Caught by disorder: lectures on bound states in random media

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To

Monika, Sonja and Maria

Introduction

Disorder is one of the predominant topics in science today. The present text is devoted to the mathematical study of some particular cases of disordered systems. Its aim is to give an elementary, yet detailed study of a phenomenon which is referred to as **localization** in the Mathematical Physics literature. By localization it is meant that waves in a sufficiently irregular environment exhibit a totally different time evolution than in a regular medium. Instead of travelling through space, they are localized to a finite region: they are **caught by disorder**.

The two basic models, we are dealing with throughout the text, describe two different kinds of waves: in a model suggested by solid state physics, the wave functions represent the electron cloud in a crystalline matter with impurities, and the time evolution is predicted by the Schrödinger equation. The second model concerns classical waves in certain disordered media.

From the mathematical point of view, we are dealing with partial differential equations with random coefficients. More precisely, we are interested in spectral theoretic properties of the partial differential operators appearing in these equations. But that's only part of the story:

the history of this subject goes back to solid state physics where the localization phenomenon – the transition from conductor (freely travelling waves) to insulator (localized waves) – has been encountered. The theoretical explanations of P.W. Anderson in the late fifties (for which he has been awarded the Nobel prize) and I.M. Lifshitz in the sixties laid the foundation for mathematically rigorous studies. But those rigorous studies came much more recently: if one focusses on localization, the breakthrough came with the work of J. Fröhlich and T. Spencer in the early eighties. By that time a systematic study of random operators had already attracted a large number of mathematical physicists, probabilists and operator theorists to whom a number of wonderful theorems can be attributed. To mention only the most important contributions would end in a rather long list. Instead, we refer the reader to either the monograph of Pastur and Figotin, which was published in 1992 or the one by Carmona and Lacroix, which was published in 1990.

What was it that triggered such an enormous interest? Well, first of all, there is the real world setting indicated above. This environment both motivates a list of relevant problems and provides their expected (either by experiment or by heuristics) solutions. Another, more abstract aspect of the intrigue has to do with a development witnessed in the (more or less) recent past which one could label "randomization of analysis". Disorder, in a rather precise mathematical form has turned out to serve as a powerful tool in analysis. Generally speaking, randomization can be used to produce rather strange and exotic examples, whose existence could not be proven on deterministic grounds. In case of the subject matter it is the randomization of spectral theory which shed new light on old truth and even caused a change of paradigm in certain respects. For instance it turned out that "pathological spectral types", like dense pure point spectrum, turned out to be generic in some sense. To give an admittedly trivial comparison, this resembles the finding that almost all numbers are transcendental, despite the fact that most numbers we meet are rational, and that it is extremely hard to track down an explicit example for which you can prove the "pathological behaviour". What we will be concerned with in this book is precisely to prove that certain quite natural models exhibit dense pure point spectrum almost surely, which is one way of giving a mathematical precise definition of **localization**. Finally, the variety of mathematical methods from diverse fields which enter the subject adds to its attractiveness. It also causes trouble: As this book is intended to provide an introduction accessible to graduate students, it would be senseless to presuppose familiarity with the theory of Schrödinger and/or random operators. On the other hand, I want to proceed as soon as possible to the real issue. To cope with these problems I have chosen to present the material in the following way:

The first section serves as an introduction to the topic. Specifically, spectral theoretic properties are translated into properties of the time evolution. This is the basis for the spectral theoretic treatment of localization and may also serve as a list of prerequisites. In an appendix, the reader will find a more detailed introduction to those parts of operator theory which are needed in the main body of the text. All the relevant notions are explained to some extent, with the specific examples of the text in mind. Although the reader will find only a few proofs (references are, of course, provided) in the appendix, it might still serve as a sufficient basis to understand the main text.

The second section contains the basic framework for random operators and summarizes that part of the abstract theory which is needed later. The fundamental idea how to incorporate disorder into mathematical models is the following: instead of one fixed model, a whole family of models is considered, where the family is indexed by the points of a probability space. This reflects the fact that disorder means that one doesn't really know what's going on exactly but one still might have some statistical knowledge, which is incorporated by the probability space and the specific random family. Consider, for instance a pure crystal. It consists of a lattice of ions distributed periodically in space. A reasonable mathematical model for the time evolution of electrons in such a crystal is given by a Schrödinger equation, in which the potential term is a periodic function, namely the sum of the potential terms of all the individual ions. To get a more realistic description, one might want to take into account the possibility that some of the ions are missing. If one has some knowledge on the probability of a defect at a particular lattice site, one can consider the family of all the potential terms which appear if at certain lattice sites the ion is missing. Note that this gives a family of equations which has the cardinality of the continuum. Typical probability measures would consist of product measures if it is reasonable to assume that defects on different lattice sites occur independently. Now the questions one wants to answer are of the type: what is the typical behaviour of the electrons.This means that one is interested in properties which hold for almost all members of the specific random family in question. Among the important results listed in Section2, one will find that almost all members of a reasonable random family of operators share the same spectral properties. This is, in a way, a justification of this new point of view: it tells us that questions concerning almost sure spectral theoretic properties make sense. Since the necessary theory is already covered by textbooks, results are explained rather than proven in this section.

The third section records some important and well–studied examples. They may serve to illustrate the rich variety of specral properties which can show up in ergodic operators. Some of these examples are in some sense "poorly disordered". E.g. almost periodic examples still have quite some order built into them. The general philosophy is that with increasing disorder, the spectrum changes from absolutely continuous to pure point. This is, of course, not a precise statement, but the examples still support it.

In the fourth section those two (families of) models whose study is the backbone of the book enter the stage. The first one describes an ordered solid with defects. The respective hamiltonian is the sum of a periodic operator and a random potential. The second one is used to describe the time evolution of classical waves in a disordered medium. Here the random perturbation concerns the principal part of the operator, which is a second order partial differential operator in divergence form. A first information we obtain in this section is a formula for the almost sure spectrum of our random operators.

In the fifth section, the integrated density of states N is introduced. It is a function $N(E)$ of the energy which measures the number of electrons per unit volume whose energy does not exceed E. It contains valuable information on the random model in question. Its asymptotic behaviour at certain extremal energies is a characteristic feature of disorder. Today it bears the name of its discoverer Lifshitz, whose heuristic argument we present.

In the sixth section we prove Lifshitz asymptotics for the first model. To keep the proof as simple as possible, we don't treat the general case but an easier special case which already captures most of the flavour of the full result.

Closely related to Lifshitz tails is an estimate on the probability that a random operator, restricted to some large cube, exhibits extremal eigenvalues. These estimates run under the header "initial length scale estimates" and are proven in Section 7. The reason for this name is that these estimates serve as an initial step in an induction procedure called multi-scale analysis.

The second main ingredient of this multi-scale analysis is a so-called Wegner estimate. It is used to deal with a small denominator problem which inevitably arises in the multi-scale analysis. Again we do not strive for the technically most advanced form for pedagogical reasons. Quite contrarily, the proof we present is particularly simple but still allows for quite general assumptions.

The estimates proved in Sections 7 and 8 are formulated in terms of the distance of certain fixed energies to the spectrum of random operators. These estimates can typically be translated into exponential decay estimates of the resolvent kernel, the

Combes-Thomas estimates. Even though estimates of this genre have been used quite heavily in various areas of Mathematical Physics, Jean-Michel Combes and Collaborators found a major improvement only recently, which we present in Section 9.

In Section 10 we collect some identities and inequalities relating the resolvents of operators restricted to different cubes. These inequalities will also play a decisive role in multi–scale analysis although they are not really related to random operators. In principle, we deal here with commutator estimates.

The multi–scale analysis is presented in Section 11. It is an induction procedure to prove exponential decay for the resolvent kernel of the operator restricted to large boxes in a certain energy regime with high probability. It is quite hard to describe it in plain words and the statement as well as the proof of the result are highly technical. However, it is worth the effort. This powerful method has applications in different areas as witnessed by the work of Bourgain [10, 11] on nonlinear pde. We have chosen to give it in a rather abstract form. In this way, we can treat simultaneously our two basic models. Moreover, in the axiomatic way we have put it, it also applies to different models. What we prove here is an improved version of the von Dreifus-Klein variable energy method. A remark for specialists is in order. There is an alternative multiscale method, the fixed energy method, which has the advantage that the combinatorics involved are somewhat easier, and so the business is somewhat more transparent. There are, however at least two disadvantages: In the final step one has to apply some sort of Kotani's trick or Simon-Wolff criterion. So it doesn't apply to our divergence form model. The reason is, that in this model the single site space is no more one-dimensional. The second shortcoming is that, to our knowledge, the fixed energy method does not provide the estimates necessary for dynamical localization. Definitely, a self–contained presentation of multi–scale analysis is one of the major goals of this book so that Section 11 contains in a sense the heart of the matter.

Our efforts bear fruit in Section 12 with a proof of exponential localization for our two basic models. We show that, in certain energy intervals, almost all operators in our random family exhibit dense pure point spectrum with exponentially decaying eigenfunctions. To this end we combine the results of Section 11 with some general knowledge on generalized eigenfunction expansions for operators of Schrödinger type. Note that in doing so we prove an "exotic spectral behaviour" which we already mentioned earlier. It is also apparent that localization is not just proven by verifying some criterion for dense pure point spectrum for almost every member of a random family. It is a genuinely random phenomenon which ultimately rests on the "noise" inherent in the random coefficients of the operators we consider. Multi–scale analysis allows us to use this noise in order to prove pure point spectrum. Exponential localization already provides one mathematically rigorous version of the phenomenon that waves are caught by disorder. Namely, for certain energy intervals we only meet bound states. Apart from that the consequences for the dynamics of waves are not satisfactory.

In Section 13 we are finally able to prove dynamical localization in a rather strong form. It is a second look at multi–scale analysis which allows a comparatively easy proof of these dynamical consequences. As a basic input we also need exponential localization from the preceding Section. The strong form of dynamical localization we give here is the following: at time 0 start with an initial wave function supported an a compact set in space and project it onto the energy range in which multi–scale analysis works. Look at the norm of the wave function at time t multiplied by a suitable positive power of the space variables. The supremum over all times of this weighted norms will still be integrable with respect to the underlying probability space. Wow!

The appendix has already been referred to. It is meant as a short story on self adjoint operators containing those parts of the theory which are needed in the text. It does not really contain proofs but rather explanations. With the appendix at hand, it should be possible to follow the main text, although one has to look at respective text books for a deeper understanding.

One of the nice features of this area of research is the number of Theorems still waiting for a proof. Some are mentioned in the Notes and Remarks at the end of Sections and some are gathered in Section 15 which also serves as kind of a "noitcudortni" (an introduction in hindsight).

The book is based on a course I taught at the Johann Wolfgang Goethe-Universität Frankfurt in 1996/97. The material has been complemented considerably. In fact, at the time I gave the course, most of the results and/or the proofs appearing in this book were not known in the present form. Nevertheless, some features of the original lectures remain: instead of giving the most economical presentation, I chose a path which over and again gives heuristic arguments which are gradually becoming more precise, and which lead the way through the admittedly large number of steps which are needed to finally reach the complete proof of localization. In the same spirit, I have not presented the most general results available in order to keep the necessary technicalities at a minimal level. At one point, however, I have tried to be rather general, namely, in the abstract and somewhat axiomatic formulation of multiscale analysis. Firstly, this does not make things harder to prove, and secondly, I hope that there are interesting models to be discovered, for which one might use the more general framework. From what has been said so far, the objective of the present book is clear: its primary aim is to invite the reader to a fascinating circle of ideas and problems and to the mathematical machinery which has recently been built to investigate it. The physical background deals with the foundations of our understanding of how disorder on a microscopical scale is reflected in macroscopical behaviour. The mathematical techniques which have been prompted by this question are definitely nontrivial. To the best of my knowledge this is the first treatment on a textbook level. A reader with some basic understanding of elementary probability theory and a decentbackground in operator theory should be able to understand it. But those prerequisites could be aquired by consulting the appendix. The text might also be useful for specialists in the topic: a number of new ideas have streamlined the exposition of the mathematics of localization.

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Finally, let me thank you, the reader, and express my hope that after reading this book you will be caught by disorder.

Contents

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