Convergence of Schrödinger operators on varying domains

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1. Introduction

Let $H = -\Delta + V$ be a Schrodinger operator on $\mathbb{R}^d$, where $V = V^+ - V^-$ is a potential with negative part in the Kato class and positive part in L^1_{loc}. For any open set G we write H_G for the corresponding selfadjoint operator in $L_2(G)$ which is defined in the following way: denote by $\mathfrak{h}$ the form associated with H and by $\mathfrak{h}_G$ the closure of $\mathfrak{h}|C_c^\infty(G)$; the associated selfadjoint operator is H_G. If $V \in L^2_{loc}$ then H_G is just the Friedrichs extension of $H|C_c^\infty(G)$.

In our talk we discuss the following question: Which kind of convergence $G_n \to G$ of domains implies the convergence $H_{G_n} \to H_G$?

In the following section we present a quite satisfactory answer in terms of strong resolvent convergence. The third section is devoted to two Theorems concerning norm resolvent convergence. In connection with Theorem 3 we use a result which we call the "local test theorem". It proved to be useful also in different situations and seems worth noting. While the results of Section 2 as well as Theorem 2 are taken from [14], Theorem 3 is new. A detailed proof will appear in [15].

We end this introductory section by recalling some potential theoretic notions which we need in the sequel: the *capacity* (more precisely the (1,2)-capacity, see [7]) of an open set is given by

$$\mathrm{cap}(U) = \inf\{\int |\nabla f|^2 + |f|^2 dx; f \in W^{1,2}, f \geq 1\!\!1_U\},$$

and for arbitrary A,

$$\mathrm{cap}(A) = \inf_{A \subset U, U\, open} \mathrm{cap}(U).$$

The importance of this set–function lies in the fact that elements in the Sobolev space $W^{1,2}$ have versions which are defined and continuous up to sets of zero capacity (as cap is larger than the measure of a set, this means additional information on elements of Sobolev space which, a priori, are only defined almost everywhere). More precisely: for $f \in W^{1,2}$ the limit

$$\tilde{f}(x) := \lim_{\varepsilon \to 0} |B_\varepsilon(x)|^{-1} \int_{B_\varepsilon(x)} f(y)dy$$

exists up to a set of zero capacity and, of course, equals f almost everywhere. Moreover, $\tilde{f}$ is *quasi–continuous*, which means that for any $\varepsilon > 0$ one can discard an open set V of capacity less than ε such that $\tilde{f}|V^c$ is continuous. It is easy to see that all quasi–continuous representatives of f coincide with $\tilde{f}$ quasi–everywhere, i.e. up to sets of capacity zero.

Let us mention here some complementary results about Dirichlet boundary conditions which are derived by similar techniques, namely [9], where approximation of Dirichlet boundary conditions by multiplication operators is investigated, and [6] which deals with the lowest eigenvalue of Dirichlet Laplacians.

2. Strong resolvent convergence

In this section we present a criterion for convergence in *strong resolvent sense*

$$H_{G_n} \xrightarrow{srs} H_G.$$

Since the operators involved are acting in different Hilbert spaces we adopt the convention of [11] and extend the resolvent $(H_{G_n} + i)^{-1}$ by zero to all of $L_2(\mathbb{R}^d)$ and use the same symbol for the extended resolvent. Then

$$H_{G_n} \xrightarrow{srs} H_G :\Leftrightarrow (H_{G_n} + i)^{-1} f \to (H_G + i)^{-1} f \quad (f \in L_2(\mathbb{R}^d)).$$

In order to state and prove the main result of this section let us look more closely at the form $\mathfrak{h}_G$ and assume $V = 0$ for simplicity of notation. Then

$$D(\mathfrak{h}_G) = W_0^{1,2}(G).$$

It is a well–known fact of potential theory (cf [7]) that

$$D(\mathfrak{h}_G) = W_0^{1,2}(G) = \{f \in W^{1,2}; \tilde{f} = 0 \text{ q.e. on } G^c\}.$$

Now the right hand side of this equations is suitable to define $\mathfrak{h}_G$ for arbitrary subsets of $\mathbb{R}^d$ and so we define, for $M \subset \mathbb{R}^d$

$$D(\mathfrak{h}_M) := W_0^{1,2}(M) := \{f \in W^{1,2}; \tilde{f} = 0 \text{ q.e. on } G^c\}$$

and denote by H_M the associated selfadjoint operator in $\overline{W_0^{1,2}(M)}^{L_2}$. (See also [4, 5] for the definition of general $W_0^{1,2}$–spaces.)

Although this is merely a matter of definition, the use of these spaces is the key to the following Theorem and its simple proof. The reason is that even if one is only interested in sequences G_n of open sets, the limit of such a sequence need not be open. It might however be equivalent to an open set in the following sense:

$$M \sim M' :\Leftrightarrow W_0^{1,2}(M) = W_0^{1,2}(M').$$

Note that by the very definition $\text{cap}(M \triangle M') = 0$ implies $M \sim M'$. The converse is not true, e.g. $M = \{(x, y); x > 0\} \subset \mathbb{R}^d$, in which case $M \sim \overline{M}$ but $\text{cap}(\overline{M} \setminus M) = \infty$.

THEOREM 1 *Let G_n, G be measurable. If $\overline{\lim}(G_n) \sim \underline{\lim}(G_n) \sim G$, then*

$$H_{G_n} \xrightarrow{srs} H_G.$$

We refer to [14] for a proof and remark that Theorem 1 covers results by Simon [11], Rauch and Taylor [10], and Weidmann [19]. As a consequence of strong resolvent convergence one has convergence of eigenvalues below the essential spectrum; for details see [19].

3. Convergence in norm resolvent sense

In this section we present two results concerning norm resolvent convergence $H_{G_n} \xrightarrow{nrs} H_G$, which means convergence in operator norm of the extended resolvents (see section 2). The next theorem is valid in a quite general framework and based on a convergence theorem for measure perturbations (see [14]):

THEOREM 2 *Let G_n, G be measurable. Assume that $\overline{\lim}(G_n) \sim \underline{\lim}(G_n) \sim G$ and that there is a $\Sigma \subset \mathbb{R}^d$ such that $G_n \triangle G \subset \Sigma$ for all $n \in \mathbb{N}$. Then*

$$H_{G_n} \xrightarrow{nrs} H_G.$$

Unfortunately, this result requires that the symmetric difference of G_n and G be "small at infinity". Thus it cannot be applied to periodic domains as, for instance $G = \mathbb{R}^d$, $G_n = \mathbb{R}^d \setminus \bigcup_{k \in \mathbb{Z}^d} n^{-1}B(k)$, where $B(k)$ denotes a ball of radius 1 centered at k. For these sets it is easy to guess that $H_{G_n} \xrightarrow{nrs} H_G$ and Theorem 2 is obviously not applicable. However one can use

THEOREM 3 *Let G_n, G be measurable and denote by Γ_L the set of all open cubes of sidelength L. If*

$$\sup_{C \in \Gamma_L} \operatorname{cap}((G \triangle G_n) \cap C) \longrightarrow 0 \textit{ for } n \to \infty,$$

then

$$H_{G_n} \xrightarrow{nrs} H_G.$$

Before indicating the ideas of the proof of Theorem 3 let us mention two facts:

- contrary to Theorems 1 and 2, this result uses the geometry of $\mathbb{R}^d$. The condition should be thought of as a uniform local convergence of the sequence G_n to G.
- Secondly, Theorem 3 does not contain Theorem 2 as one can see from the following simple example: $G_n = (1 - n^{-1})B(0), G = B(0)$. In this case $\inf_{n \in \mathbb{N}} \operatorname{cap}(G \setminus G_n) > 0$ while Theorem 2 implies norm resolvent convergence.

For the proof of Theorem 3 we use two main ingredients:

The first one is the following result, parts of which are certainly known to many specialists (see e.g. [8]). Nevertheless we think it is worthwile to isolate it in form of a Theorem. Since we want to apply it to perturbations by potentials and Dirichlet boundary conditions we have chosen to present it in a general version using measure perturbations. To this end we recall that M_0 denotes the class of all non–negative measures which do not charge sets of capacity zero (see [2, 18, 12, 17]) while S_K is the set of measures which satisfy a Kato condition (see [1, 17]). For a measure $\mu \in M_0 - S_K$, i.e. $\mu = \mu^+ - \mu^-, \mu^+ \in M_0, \mu^- \in S_K$ one can define $-\Delta + \mu$ by its quadratic form. Moreover, certain properties of $\exp(-\Delta + \mu)$ only depend on $c_E(\mu)$, which is defined by

$$c_E(\mu) := \sup_{f \in L_1, \|f\| \leq 1} \int (-\Delta + E)^{-1} f d\mu.$$

Those readers who are not familiar with measure perturbations should simply read the μ as a V. Recall that $(-\Delta + \mu)_G$ is the operator on $L_2(G)$ with Dirichlet boundary conditions as defined in Section 2.

THEOREM 4 *(Local test) Let $\mu_n, \mu \in M_0 - S_K$ satisfy $c_E(\mu_n) \leq \gamma$ for all $n \in \mathbb{N}$ and some fixed $\gamma < 1/2, E > 0$. Denote by Γ_L the set of cubes of sidelength L. Consider the conditions*

(i) $-\Delta + \mu_n \xrightarrow{nrs} -\Delta + \mu$.

(ii) *For all $L > 0$:* $\sup_{C \in \Gamma_L} \|(-\Delta + 1\!\!1_C \mu_n + i)^{-1} - (-\Delta + 1\!\!1_C \mu + i)^{-1}\| \to 0$ *for* $n \to \infty$.

(ii') *For all $L > 0$:* $\sup_{C \in \Gamma_L} \|(-\Delta + \mu_n + i)_C^{-1} - (-\Delta + \mu + i)_C^{-1}\| \to 0$ *for* $n \to \infty$.

(iii) *For all (some) $L > 0$:* $\sup_{C \in \Gamma_L} \|(-\Delta + \mu_n + i)^{-1} - (-\Delta + \mu + i)^{-1} 1\!\!1_C\| \to 0$ *for* $n \to \infty$.

Then we have the following implications: (ii)$\Longrightarrow$ (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) *and* (ii') $\Longrightarrow$ (iii). *If, for every $n \in \mathbb{N}$ either $\mu_n \leq \mu$ or $\mu_n \geq \mu$ then* (i)-(iii) *are equivalent.*

Let us sketch the main ideas needed for the proof of the local test theorem:
(ii) $\Longrightarrow$ (iii):
Fix C and let C' be a much bigger cube with the same center. Then $\mu_n \neq 1\!\!1_{C'} \mu_n$ only outside C' so the difference

$$\|(-\Delta + 1\!\!1_{C'} \mu_n + i)^{-1} 1\!\!1_C - (-\Delta + \mu_n + i)^{-1} 1\!\!1_C\| \to 0$$

exponentially in $\mathrm{dist}(C, \mathbb{R}^d \setminus C')$. This can be seen by using techniques from [16]. Moreover the exponential estimate is uniform in n by the assumption on $c_E(\mu_n)$.
(ii') $\Longrightarrow$ (iii) follows by the same arguments.
(iii) $\Longrightarrow$ (i):

Consider a partition of $\mathbb{R}^d$ into unit cubes C_k centered at $k \in \mathbb{Z}^d$ and denote $1\!\!1_k := 1\!\!1_{C_k}$. Then

$$(-\Delta + \mu_n + i)^{-1} - (-\Delta + \mu + i)^{-1} = \sum_{l,k \in \mathbb{Z}^d} 1\!\!1_l [(-\Delta + \mu_n + i)^{-1} - (-\Delta + \mu + i)^{-1}] 1\!\!1_k.$$

Now the terms of the sum converge to zero uniformly in $k, l \in \mathbb{Z}^d$ and moreover we have exponential decay for these terms if k, l are far apart. Using the Cotlar–Stein lemma (see [3]) or direct estimates, the asserted convergence follows (for a similar argument, see [8]).
If $\mu_n \leq \mu$ or vice versa, one can use monotonicity arguments based on the Feynman–Kac formula to derive (ii) and (ii') from (i). $\square$

At this point let me acknowledge helpful comments of E. Mourre who gave me a decisive hint concerning a direct proof of (iii) $\Longrightarrow$ (i) and B. Helffer, who referred me to the Cotlar–Stein lemma.

The interesting implication in the above theorem is of course from (ii) or (ii') to (i). Moreover one is tempted to think that (ii),(ii') and (iii) should easily follow from (i). For (iii) this is obvious, for (ii') we don't know it, but for (ii) it is false, as can be seen from the following example:
Take $V_n := (-1)^n n(1\!\!1_{(-n^{-1},0)} + 1\!\!1_{(0,n^{-1})})$ in $\mathbb{R}$. Then one can check that $-\Delta + V_n \xrightarrow{nrs} -\Delta$. If we take $C = (0,1)$, then $-\Delta + 1\!\!1_C V_n$ won't converge: as $1\!\!1_C V_n = (-1)^n n 1\!\!1_{(0,n^{-1})} =: W_n$ it is easy to see that $-\Delta + W_{2n+k} \xrightarrow{nrs} -\Delta + (-1)^k \delta$ for $k = 0, 1$.

To deduce Theorem 3 with the help of the local test we recall first, that there is a measure $\infty_{G_n^c}$ such that $-\Delta + V + \infty_{G_n^c} = (-\Delta + V)_{G_n} = H_{G_n}$. Let $\mu_n := V + \infty_{G_n^c}$ and $\mu := \infty_{G^c}$. Then $1\!\!1_C \mu_n$ and $1\!\!1_C \mu$ agree outside the set $(G \triangle G_n) \cap C$. The estimate on the capacity of this set allows one to check condition (ii) of Theorem 4 .

This is the second main point in the proof of Theorem 3: obstacles of finite capacity produce perturbations of the semigroup which can be controlled (even with respect to the Hilbert–Schmidt norm) by the capacity. This is the theme of [13] and was also used in [14].

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