_P to be the set of all occurrences of P in $\mathbb{R}^d$, i.e.

$$\omega_P \equiv \{t \in \mathbb{R}^d : [B(t, s(P)) \wedge \omega] = P\}. \tag{1}$$

Now, let (Ω, T) be a DDSF and $r < r(\omega)$ for all $\omega \in \Omega$, G be a finite graph with V_G the set of vertices of G contained in $\mathbb{R}^d$. Furthermore let $\mathrm{diam}(G) = \frac{r}{21}$ and $V_G \subset B(0, \frac{r}{42})$. We use this finite graph to define a local derivation map by setting $D_{P,V_G}(\omega) \equiv \omega \cup \{t + V_G : t \in \omega_P\}$ for $\omega \in \Omega$ and $\Omega^b := \{D_{P,V_G}(\omega) : \omega \in \Omega\}$. Then,

$$D_{P,V_G} : \Omega \to \Omega^b, \quad \omega \mapsto D_{P,V_G}(\omega)$$

is a local derivation with inverse given by the local derivation map

$$H_{P,V_G} : \Omega^b \to \Omega, \quad H_{P,V_G}(\omega^b) = \{x \in \omega^b : \omega^b \cap B(x, \tfrac{r}{3}) = \omega^b \cap B(x, \tfrac{r}{42})\}.$$

Note that (Ω, T) is also a local derivation of (Ω^b, T). Thus, (Ω, T) and (Ω^b, T) are mutually locally derivable.

Remarks 3.3. Let Ω and Ω^b be as above. Then

(1) As (Ω, T) is a DDSF so is (Ω^b, T).
(2) If (Ω, T) is a uniquely ergodic DDSF, the same holds for (Ω^b, T).
(3) The frequency of G in Ω^b is the same as the frequency $\nu(P)$ of P in Ω.

The following two ingredients are essential for the proof of Theorem 2. The first one is one of the main results from [LS3]. It relies on a strong ergodic type theorem proven there (see [Len] for a study of uniform ergodic theorems in the one dimensional case).

Theorem 3.4. *Let (Ω, T) be a strictly ergodic DDSF. Let A be a selfadjoint operator of finite range and (Q_k) be a van Hove sequence in $\mathbb{R}^d$. Then, the distribution functions of $\rho^{A_\omega}_{Q_k}$ converge uniformly to the distribution function of the measure ρ^A and this convergence is uniform in $\omega \in \Omega$.*

The second one is a well known dimension argument from linear algebra which we will state for completeness reasons.

Proposition 3.5. *Let H be a finite dimensional Hilbert space, U, V subspaces of H with* $\dim U > \dim V$, *then* $\dim V^{\perp} \cap U > 0$.

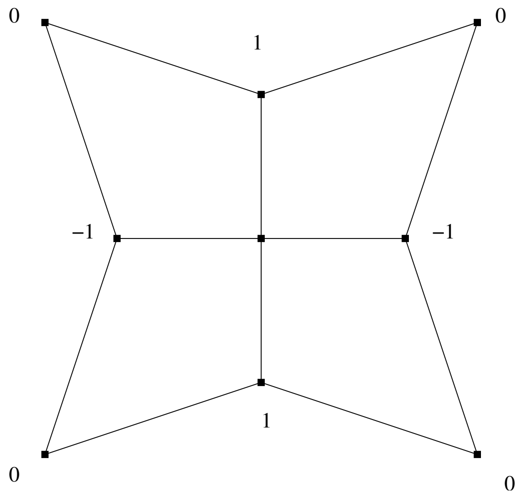

Fig. 1. The finite graph G_{fin}

4. Proofs

In this section we prove Theorem 1 and Theorem 2.

Proof of Theorem 1. To prove the theorem we start with the construction of a DDSF where a random operator of finite range (A_ω) exists which has locally supported eigenfunctions. The starting point is a small graph $G_{fin} = (V_{fin}, E_{fin})$ and an eigenfunction u_{fin} of the associated nearest neighbor Laplacian. For definiteness sake consider Fig. 1. The values of u_{fin} are indicated near the corresponding vertices. Here the eigenvalue is $E = 0$.

It is clear that whatever edges reach out of the four corners in a larger graph extending G_{fin}, the extension of u_{fin} by 0 to the larger vertex set will still constitute an eigenfunction of the Laplacian on the large graph. It is now easy to implement this picture into a given DDSF. In fact, let (Ω, T) be a DDSF and P be a ball pattern. We use the local derivation map $D_{P,V_{fin}}$ discussed in the last section to put in V_{fin} from above, scaled properly, whenever P appears. It is obvious by the definition of $D_{P,V_{fin}}$, that this gives rise to a DDSF (Ω^b, T) which is locally derivable from (Ω, T) and vice versa.

Obviously we get a random operator A^b with locally supported eigenfunctions by taking for A^b_ω the nearest neighbor Laplacian on the copies of E_{fin} in ω and consistent matrix elements otherwise. □

Remarks 4.1. (1) The simplest case of the construction made above is of course given by choosing $P = (\{x\}, B(x, r))$ with $r < r(\omega)$. Then the graph G_{fin} is glued at any point of the underlying Delone set. The corresponding ℓ^2 space is just a direct sum (or tensor product) and that applies to the operators as well. Related constructions have been considered by [SA] in the context of creation of spectral gaps.

(2) For those who prefer tiling examples we now indicate how to view the construction above in this framework. Take a tiling dynamical system (see [LS2, Sol2]) and replace one given tile T by a suitable homeomorphic image of T^b indicated in Fig. 2.

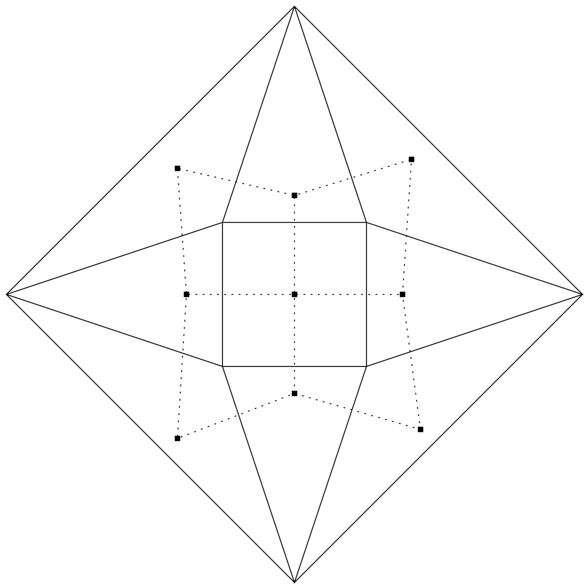

Fig. 2. The tiling of G_{fin}

We also indicated the next neighbor relations, showing that the resulting graph is just G_{fin} above.

Proof of Theorem 2. We first show that the condition is sufficient. Let u be an eigenfunction of A_{ω_0} associated to an eigenvalue E with $\operatorname{supp} u \subset B(x,r)$ and $x \in \omega_0$. Then for any $\omega \in \Omega$ every copy of $P = B(x,r) \wedge \omega_0$ in $Q \wedge \omega$ adds a dimension to the eigenspace of $A_\omega|Q$ belonging to the eigenvalue E. Let $\dot{\sharp}_P Q \wedge \omega$ be the maximal number of disjoint copies of P in $Q \wedge \omega$. Note that $\frac{|B(0,3r+r(\omega))|}{|B(0,r(\omega))|} =: C$ is an upper bound for the number of points (and therefore the maximal number of copies of P) in $B(0,3r) \cap \omega$.

This gives by a direct combinatorial argument that

$$\dot{\sharp}_P Q \wedge \omega \geq \frac{1}{C} \sharp_P Q \wedge \omega.$$

Thus, for arbitrary $\epsilon > 0$,

$$\frac{\operatorname{tr}(\chi_{(-\infty,E-\epsilon)}(A_\omega|_Q))}{|Q|} \leq \frac{\operatorname{tr}(\chi_{(-\infty,E+\epsilon)}(A_\omega|_Q))}{|Q|} - \frac{1}{C}\frac{\sharp_P \omega \wedge Q}{|Q|}.$$

Setting $Q = Q_k$ with Q_k from a van Hove sequence and letting k tend to infinity, we get that $\rho^A(E-\epsilon) \leq \rho^A(E+\epsilon) - \frac{\nu(P)}{C}$. As $\epsilon > 0$ is arbitrary and $\nu(P) > 0$ the desired implication follows.

Next we show the converse implication. Let $\tilde{E}$ be a point of discontinuity of the function $E \mapsto \rho_A((-\infty, E])$ and (Q_k) an arbitrary van Hove sequence. We consider the distribution function $\frac{1}{|Q|} n(A_\omega, Q)$ of the measure $\rho_{Q_k}^{A_\omega}$. Proposition 3.4 shows that $\frac{1}{|Q_k|} n(A_\omega, Q_k)$ converges w.r.t. the supremum norm to the function $E \mapsto \rho^A((-\infty, E])$.

Thus, for large k the jump at $\tilde{E}$ of the function $\frac{1}{|Q_k|} n(A_\omega, Q_k)(E)$ does not become small. More precisely we get

$$\dim\left(\ker\left(A_\omega|_{Q_k} - \tilde{E}\right)\right) = \lim_{\epsilon \to 0} (n(A_\omega, Q_k)(\tilde{E} + \epsilon) - n(A_\omega, Q_k)(\tilde{E} - \epsilon)) \geq c|Q_k|$$

for a $c > 0$ and all $k \in \mathbb{N}$. Now let $\partial_{2r_A} Q_k \equiv \partial^{2r_A} Q_k \cap Q_k$ denote the inner boundary of range $2r_A$ of Q_k. For a van Hove sequence (Q_k) we have

$$\begin{aligned} \dim \quad \ell^2(\partial_{2r_A} Q_k \cap \omega) &= \sharp\{x \in \mathbb{R}^d : x \in \partial_{2r_A} Q_k \cap \omega\} \\ &\leq \frac{|\partial^{2r_A + r(\omega)} Q_k|}{|B(0, r(\omega))|} \\ &= \epsilon_k \cdot \frac{1}{|B(0, r(\omega))|} \cdot |Q_k| \end{aligned}$$

with a suitable ϵ_k which tends to 0 for $k \to \infty$. For k large enough we get that $c > \epsilon_k \cdot \frac{1}{|B(0,r(\omega))|}$. Thus, for large k the inequality

$$\dim\left(\ker\left(A_\omega|_{Q_k} - \tilde{E}\right)\right) > \dim \quad \ell^2(\partial_{2r_A} Q_k \cap \omega)$$

holds. Now let W_k be the projection onto the inner boundary of range $2r_A$ of Q_k. Then Propositon 3.5 shows that there exists an eigenfunction f of A_ω such that $W_k f = 0$ for k large enough. □

Remark 4.2. Let the conditions be as above. Then E is an infinitely degenerate eigenvalue of A_ω for every $\omega \in \Omega$. The integrated density of states has a jump at E whose height is at least $C^{-1}\nu(P)$.

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