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The Feller Property for Absorption Semigroups

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Let $U = (U(t); t \geqslant 0)$ be a substochastic strongly continuous semigroup on $L_1(X, m)$ where X is locally compact and m a Borel measure on X. We give conditions on absorption rates V implying that the (strong) Feller property carries over from U^* to U_V^*. These conditions are essentially in terms of the Kato class associated with U. Preparing these results we discuss the perturbation theory of strongly continuous semigroups and properties of one-parameter semigroups on $L_\infty(m)$. In the symmetric case of Dirichlet forms we generalize the results to measure perturbations. For the case of the heat equation on $\mathbb{R}^d$ we show that the results are close to optimal. © 1996 Academic Press, Inc.

INTRODUCTION

Let X be a locally compact space, m a Radon measure on X, $U = (U(t); t \geqslant 0)$ a strongly continuous symmetric sub-Markov semigroup on $L_2(m)$ (i.e., a semigroup associated with a Dirichlet form in $L_2(m)$.) Assume further that the semigroup U_∞ induced by U on $L_\infty(m)$ satisfies the Feller property, i.e., the restriction of U_∞ to $C_0(X)$ (the continuous functions on X vanishing at infinity) exists and is a strongly continuous

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semigroup. The basic issue of this paper is to present conditions on perturbations of the generator by multiplication operators such that the perturbed semigroup still has the Feller property. In fact the situation described so far is generalized in various ways, and other properties are discussed as well.

As a particular example we mention the semigroup associated with the heat equation on $\mathbb{R}^d$. Adding a suitable (but rather general) absorption rate V one is led to treating Schrödinger operators $-\frac{1}{2}\Delta + V$. It is known that the spectrum of the Schrödinger operator as an operator in $L_p(\mathbb{R}^d)$ does not depend on $p \in [1, \infty)$ (cf. [HV1], [ScV], [A], [D]). Here the Schrödinger operator in $L_p(\mathbb{R}^d)$ is defined as the negative generator of the strongly continuous semigroup associated with the heat equation with absorption

$$u_t = \frac{1}{2}\Delta u - Vu.$$

Recently it was shown in [HV2] that the spectrum of the corresponding operator in $C_0(\mathbb{R}^d)$ coincides with the L_p-spectrum as well whenever it makes sense to speak of this operator, i.e., if the L_p-semigroup induces a strongly continuous semigroup on $C_0(\mathbb{R}^d)$. It was one of the motivations of this paper to investigate circumstances for this to occur. We mention here that the results for this case are contained to a certain extent in [Si], and several of the methods we present here are abstractions or generalizations of ideas contained in that paper.

Investigating the above problem we found that it requires no additional effort to treat the non-symmetric situation. In fact, the general case more clearly exposes the involved structure of the problem. Thus we assume U to be a substochastic strongly continuous semigroup on $L_1(m)$ and treat perturbations by multiplication operators of its generator. The question is then for which absorptions the Feller property carries over from the adjoint semigroup U^* to the adjoint U_V^* of the perturbed semigroup.

As a particular feature of this paper we point out that we do not require any kind of separability of X. Therefore, even in the symmetric case of Dirichlet forms, there is no general construction of a Hunt process associated with the semigroup. The reason that we can dispense with the existence of a process is that our methods are purely functional analytic—although they are influenced to a substantial degree by path integral methods. (We note, however, that a certain process is associated with any substochastic semigroup on any L_1-space; cf. [Sto].)

In Section 1 we give a perturbation theorem for semigroup generators asserting norm convergence of the perturbed semigroups (Theorem 1.2). The important quantity is the Miyadera norm which for potential perturbations of the Laplace operator corresponds to the Kato class norm.

In Section 2 we define several smoothing and localization properties of one-parameter semigroups $W=(W(t); t\geqslant 0)$ on $L_\infty(m)$ where $(X, \mathfrak{B}, m)$ is as above. We delimit several of these properties by examples, and we derive implications between these properties and other notions.

Section 3 contains the main result concerning the Feller property for the general case. At the beginning we recall shortly how, for a positive semigroup U on L_1 and an absorption rate V, the perturbed semigroup U_V is constructed. This construction uses strong convergence of semigroups. Since strong convergence does not dualize we use the results of Section 1 in order to show that the Feller property carries over from U^* to U_V^* if V can be approximated from $C_b(X)$ with respect to the Miyadera norm (Proposition 3.2). The class obtained in this step is too narrow, however, since the conditions are global. In a second step we prove that, in fact, global conditions are only needed for the negative part of V (Theorem 3.3).

In Section 4 we investigate how the smoothing property, i.e., the property that $U^*(t)$ maps $L_\infty(m)$ to $C_b(X)$ for $t>0$, carries over to the perturbed semigroup. It turns out that, assuming the smoothing property, the Feller property carries over under more general conditions than in Section 3 (Theorem 4.5), whereas for the smoothing property to carry over one needs an additional localization property (Theorem 4.6).

In Section 5 we return to the symmetric case. The trade-off for this restriction consists in being able to incorporate perturbations by measures. The method is to approximate measures by functions and to use estimates which are independent of the approximation. In this case we cannot transfer the Feller property without assuming the smoothing property since in general measures cannot be approximated by functions with respect to the Kato class norm. We obtain, however, results completely analogous to those of Section 4 (Theorem 5.5).

In Section 6 we show that the Feller property for the Schrödinger semigroup implies that the perturbation is locally in the Kato class (of measures) if the perturbation is of one sign.

1. On Miyadera Perturbations for Strongly Continuous Semigroups

Let E be a Banach space, $U=(U(t); t\geqslant 0)$ a strongly continuous semigroup on E, with generator T. We call an operator B in E a *Miyadera perturbation* of T if B is T-bounded and there exists $c\geqslant 0$ such that

$$\int_0^1 \|BU(t)x\|\, dt \leqslant c\, \|x\| \qquad (x\in D(T)).$$

We shall denote by

$$\|B\|_U := \sup_{\substack{x \in D(T) \\ \|x\| \leqslant 1}} \int_0^1 \|BU(t)x\|\, dt$$

the *Miyadera norm* of B (with respect to U). Further, for $\alpha > 0$ we introduce

$$c'_\alpha(B) := \sup_{\substack{x \in D(T) \\ \|x\| \leqslant 1}} \int_0^\alpha \|BU(t)x\|\, dt$$

We refer to [Mi1], [Mi2], [Vo1] for this kind of perturbation. In particular we recall that $c'_\alpha(B) < 1$ for some $\alpha > 0$ implies that $T + B$ is a generator ([Mi2], [Vo1]).

1.1. LEMMA. *Let U be a C_0-semigroup, T its generator, B, B_1 Miyadera perturbations of T,*

$$\int_0^\alpha \|BU(t)x\|\, dt \leqslant \gamma\, \|x\|,$$

$$\int_0^\alpha \|B_1 U(t)x\|\, dt \leqslant \gamma_1\, \|x\| \qquad (x \in D(T)),$$

where $\alpha > 0$, $0 \leqslant \gamma < 1$, $0 \leqslant \gamma_1$, and denote by $\tilde{U}$ the C_0-semigroup generated by $T + B$.

Then B_1 is a Miyadera perturbation of $T + B$, and

$$\int_0^\alpha \|B_1 \tilde{U}(t)x\|\, dt \leqslant \frac{\gamma_1}{1-\gamma}\, \|x\| \qquad (x \in D(T)).$$

Proof. Let $x \in D(T)$. Then the Duhamel formula

$$\tilde{U}(t)x = U(t)x + \int_0^t U(t-s)\, B\tilde{U}(s)x\, ds$$

holds for $t \geqslant 0$. For λ larger than the type of U we define $C_\lambda := \lambda(\lambda - T)^{-1}$ and conclude

$$\int_0^\alpha \|B_1 C_\lambda \tilde{U}(t) x\| \, dt$$

$$\leqslant \int_0^\alpha \|B_1 C_\lambda U(t) x\| \, dt + \int_0^\alpha \int_s^\alpha \|B_1 U(t-s) \, C_\lambda B \tilde{U}(s) x\| \, dt \, ds$$

$$\leqslant \int_0^\alpha \|B_1 C_\lambda U(t) x\| \, dt + \gamma_1 \int_0^\alpha \|C_\lambda B \tilde{U}(s) x\| \, ds.$$

For $\lambda \to \infty$ the operators C_λ converge strongly to the identity on E as well as on $D(T)$ (with graph norm). Therefore

$$\int_0^\alpha \|B_1 \tilde{U}(t) x\| \, dt \leqslant \gamma_1 \|x\| + \gamma_1 \int_0^\alpha \|B \tilde{U}(s) x\| \, ds.$$

Exploiting this inequality first with $B_1 = B$ we obtain

$$\int_0^\alpha \|B \tilde{U}(t) x\| \, dt \leqslant \frac{\gamma}{1-\gamma} \|x\|.$$

Inserting this into the previous inequality we obtain the assertion. ∎

1.2. THEOREM. *Let U be a C_0-semigroup, T its generator, B, B_j $(j \in \mathbb{N})$ Miyadera perturbations of T, $\alpha > 0$, $0 \leqslant \gamma < 1$,*

$$c'_\alpha(B_j) \leqslant \gamma \qquad (j \in \mathbb{N}),$$

$$\|B_j - B\|_U \to 0 \qquad (j \to \infty).$$

(Note that this implies $c'_\alpha(B) \leqslant \gamma$.) Denote by $\tilde{U}$, $\tilde{U}_j$ the C_0-semigroups generated by $T + B$, $T + B_j$ $(j \in \mathbb{N})$, respectively. Then $\sup_{0 \leqslant t \leqslant 1} \|\tilde{U}_j(t) - \tilde{U}(t)\| \to 0$ $(j \to \infty)$.

Proof. In view of Lemma 1 we may assume $B = 0$. There exists $M \geqslant 1$ such that $\|\tilde{U}_j(t)\| \leqslant M$ $(j \in \mathbb{N}, \ 0 \leqslant t \leqslant 1)$; cf. [Vo1; Theorem 1]. Now the Duhamel formula, for $x \in D(T)$,

$$\tilde{U}_j(t) x - U(t) x = \int_0^t \tilde{U}_j(t-s) \, B_j U(s) x \, ds,$$

implies

$$\|(\tilde{U}_j(t) - U(t)) x\| \leqslant M \int_0^t \|B_j U(s) x\| \, ds,$$

and therefore the assertion is obtained. ∎

2. Some Properties of Semigroups on L_∞

In this section let X be a locally compact space, m a measure on the Borel σ-algebra $\mathfrak{B}$ of X having the properties that $C_0(X)$ separates the functions of $L_1(m)$ and that $C_0(X) \to L_\infty(m)$ is injective. For notational convenience we also assume that $L_\infty(m)$ is the dual of $L_1(m)$. These assumptions are satisfied if m is a Radon measure having supp $m = X$.

By $\mathscr{K}$ we shall denote the system of compact subsets of X. Further, $C_c(X)$ will be the space of continuous functions with compact support, $C_0(X)$ the space of continuous functions vanishing at infinity, and $C_b(X)$ the space of bounded continuous functions.

We assume that $W = (W(t); t \geqslant 0)$ is a one-parameter semigroup of positive operators on $L_\infty(m)$, $\sup_{0 \leqslant t \leqslant 1} \|W(t)\| < \infty$. First we define several possible properties which W might have.

(F) (*Feller property*) $W(t)(C_0(X)) \subset C_0(X)$ for all $t > 0$, and the restriction of W to $C_0(X)$ is strongly continuous.

(WL) (*weak localization property*) $\forall K \in \mathscr{K}\ \exists K' \in \mathscr{K}$:

$$\|1_K W(t)\, 1_{X \setminus K'}\|_{\infty,\infty} \to 0 \qquad (t \to 0).$$

(L) (*localization property*) $\forall K \in \mathscr{K}\ \exists \alpha > 0\ \forall \varepsilon > 0\ \exists K' \in \mathscr{K}$:

$$\|1_K W(t)\, 1_{X \setminus K'}\|_{\infty,\infty} \leqslant \varepsilon \qquad (0 \leqslant t \leqslant \alpha).$$

(SL) (*strong localization property*) $\forall K \in \mathscr{K}\ \forall \varepsilon > 0\ \exists K' \in \mathscr{K}$:

$$\|1_K W(t)\, 1_{X \setminus K'}\|_{\infty,\infty} \leqslant \varepsilon \qquad (0 \leqslant t \leqslant 1).$$

(S) (*smoothing property*) $W(t)(L_\infty(m)) \subset C_b(X)$ $(t > 0)$.

(SF) (*strong Feller property*) W possesses properties (F) and (S).

2.1. *Remarks.* (a) If W is the adjoint semigroup U^* of some positive strongly continuous semigroup U on $L_1(m)$, then the norm in the localization properties can be written in the form $\|1_{X \setminus K'} U(t) 1_K\|_{1,1}$, and these properties express, in a certain way, that the mass transport described by U does not transport mass too far away in short times.

(b) In (SL) it is equivalent to require that the inequality should hold on any fixed t-interval. E.g., given $K \in \mathscr{K}$, $\varepsilon > 0$, then choosing K' such that $\|1_K W(t)\, 1_{X \setminus K'}\|_{\infty,\infty} \leqslant \varepsilon$ $(0 \leqslant t \leqslant 1)$ and K'' such that $\|1_{K'} W(t)\, 1_{X \setminus K''}\|_{\infty,\infty} \leqslant \varepsilon$ $(0 \leqslant t \leqslant 1)$ we find, for $0 \leqslant t \leqslant 2$,

$$\begin{aligned}\|1_K W(t)\, 1_{X \setminus K''}\|_{\infty,\infty} &= \|1_K W(t/2)(1_{K'} + 1_{X \setminus K'})\, W(t/2)\, 1_{X \setminus K''}\|_{\infty,\infty} \\ &\leqslant 2\varepsilon \sup_{0 \leqslant s \leqslant 1} \|W(s)\|.\end{aligned}$$

(c) Clearly, there are the implications (SL) $\Rightarrow$ (L) $\Rightarrow$ (WL). In the following sections, property (L) will be used as an hypothesis. On the other hand, the stronger property (SL) can by characterized in different other ways; this will be presented in Theorem 2.4 below.

We include several examples illustrating the properties defined before.

2.2. Examples. (a) X not discrete, $W(t) = id_{L_\infty}$ $(t \geqslant 0)$. Then (F), (SL), not (S).

(b) $X = \mathbb{R}^d$ with Borel–Lebesgue measure, W Brownian semigroup on L_∞. Then (SF), (SL).

(c) $X = \mathbb{R}^d \setminus \{0\}$, W as in (b). Then (S), (SL), not (F) $(W(t)(C_0(X)) \not\subset C_0(X))$.

(d) $X = \mathbb{R}^d$, U the semigroup generated by $\frac{1}{2}\Delta - 1/|x|^2$ on $L_1(\mathbb{R}^d)$ (cf. [Vo2]), W the adjoint semigroup on $L_\infty(\mathbb{R}^d)$. Then W satisfies (S), (SL), not (F). In fact $W(t)(C_0(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d)$, but W is not strongly continuous on $C_0(\mathbb{R}^d)$. In order to show this we note first that the expression given by the Feynman–Kac formula

$$E_x\left[\exp\left(-\int_0^t V(b(s))\,ds\right)\varphi(b(t))\right]$$

$$= E_0\left[\exp\left(-\int_0^t V(x+b(s))\,ds\right)\varphi(x+b(t))\right]$$

is continuous if $V \in C_b(\mathbb{R}^d)$, $\varphi \in C_0(\mathbb{R}^d)$, as can be seen from the dominated convergence theorem applied in Wiener space. Here E_x denotes expectation with respect to the Wiener measure P_x for Brownian motion starting at $x \in \mathbb{R}^d$.

Let $V(x) := 1/|x|^2$, $V^{(n)} := V \wedge n$. Let $\varphi \in C_0(\mathbb{R}^d)$, $\varphi \geqslant 0$, $u(t) = e^{t((1/2)\Delta - V)}\varphi$. Then u is continuous on $[0, \infty) \times (\mathbb{R}^d \setminus \{0\})$, by parabolic regularity (cf. [ArSe; Theorem 4]). Monotonicity implies

$$0 \leqslant \varlimsup_{x \to 0} u(t, x) \leqslant e^{t((1/2)\Delta - V^{(n)})}\varphi(0)$$

$$= E_0\left[\exp\left(-\int_0^t V^{(n)}(b(s))\,ds\right)\varphi(b(t))\right].$$

Now, $\int_0^t (1/b(s)^2)\,ds = \infty$ P_0-*a.s.* (cf. [FOT; Example 5.1.1]), and so the right hand side of the last inequality goes to 0 for $n \to \infty$. It follows that $W(t)\varphi$ is continuous and $W(t)\,\varphi(0) = 0$ for $t > 0$. Therefore $W(t)\varphi$ cannot converge to φ for $t \to 0$ if $\varphi(0) \neq 0$.

(e) $X=(0,1)$, $W(t)\,f(x):=\int_0^1 f(y)\,dy$ $(x\in X,\ f\in L_\infty,\ t>0)$. Then (S), not (F), not (WL).

(f) $X=(0,1)$, $$W(t)\,f(x):=\begin{cases} f(x-t) & \text{if } x>t,\\ 0 & \text{if } x\leqslant t.\end{cases}$$

Then (L), not (SL), not (F), not (S).

(g) $X=(0,\infty)$, $$W(t)\,f(x):=\begin{cases} f(x-t) & \text{if } x>t,\\ 0 & \text{if } x\leqslant t.\end{cases}$$

Then (F), (L), not (SL), not (S).

(h) Let $X=\mathbb{T}^{\mathbb{R}}$ with the product topology, where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is the torus. Then X is not separable.

Choose $y\in X$ and define the semigroup of translations in direction y on $L_\infty(X)$ by

$$W(t)\,f(x):=f(x-ty).$$

($x-ty$ is to be understood coordinate-wise and modulo 1.)

Then (F), (SL), not (S). (For (F) note that $x\mapsto x-ty$ is continuous and therefore $W(t)\,C(X)=C(X)$. Further $\|W(t)\,f-f\|_\infty\to 0$ $(t\to 0)$ since this is true for functions depending only on finitely many coordinates, and the set of these functions is dense in $C(X)$. If W is extended to a one-parameter group—by the above expression for $W(t)$—then $W(t)\,C(X)=C(X)$ holds for all $t\in\mathbb{R}$. This implies that (S) does not hold.)

(i) Let X be as in (h), and let $m=(\lambda_1\mid[0,1])^{\mathbb{R}}$. Define $(U(t);\,t\geqslant 0)$ on $L_1(m)$ as the tensor product of periodic Brownian motions on all the coordinates. Then U is associated with a Dirichlet form in $L_2(m)$. For U^* one has (SF), (SL).

2.3. Proposition. *Assume that W satisfies* (F). *Then*:

(a) (WL) $\Leftrightarrow \|1_K(W(t)\,f-f)\|_\infty\to 0$ $(t\to 0)$ *for all* $f\in C_b(X)$, $K\in\mathscr{K}$.

(b) *If* $\limsup_{t\to 0}\|W(t)\|\leqslant 1$ *then* (WL) *holds for* W.

Proof. (a) "$\Rightarrow$" Let $f\in C_b(X)$, $K\in\mathscr{K}$, and choose K' according to (WL). (Note that automatically $K\subset K'$.) Further, choose $\psi\in C_c(X)$, $1_{K'}\leqslant\psi\leqslant 1$. Then

$$1_K W(t)\,f=1_K W(t)(\psi f)+1_K W(t)((1-\psi)\,f)\to 1_K\psi f+0=1_K f \qquad (t\to 0)$$

uniformly.

"$\Leftarrow$" Let $K \in \mathscr{K}$, and $K' \in \mathscr{K}$ such that $K \subset \mathring{K}'$. There exists $\psi \in C_c(X)$, $1_K \leqslant \psi \leqslant 1_{K'}$. Then

$$\|1_K W(t)\, 1_{X \setminus K'}\|_{\infty,\infty} \leqslant \|1_K W(t)(1-\psi)\|_\infty \to \|1_K(1-\psi)\|_\infty = 0 \qquad (t \to 0).$$

(b) Let $K \in \mathscr{K}$, and $K' \in \mathscr{K}$ with $K \subset \mathring{K}'$, $\psi \in C_c(X)$ with $1_K \leqslant \psi \leqslant 1_{K'}$. Then

$$W(t)\psi + W(t)(1-\psi) = W(t)1 \leqslant \|W(t)\|,$$
$$0 \leqslant W(t)(1-\psi) \leqslant \|W(t)\| - W(t)\psi \to 0 \qquad (t \to 0)$$

uniformly on K, and therefore

$$\|1_K W(t)\, 1_{X \setminus K'}\|_{\infty,\infty} \leqslant \|1_K W(t)(1-\psi)\|_\infty \to 0 \qquad (t \to 0). \quad \blacksquare$$

If W satisfies (F) then for all $t \geqslant 0$, $x \in X$ there is a positive Radon measure $p(t, x, \cdot)$ on X such that

$$W(t)\, \varphi(x) = \int \varphi(y)\, p(t, x, dy)$$

for all $\varphi \in C_0(X)$. For $t \geqslant 0$, $f \in C_b(X)$ one can then define

$$\widetilde{W}(t)\, f(x) := \int f(y)\, p(t, x, dy) \qquad (x \in X).$$

2.4. Theorem. *Assume that W satisfies* (F), *and let $p(\cdot, \cdot, \cdot)$ be as just defined. Then the following properties are equivalent*:

(i) *W satisfies* (SL).

(ii) *On $\|\cdot\|_\infty$-bounded subsets of $C_0(X)$ the set $\{W(t); 0 \leqslant t \leqslant 1\}$ is equicontinuous for the topology of compact convergence.*

(iii) $\forall K \in \mathscr{K}$, $\varepsilon > 0$ $\exists K' \in \mathscr{K}$: *$p(t, x, X \setminus K') \leqslant \varepsilon$ for all $0 \leqslant t \leqslant 1$, $x \in K$.*

(iv) *The function F, defined by $F(t, x) := p(t, x, X)$, is continuous on $[0, 1] \times X$.*

If these conditions are satisfied then

$$\widetilde{W}(t)(C_b(X)) \subset C_b(X) \qquad (t \geqslant 0).$$

Proof. (i) $\Rightarrow$ (ii). Let $c > 0$, $K \in \mathscr{K}$, $\varepsilon > 0$. Then there exists $K' \in \mathscr{K}$ such that $\|1_K W(t)\, 1_{X \setminus K'}\|_{\infty,\infty} \leqslant \varepsilon$ $(0 \leqslant t \leqslant 1)$. For $\varphi \in C_0(X)$, $\|\varphi\|_\infty \leqslant c$, $\|1_{K'}\varphi\|_\infty \leqslant \varepsilon$ we conclude

$$\begin{aligned}\|1_K W(t)\varphi\| &\leqslant \|1_K W(t)(1_{K'}\varphi)\| + \|1_K W(t)(1_{X \setminus K'}\varphi)\| \\ &\leqslant \varepsilon \sup_{0 \leqslant t \leqslant 1} \|W(t)\| + \varepsilon c.\end{aligned}$$

(ii) $\Rightarrow$ (iii). Let $K \in \mathscr{K}$, $\varepsilon > 0$. Then there exist $K' \in \mathscr{K}$, $\delta > 0$ such that $\varphi \in C_0(X)$, $\|1_{K'} \varphi\|_\infty \leqslant \delta$, $\|\varphi\|_\infty \leqslant 1$ imply $\|1_K W(t) \varphi\| \leqslant \varepsilon$ $(0 \leqslant t \leqslant 1)$. Let $0 \leqslant t \leqslant 1$, $x \in K$, $\varphi \in C_0(X)$, $\operatorname{supp} \varphi \subset X \backslash K'$, $\|\varphi\|_\infty \leqslant 1$. Then

$$\left| \int \varphi(y)\, p(t, x, dy) \right| = |W(t)\, \varphi(x)| \leqslant \varepsilon.$$

Taking the supremum over the φ on the left hand side we obtain $p(t, x, X \backslash K') \leqslant \varepsilon$.

(iii) $\Rightarrow$ (iv). Let $K \in \mathscr{K}$. For $\varepsilon > 0$ there exists $K' \in \mathscr{K}$ such that $p(t, x, X \backslash K') \leqslant \varepsilon$ for all $0 \leqslant t \leqslant 1$, $x \in K$. Choose $\psi \in C_0(X)$, $1_{K'} \leqslant \psi \leqslant 1$. Then $|p(t, x, X) - W(t)\, \psi(x)| \leqslant p(t, x, X \backslash K') \leqslant \varepsilon$ for all $(t, x) \in [0, 1] \times K$. Since $(t, x) \mapsto W(t)\, \psi(x)$ is continuous we obtain that $p(\cdot, \cdot, X)$ is continuous on $[0, 1] \times K$.

(iv) $\Rightarrow$ (i). The set $\mathscr{F} := \{\psi \in C_c(X);\, 0 \leqslant \psi \leqslant 1\}$ is directed $(\leqslant)$ by the order of functions. For each $(t, x) \in [0, 1] \times X$ the net $(W(t) \psi)_{\psi \in \mathscr{F}}$ converges to $F(t, x)$. Dini's theorem (cf. [Bou; X, 34]) implies that this convergence is uniform on compact sets.

Let $K \in \mathscr{K}$, $\varepsilon > 0$. Then there exists $\psi \in \mathscr{F}$ such that

$$\begin{aligned} \|1_K W(t)\, 1_{X \backslash \operatorname{supp} \psi}\|_{\infty, \infty} &\leqslant \|1_K W(t)(1 - \psi)\|_{\infty, \infty} \\ &= \|1_K (F(t, \cdot) - W(t) \psi)\|_\infty \\ &\leqslant \varepsilon \qquad (0 \leqslant t \leqslant 1). \end{aligned}$$

In order to prove the last assertion of the theorem we use property (iii). Let $f \in C_b(X)$. For $K \in \mathscr{K}$, $\varepsilon > 0$ there exists $K' \in \mathscr{K}$ as indicated. Let $\psi \in C_0(X)$, $1_{K'} \leqslant \psi \leqslant 1$. For $(t, x) \in [0, 1] \times K$ we obtain

$$\begin{aligned} |W(t)\, f(x) - W(t)(\psi f)(x)| &= \left| \int f(y)(1 - \psi(y))\, p(t, x, dy) \right| \\ &\leqslant \|f\|_\infty\, p(t, x, X \backslash K') \\ &\leqslant \varepsilon\, \|f\|_\infty. \end{aligned}$$

Thus $W(t) f$ is approximated uniformly on K by continuous functions. ∎

2.5. *Remark*. Assume that W satisfies (F) and (SL) and that the operators $W(t)$ are order continuous, i.e., $\sup W(t) \mathscr{F} = W(t) \sup \mathscr{F}$ for all sets $\mathscr{F} \subset L_\infty(m)_+$ which are bounded and directed by $\leqslant$, for all $t \geqslant 0$. Then one has $\widetilde{W} = W$, and therefore the last statement of Theorem 2.4 applies to W.

In order to see this, let $f \in C_b(X)_+$. Then $\mathscr{F} := \{\varphi \in C_c(X); 0 \leqslant \varphi \leqslant f\}$ is directed by $\leqslant$, and $\sup \mathscr{F} = f$ in $L_\infty(m)$. Consequently $W(t) f = \sup_{\varphi \in \mathscr{F}} W(t)\varphi = \sup_{\varphi \in \mathscr{F}} \tilde{W}(t)\varphi = \tilde{W}(t) f$, where the last equality follows from the fact that $\tilde{W}(t) f = \lim_{\varphi \in \mathscr{F}} \tilde{W}(t)\varphi$ in the topology of compact convergence (property (ii) of Theorem 2.3).

If U is a positive C_0-semigroup on $L_1(m)$ and $W = U^*$, then it is easy to see that the operators $W(t)$ are order continuous.

3. The Feller Property for Adjoint Absorption Semigroups

Let X and m be as in Section 2. We assume that $U = (U(t); t \geqslant 0)$ is a positive C_0-semigroup of contractions on $L_1(m)$, with generator T.

The *enlarged Kato class* $\hat{\mathbf{K}}$ associated with U is defined as

$$\hat{\mathbf{K}} := \{ V\colon X \to [-\infty, \infty] \text{ locally measurable; } V(1-T)^{-1} \text{ bounded}\}.$$

For $V \in \hat{\mathbf{K}}$, $\beta > 0$ we define

$$c_\beta(V) := \| V(\beta - T)^{-1} \|,$$

$$c(V) := \lim_{\beta \to \infty} c_\beta(V) = \inf_{\beta > 0} c_\beta(V).$$

Then

$$\mathbf{K} := \{ V \in \hat{\mathbf{K}}; c(V) = 0\}$$

is the *Kato class* associated with U.

We refer to [Vo2; Proposition 5.1] for the proof that the Kato class as defined above coincides with the usual Kato class if U is the unperturbed Schrödinger semigroup. Further, we note that

$$\hat{\mathbf{K}} = \left\{ V; \|V\|_U = \sup_{\substack{f \in D(T) \\ \|f\| \leqslant 1}} \int_0^1 \| VU(t) f \| \, dt < \infty \right\}$$

which means that $\hat{\mathbf{K}}$ consists of Miyadera perturbations of T. Recall the definition of the norm $\|\cdot\|_U$ from Section 1; this norm will be used as the norm on $\hat{\mathbf{K}}$. It is easy to verify that $\mathbf{K}$ is a closed subspace of $\hat{\mathbf{K}}$. Moreover it is shown in [Vo4; Proposition 3.2] that $\hat{\mathbf{K}}$ is complete. With c'_α from Section 1 one obtains

$$c(V) = \lim_{\alpha \to 0} c'_\alpha(V) = \inf_{\alpha > 0} c'_\alpha(V).$$

For these statements we refer to [Vo2; Proposition 4.7]. The condition that U^* satisfies the Feller property implies in particular that $C_0(X)$ is contained in the sun dual $L_1(m)^{\odot}$ associated with U. (For the theory of adjoint semigroups we refer to [HP], [vN].)

The following examples illustrate typical situations covered by the previous assumptions.

3.1. EXAMPLES. (a) $(X, \mathfrak{B}, m)$ as above, and $U(t) = \mathrm{id}_{L_1(m)}$, $U^*(t) = \mathrm{id}_{L_\infty(m)}$ $(t \geqslant 0)$. In this case

$$\hat{\mathbf{K}} = \mathbf{K} = L_\infty(m), \qquad \|\cdot\|_U = \|\cdot\|_\infty.$$

(b) $X = \mathbb{R}$ with Borel–Lebesgue measure, $U(t)$ right translation by t, U^* left translation by t $(t \geqslant 0)$. Here

$$\hat{\mathbf{K}} = L_{1,\, loc,\, unif}(\mathbb{R}),$$

$$\|V\|_U = \sup_{x \in \mathbb{R}} \int_x^{x+1} |V(y)|\, dy,$$

$$\mathbf{K} = \left\{ V \in L_{1,\, loc,\, unif}(\mathbb{R});\ \sup_x \int_x^{x+\alpha} |V(y)|\, dy \to 0\ (\alpha \to 0) \right\}.$$

(c) $X = \mathbb{R}^d$ with Borel–Lebesgue measure, U the strongly continuous semigroup generated by $\frac{1}{2}\Delta$. Then $U^*(t) = U(t)$ on $L_1 \cap L_\infty(\mathbb{R}^d)$ $(t \geqslant 0)$. In this case $\hat{\mathbf{K}}$ and $\mathbf{K}$ are the known (enlarged) Kato classes $\hat{\mathbf{K}}_d$ and $\mathbf{K}_d$; cf. [AS], [Si], [Vo2]. For $d \geqslant 3$ they are as indicated in (c) below, with $\alpha = 2$.

(d) $X = \mathbb{R}^d$ as in (c), U the semigroup generated by $-|-\Delta|^{\alpha/2}$, where $0 < \alpha < 2$; this semigroup is called the symmetric stable semigroup (of index α) (cf. [FOT]). For $\alpha < d$ one has

$$\hat{\mathbf{K}} = \left\{ V \in L_{1,\, loc}(\mathbb{R}^d);\ \sup_x \int_{|x-y| \leqslant 1} \frac{|V(y)|}{|x-y|^{d-\alpha}}\, dy < \infty \right\},$$

$$\mathbf{K} = \left\{ V \in \hat{\mathbf{K}};\ \lim_{r \downarrow 0} \left[\sup_x \int_{|x-y| \leqslant r} \frac{|V(y)|}{|x-y|^{d-\alpha}}\, dy \right] = 0 \right\};$$

cf. [Zh; Theorem 2]. The expression in the above description of $\hat{\mathbf{K}}$ is not the norm $\|\cdot\|_U$, but equivalent to it. This is an example with a non-local generator (and with the underlying process having discontinuous paths).

If $V\colon X \to [0, \infty]$ is locally measurable, $V^{(n)} := V \wedge n$ $(n \in \mathbb{N})$, then

$$U_V(t) := s - \lim_{n \to \infty} e^{t(T - V^{(n)})}$$

exists for all $t > 0$, and has the semigroup property. It was shown in [AB; Corollary 3.3] that $U_V(0) := s - \lim_{t \to 0} U_V(t)$ exists and is a band projection, and therefore U_V is a strongly continuous semigroup on $L_1(\check{X})$ for suitable $\check{X} \subset X$, and U_V vanishes on $L_1(X \setminus \check{X})$. The function V is called U-admissible if $U_V(0) = I$.

On the other hand, $-V$ is called U-admissible if

$$U_{-V}(t) := s - \lim_{n \to \infty} e^{t(T + V^{(n)})}$$

exists for all $t \geqslant 0$ and $\sup_{0 \leqslant t \leqslant 1} \|U_V(t)\| < \infty$. (Then U_{-V} is strongly continuous by [Vo3; Proposition A.1].)

If $V_\pm : X \to [0, \infty]$ are locally measurable and $-V_-$ is U-admissible then

$$(U_{V_+})_{-V_-} = (U_{-V_-})_{V_+} =: U_V \tag{3.1}$$

(where in the first term one considers U_{V_+} as a strongly continuous semigroup on $L_1(\check{X})$). For the case that V_+ is also U-admissible this statement is shown in [Vo3; Theorem 2.6]. The proof given there carries over as soon as one knows that it is true for bounded V^-. For this case, however, (3.1) is an easy consequence of [AB; Proposition 4.6]. Note that $V_\pm$ are not necessarily the positive and negative parts of $V = V_+ - V_-$, where (3.1) together with [Vo2; Theorem 2.6] also implies that it is irrelevant how V is defined at those points where V_+ as well as V_- are infinite.

For the above definitions and for further information we refer the reader to [Vo2], [Vo3], [AB].

3.2. Proposition. *Assume that* U^* *has the Feller property. Let* $V \in \overline{C_b(X)}^{\mathbf{K}}$. *Then* U_V^* *has the Feller property.*

Proof. Note first that $V \in \mathbf{K}$ implies $c(V) = 0$, $c'_\alpha(V) < 1$ for some $\alpha > 0$. Denote the restriction of U^* to $C_0(X)$ by U_0^*, and its generator by T_0^* (the restriction of T^* to $C_0(X)$). For $V \in C_b(X)$ the operator V is continuous in $C_0(X)$, and therefore $T_0^* - V$ is the generator of a strongly continuous semigroup $U_{0,V}^*$ which is adjoint to U_V, and therefore $U_{0,V}^*$ is the restriction of U_V^* to $C_0(X)$.

Let $V \in \overline{C_b(X)}^{\mathbf{K}}$, (V_n) in $C_b(X)$, $\|V - V_n\|_U \to 0$. Theorem 1.2 implies

$$\sup_{0 \leqslant t \leqslant 1} \|U_{V_n}^*(t) - U_V^*(t)\|_{\infty,\infty} = \sup_{0 \leqslant t \leqslant 1} \|U_{V_n}(t) - U_V(t)\|_{1,1} \to 0 \quad (n \to \infty). \qquad \blacksquare$$

The following theorem is the main result of this section.

3.3. THEOREM. *Assume that* U^* *satisfies* (F). *Let* $V = V_+ - V_-$, $V_\pm \geqslant 0$ *be measurable functions on* X *such that* ψV *is U-admissible and* $U^*_{\psi V}$ *satisfies* (F) *for all* $\psi \in C_c(X)_+$. (*By Proposition* 3.2, *this holds if* $\psi V \in \overline{C_b(X)}^{\mathbf{K}}$ *for all* $\psi \in C_c(X)_+$.) *Furthermore assume that* $V_- \in \hat{\mathbf{K}}$, $c(V_-) < 1$. *Then* V *is U-admissible, and* U^*_V *satisfies* (F).

Note that $V \in \overline{C_b(X)}^{\mathbf{K}}$ clearly implies $\psi V \in \overline{C_b(X)}^{\mathbf{K}}$ for all $\psi \in C_b(X)$. This property, however, does not carry over to all $\psi \in L_\infty(m)$, as can be seen easily by considering Example 3.1(a). This is the reason why localization has to be carried out more cautiously than by simply cutting off by indicator functions.

For the proof of Theorem 3.3 we shall need several auxiliary results which will be presented next.

3.4. LEMMA. *Let* $V = V_+ - V_-$ *where* $V_\pm := X \to [0, \infty]$ *are locally measurable, and* $-V_-$ *is U-admissible. Let* $\mathscr{F} \subset L_\infty(m)_+$ *be directed by* $\leqslant$, *with* $\sup \mathscr{F} = 1$. *Then*

$$U_V(t) = s - \lim_{\psi \in \mathscr{F}} U_{\psi V}(t)$$

for all $t > 0$.

Proof. (i) Assume first $V_- = 0$. Fix $t > 0$. The net $(U_{\psi V}(t))_{\psi \in \mathscr{F}}$ is decreasing, and therefore $\tilde{U}_V(t) := s - \lim_{\psi \in \mathscr{F}} U_{\psi V}(t)$ exists. From $U_V(t) \leqslant U_{\psi V}(t)$ $(\psi \in \mathscr{F})$ we obtain $U_V(t) \leqslant \tilde{U}_V(t)$. On the other hand, for $n \in \mathbb{N}$ one has

$$U_{V^{(n)}}(t) = s - \lim_{\psi \in \mathscr{F}} U_{\psi V^{(n)}}(t)$$

by the Trotter convergence theorem (cf. [Pa]; this theorem is usually stated for sequences of semigroups but holds similarly in the above situation). Because of $U_{\psi V^{(n)}}(t) \geqslant U_{\psi V}(t)$ we obtain $U_{V^{(n)}}(t) \geqslant \tilde{U}_V(t)$ $(n \in \mathbb{N})$, hence $U_V(t) \geqslant \tilde{U}_V(t)$.

(ii) Assume $V_+ = 0$. Reversing the inequality signs in (i) yields the statement in this case.

(iii) General case. Fix $t > 0$. For $\psi \in \mathscr{F}$ we note the inequalities

$$-V_- + \psi V_+ \leqslant \psi V \leqslant -\psi V_- + V_+,$$

which imply (recall (3.1))

$$(U_{-V_-})_{\psi V_+}(t) \geqslant U_{\psi V}(t) \geqslant (U_{V_+})_{-\psi V_-}(t).$$

Since the outer terms converge strongly to $U_V(t)$ by (i), (ii) and (3.1) we obtain the desired convergence. ∎

3.5. LEMMA. *Let $V = V_+ - V_-$ where $V_\pm : X \to [0, \infty]$, V_+ U-admissible, V_- a Miyadera perturbation of T, with $\alpha > 0$, $\gamma \in [0, 1)$ such that*

$$\left\| \int_0^\alpha V_- U(t)\, dt \right\|_{1,1} \leqslant \gamma.$$

Then $V_+ + V_-$ is a Miyadera perturbation of T_V (the generator of U_V), with

$$\left\| \int_0^\alpha (V_+ + V_-)\, U_V(t)\, dt \right\|_{1,1} \leqslant \frac{1+\gamma}{1-\gamma}.$$

Proof. From [Vo2; Lemma 4.1] we know

$$\left\| \int_0^\alpha V_+ U_{V_+}(t)\, dt \right\|_{1,1} \leqslant 1.$$

Therefore Lemma 1.1 implies

$$\left\| \int_0^\alpha V_+ (U_{V_+})_{-V_-}(t)\, dt \right\|_{1,1} \leqslant \frac{1}{1-\gamma}.$$

Also, Lemma 1.1 implies

$$\left\| \int_0^\alpha V_- U_V(t)\, dt \right\|_{1,1} \leqslant \left\| \int_0^\alpha V_- U_{-V_-}(t)\, dt \right\|_{1,1} \leqslant \frac{\gamma}{1-\gamma}. \quad ∎$$

The following lemma is analogous to [Si; Lemma B.4.1].

3.6. LEMMA. *Let $V \leqslant 0$, and assume that there exists $c > 1$ such that cV is U-admissible. Let $f, g \in L_\infty(m)_+$. Then, for all $t \geqslant 0$,*

$$\| f U_V(t)\, g \|_{1,1} \leqslant \| f U_{cV}(t)\, g \|_{1,1}^{1/c}\ \| f U(t)\, g \|_{1,1}^{1-1/c}.$$

Proof. Stein interpolation. ∎

Proof of Theorem 3.3. The admissibility of V will be shown at the end of the proof.

Let $\alpha > 0$, $\gamma < 1$ be chosen such that $c'_\alpha(V_-) \leqslant \gamma$.

The set $\mathscr{F} := \{\psi \in C_c(X); 0 \leqslant \psi \leqslant 1\}$ is directed by $\leqslant$, with $\sup \mathscr{F} = 1$. Let $\varphi \in C_0(X)$. We show that the net $(t \mapsto U^*_{\psi V}(t)\varphi)_{\psi \in \mathscr{F}}$ is a Cauchy net in $C([0, \alpha]; C_0(X))$.

Let $\varepsilon > 0$. From (F) we obtain $K_0 \in \mathscr{K}$ such that

$$\|1_{X\setminus K_0} U^*(s)\varphi\|_\infty \leqslant \varepsilon$$

for $0 \leqslant s \leqslant \alpha$. Choose $\psi_0 \in \mathscr{F}$ such that $1_{K_0} \leqslant \psi_0$. In order to apply Lemma 3.6 we choose $c > 1$ such that $c\gamma < 1$. Taking adjoints one obtains

$$\|gU_V^*(t) f\|_{\infty,\infty} \leqslant C \|f\|_\infty^{1/c} \|g\|_\infty^{1/c} \|gU^*(t) f\|_{\infty,\infty}^{1-1/c}$$

for $0 \leqslant t \leqslant \alpha$, where the constant C with $\sup_{0 \leqslant t \leqslant \alpha} \|U_{-cV_-}(t)\|_{1,1}^{1/c} \leqslant C$ only depends on α, γ, c. Choosing $f = |\varphi|$ one obtains

$$\|1_{X\setminus K_0} U_V^*(t)\varphi\|_\infty \leqslant C\varepsilon.$$

(We emphasize the fact that C depends only on the mentioned constants. We shall use the last inequality with V replaced by $\psi_0 V$, in the sequel.)

Let $\psi \in \mathscr{F}$, $\psi \geqslant \psi_0$. Then, having in mind the Duhamel formula

$$U_{\psi V}^*(t)\varphi - U_{\psi_0 V}^*(t)\varphi = \int_0^t U_{\psi V}^*(t-s)(\psi_0 - \psi)\, VU_{\psi_0 V}^*(s)\varphi\, ds \tag{3.2}$$

we estimate, for $0 \leqslant t \leqslant \alpha$,

$$\left\| \int_0^t U_{\psi V}^*(t-s)\, |(\psi_0 - \psi)\, V|\, ds \right\|_{\infty,\infty} \leqslant \left\| \int_0^t \psi\, |V|\, U_{\psi V}(t-s)\, ds \right\|_{1,1} \leqslant \frac{1+\gamma}{1-\gamma} \tag{3.3}$$

by Lemma 3.5. (The first integral in the previous estimate should be understood in the w^*-sense.) Hence, using (3.2) one obtains

$$\begin{aligned} \sup_{0 \leqslant t \leqslant \alpha} \|U_{\psi V}^*(t)\varphi - U_{\psi_0 V}^*(t)\varphi\|_\infty &\leqslant \frac{1+\gamma}{1-\gamma} \sup_{0 \leqslant s \leqslant \alpha} \|1_{X\setminus K_0} U_{\psi_0 V}^*(s)\varphi\|_\infty \\ &\leqslant \frac{1+\gamma}{1-\gamma} C\varepsilon. \end{aligned}$$

For $t > 0$ Lemma 3.4 implies $U_{\psi V}(t) \to U_V(t)$ strongly, and therefore $U_{\psi V}^*(t)\varphi \to U_V^*(t)\varphi$ in the w^*-sense. This shows $U_{\psi V}^*(t)\varphi \to U_V^*(t)\varphi$ uniformly for $t \in (0, \alpha]$.

Thus (F) for U_V^* is shown if we show that V is admissible, i.e., $U_V(t) \to I$ strongly for $t \to 0$. We know that $P = s - \lim_{t \to 0} U_V(t)$ exists and is a band projection; we have to show $P=I$. In order to do so let $f\in L_1(m)$. Because of

$$\int Pf\varphi\, dm = \lim_{t\to 0}\int (U_V(t)f)\varphi\, dm$$

$$= \lim_{t\to 0}\int f U_V^*(t)\varphi\, dm$$

$$= \int f\varphi\, dm$$

for all $\varphi\in C_0(X)$ we obtain $Pf=f$. ∎

3.7. *Remarks.* (a) The hypothesis "$V_-\in\hat{\mathbf{K}}$, $c(V_-)<1$" in Theorem 3.3 can be weakened to the requirement that $-V_-$ is U-admissible and a Miyadera perturbation of the generator T_{-V_-} of U_{-V_-}.

In order to see this note first that it is sufficient to treat the case $V_+=0$. (This is because Theorem 3.3 implies (F) for $U_{V_+}^*$.)

Now we assume $V=-V_-$ and use the proof of Theorem 3.3. The hypothesis implies that there exists $c>1$ such that $-(c-1)V_-$ is U_{-V_-}-admissible which in turn implies that $-cV_-$ is U-admissible, and Lemma 3.6 can be applied. Further, the hypothesis yields directly an estimate for the term estimated in (3.2).

(b) If $U(t)$ is stochastic for all $t\geqslant 0$ (i.e., $\|U(t)f\|=\|f\|$ for all $f\in L_1(m)_+$) and $-V_-$ is U-admissible then it follows from [Vo2; Proposition 4.6] that V_- is a Miyadera perturbation of $U_{-V_-}(\cdot)$, and therefore the hypothesis made in (a) is satisfied.

We conclude this section with a "noncanonical" application of Theorem 3.3.

3.8. Example. We consider the semigroup U associated with the heat equation $u_t=\frac{1}{2}\Delta u$ and want to find a class of V's such that U_V acts as a C_0-semigroup on $C_{b,u}(\mathbb{R}^d)$, the bounded uniformly continuous functions.

This question enters the framework treated so far if we note that $C_{b,u}(\mathbb{R}^d)$ is a commutative C^*-algebra, and the Gelfand space X of $C_{b,u}(\mathbb{R}^d)$ is a compactification of $\mathbb{R}^d$. Then $C(X)=C_{b,u}(\mathbb{R}^d)$. Also, U^* satisfies (F) on X. Therefore, Theorem 3.3 yields that U_V^* acts as a C_0-semigroup on $C_{b,u}(\mathbb{R}^d)$ if $V\in\overline{C_{b,u}(\mathbb{R}^d)}^{\mathbf{K}}$. (In fact, note that for compact X Theorems 3.2 and 3.3 coincide.)

We mention that the inclusion $\overline{C_{b,u}(\mathbb{R}^d)}^{\mathbf{K}}\subset\mathbf{K}$ is strict since even $\overline{L_\infty(\mathbb{R}^d)}^{\mathbf{K}}$ is strictly contained in $\mathbf{K}$.

4. The Smoothing Property for Adjoint Absorption Semigroups

In this section we assume the same setup concerning $(X, \mathfrak{B}, m)$ and U as in Section 3.

4.1. Theorem. *Assume that* U^* *satisfies* (S). *Let* $V \in \mathbf{K}$. *Then* U_V^* *satisfies* (S).

The following lemma is a preparation for the proof.

4.2. Lemma. *Let* $V \in \mathbf{K}$. *Then*

$$\| U_V^*(t) - U^*(t) \|_{\infty, \infty} \to 0 \qquad (t \to 0).$$

More precisely, let $\alpha > 0$, $0 \leqslant \gamma < 1$ *be such that* $c'_\alpha(V) \leqslant \gamma$. *Then there exists* M *depending only on* α, γ *such that*

$$\| U_V(t) - U(t) \|_{1,1} \leqslant M c'_t(V).$$

(Recall the definition and properties of $c'_\alpha(V)$ from Section 3.)

Proof. It is sufficient to prove the second statement. There exists M only depending on α, γ such that $\| U_V(t) \| \leqslant M$ for $0 \leqslant t \leqslant \alpha$. Now the Duhamel formula

$$U_V(t) f - U(t) f = -\int_0^t U_V(t-s)\, V U(s) f \, ds \qquad (f \in D(T))$$

implies the desired estimate. ∎

Proof of Theorem 4.1. Let $f \in L_\infty(m)$, $t > 0$. For $0 < s \leqslant t$ we have $U^*(s)\, U_V^*(t-s) f \in C_b(X)$, by (S). Further

$$\begin{aligned} &\| U_V^*(t) f - U^*(s)\, U_V^*(t-s) f \|_\infty \\ &\qquad \leqslant \| U_V^*(s) - U^*(s) \|_{\infty,\infty} \, \| U_V^*(t-s) f \|_\infty \to 0 \end{aligned}$$

for $s \to 0$ by Lemma 4.2. This implies $U_V^*(t) f \in C_b(X)$. ∎

4.3. Proposition. *Assume that* U^* *satisfies* (F), *and let* $V_\pm \geqslant 0$, $V = V_+ - V_-$, $-cV_-$ *admissible for some* $c > 1$. *Then* $U_V^*(t)(L_{\infty,0}(m)) \subset L_{\infty,0}(m)$ *for all* $t \geqslant 0$.

(Here $L_{\infty,0}(m) := \overline{L_{\infty,c}(m)}^{L_\infty(m)}$, where

$$L_{\infty,c}(m) := \{ f \in L_\infty(m);\ \operatorname{supp} f \text{ compact} \}.)$$

Proof. It is sufficient to treat the case $V_+=0$. We have to show that, given $K\in\mathscr{K}$, $\varepsilon>0$, there exists $K'\in\mathscr{K}$ such that

$$\|1_{X\setminus K'}U_V^*(t)1_K\|_{\infty,\infty}\leqslant\varepsilon \qquad (0\leqslant t\leqslant 1).$$

This, however, follows from (F) for U^* together with Lemma 3.6. ∎

4.4. COROLLARY. *Assume that U^* satisfies* (SF), *and let $V\in\mathbf{K}$. Then U_V^* satisfies* (F).

Proof. This is a straight-forward combination of Theorem 4.1, Lemma 4.2 and Proposition 4.3. (Note $C_0(X)=C_b(X)\cap L_{\infty,0}(X)$.) ∎

In the following we shall need the local Kato class

$$\mathbf{K}_{loc}:=\{V\colon X\to[-\infty,\infty];\ 1_K V\in\mathbf{K}\text{ for all }K\in\mathscr{K}\}$$
$$(=\{V\colon X\to[-\infty,\infty];\ \varphi V\in\mathbf{K}\text{ for all }\varphi\in C_c(X)\}).$$

4.5. THEOREM. *Assume that U^* satisfies* (SF). *Let $V=V_+-V_-$, $V_\pm\geqslant 0$, $V_\pm\in\mathbf{K}_{loc}$, $V_-\in\hat{\mathbf{K}}$, $c(V_-)<1$. Then U_V^* satisfies* (F).

Proof. This follows from Theorem 3.3 in combination with Corollary 4.4. ∎

In order to extend Theorem 4.1 to more general absorption rates we need the localization property (L) defined in Section 2.

4.6. THEOREM. *Assume that U^* satisfies* (SF) *and* (L). *Let $V=V_+-V_-$, $V_\pm\geqslant 0$, $V_\pm\in\mathbf{K}_{loc}$, $V_-\in\hat{\mathbf{K}}$, $c(V_-)<1$. Then U_V^* satisfies* (SF).

Proof. In view of Theorem 4.5 it remains to show (S).

Let $\alpha'>0$, $\gamma<1$ be chosen such that $\|\int_0^{\alpha'}V_-U(t)\,dt\|_{1,1}\leqslant\gamma$.

Let $K\in\mathscr{K}$, and choose $\alpha>0$ according to (L); without restriction $\alpha\leqslant\alpha'$. Let $\varepsilon>0$, and choose K' according to (L). Choosing $c>1$ such that $c\gamma<1$ we obtain

$$\|1_K U_V^*(t)\,1_{X\setminus K'}\|_{\infty,\infty}\leqslant C\varepsilon \qquad (0\leqslant t\leqslant\alpha),$$

as in the proof of Theorem 3.3. The Feller property implies that there exists $K''\in\mathscr{K}$ such that

$$\|1_{X\setminus K''}U^*(s)\,1_{K'}\|_{\infty,\infty}\leqslant\varepsilon \qquad (0\leqslant s\leqslant\alpha).$$

As above we obtain

$$\|1_{X\setminus K''}U_V^*(s)\,1_{K'}\|_{\infty,\infty}\leqslant C\varepsilon \qquad (0\leqslant s\leqslant\alpha).$$

We conclude

$$\begin{aligned}
&\|(U_V^*(t) - U_{1_{K''}V}^*(t))\, 1_{K'}\|_{\infty,\infty}\\
&\quad = \left\| \int_0^t U_V^*(t-s)\, V(1-1_{K''})\, U_{1_{K''}V}^*(s))\, 1_{K'}\, ds \right\|_{\infty,\infty}\\
&\quad \leqslant \left\| \int_0^t U_V^*(t-s)\, V\, ds \right\|_{\infty,\infty} C\varepsilon\\
&\quad \leqslant \frac{1+\gamma}{1-\gamma}\, C\varepsilon
\end{aligned}$$

for $0 \leqslant t \leqslant \alpha$, where Lemma 3.5 has been used in the last estimate.

Now let $f \in L_\infty(m)$. Then $U_{1_{K''}V}(t)\, f \in C_b(X)$ by Theorem 4.1, and, for $0 \leqslant t \leqslant \alpha$,

$$\begin{aligned}
\|1_K(U_V^*(t)\, f - U_{1_{K''}V}^*(t)\, f)\|_\infty &\leqslant \|1_K(U_V^*(t) - U_{1_{K''}V}^*(t))(1-1_{K'})\, f\|_\infty\\
&\quad + \|(U_V^*(t) - U_{1_{K''}V}^*(t))\, 1_{K'}\, f\|_\infty\\
&\leqslant 2C\varepsilon\, \|f\|_\infty + \frac{1+\gamma}{1-\gamma}\, C\varepsilon\, \|f\|_\infty,
\end{aligned}$$

by the previous inequalities. Therefore, on K, $U_V^*(t)\, f$ is uniformly approximated by continuous functions, and thereby is itself continuous on K, for $0 < t \leqslant \alpha$. The formula $U(t)\, f = U(s)\, U(t-s)\, f$ (with $0 < s \leqslant \alpha$, and where $U(t-s)\, f \in L_\infty(m)$) implies that $U(t)\, f$ is continuous on K for all $t > 0$. ∎

4.7. *Remark.* The proof of Theorem 4.6 is, in a sense, an abstract version of [Si; Proof of Theorem B.10.2], and in fact is modelled after this proof.

5. Dirichlet Forms and Measure Perturbations

In this section we are going to extend the results of the previous sections to measure perturbations of Dirichlet forms.

Let X, m and U be as in the previous sections, with the additional requirement that m is a Radon measure on X satisfying $\operatorname{supp} m = X$. Moreover assume that $U(t) = U^*(t)$ on the intersection $L_1(m) \cap L_\infty(m)$, and denote by $-H$ the generator of the C_0-semigroup induced on $L_2(m)$. The form $\mathfrak{h}$ associated with H is then a *Dirichlet form.* We further assume that $\mathfrak{h}$ is *regular*, i.e., $D(\mathfrak{h}) \cap C_c(X)$ is a core for $\mathfrak{h}$, and $D(\mathfrak{h}) \cap C_c(X)$ is dense in $C_c(X)$ with respect to the supremum norm.

For measure perturbations of Dirichlet forms we refer to [AM], [SV2]. For the definition of the classes $\mathbf{M}_0$ (capacity-absolutely continuous measures), $\mathbf{S}$ (smooth measures), $\mathbf{S}_0$ (finite energy integral measures), $\hat{\mathbf{S}}_K$ (extended Kato class), $\mathbf{S}_K$ (Kato class) we refer to [SV2].

Since in general measures in $\hat{\mathbf{S}}_K$ cannot be approximated by functions in the norm of $\hat{\mathbf{S}}_K$ (for the Schrödinger semigroup, e.g., the Kato class $\hat{\mathbf{K}}$ is complete), we do not obtain results for measure perturbations of U which are analogous to the results of Section 3. We rather follow the treatment given in Section 4, using that the estimates obtained there carry over.

5.1. THEOREM. *Assume that* U^* *satisfies* (S), *and let* $\mu_\pm \in \mathbf{S}_K$, $\mu = \mu_+ - \mu_-$. *Then* U_μ^* *satisfies* (S).

This can be proved as Theorem 4.1 once the following lemma is established.

5.2. LEMMA. *Let* U *and* μ *be as in Theorem* 5.1. *Then* $\|U_\mu^*(t) - U^*(t)\|_{\infty,\infty} \to 0$ $(t \to 0)$.

Proof. We only treat the case $\mu_+ = 0$ and refer to the end of this proof for the general case.

Let $\varepsilon > 0$. There exists $\beta > 0$ such that $c_\beta(\mu) < \varepsilon$. By [SV2; Theorem 3.5] there exists a net $(V_\iota)_{\iota \in I}$ of functions $V_\iota \in L_2 \cap L_\infty(m)$ such that $c_\beta(V_\iota) \leqslant c_\beta(\mu)$, $e^{-t(H - V_\iota)} \to e^{-tH_\mu}$ strongly, for all $t \geqslant 0$. There exists $\alpha > 0$ such that $c'_\alpha(V) \leqslant \varepsilon$ for any $V \in \hat{\mathbf{K}}$ with $c_\beta(V) \leqslant c_\beta(\mu)$ $(< \varepsilon)$; cf. [Vo2; Proposition 4.7]. Therefore

$$\|U_{-V_\iota}(t) - U(t)\|_{1,1} \leqslant M\varepsilon \qquad (0 \leqslant t \leqslant \alpha)$$

for all $\iota \in I$; by Lemma 4.2. The strong convergence in $L_2(m)$ implies, by Fatou's lemma, that the last inequality carries over to the limit,

$$\|U_\mu(t) - U(t)\|_{1,1} \leqslant M\varepsilon \qquad (0 \leqslant t \leqslant \alpha).$$

The general case is proved in the same way if one observes that the convergence theorem [SV2; Theorem A.1] yields a statement analogous to [SV2; Theorem 3.5] if $\mu = \mu_+ - \mu_-$ as assumed in Theorem 5.1 is allowed. ∎

5.3. PROPOSITION. *Assume that* U^* *satisfies* (F), *and let* $\mu_- \in \hat{\mathbf{S}}_K$, $c(\mu_-) < 1$, $\mu_+ \in \mathbf{M}_0$. *Then* $U_{-\mu_- + \mu_+}(t)(L_{\infty,0}(m)) \subset L_{\infty,0}(m)$ *for all* $t \geqslant 0$.

(We recall from [SV2; Section 4] that, in general, $U_{-\mu_- + \mu_+}$ is not strongly continuous.)

Proof. Because of monotonicity it is sufficient to treat the case $\mu_+ = 0$. It is sufficient to show that, given $K \in \mathscr{K}$, $\varepsilon > 0$, there exists $K' \in \mathscr{K}$ such that

$$\| 1_{X \setminus K'} U^*_{-\mu_-}(t) 1_K \|_{\infty, \infty} \leqslant \varepsilon \qquad (0 \leqslant t \leqslant 1).$$

This, however, follows using (F) for U^* as well as Lemma 3.6 together with a suitable approximation procedure (as in the proof of Lemma 5.2). ∎

5.4. COROLLARY. *Assume that* U^* *satisfies* (SF), *and let* $\mu_\pm \in \mathbf{S}_K$, $\mu = \mu_+ - \mu_-$. *Then* U^*_μ *satisfies* (SF).

Proof. This is a straight-forward combination of Theorem 5.1, Lemma 5.2, and Proposition 5.3. ∎

Similarly to the local Kato class the local Kato class of measures is defined by

$$\mathbf{S}_{K,\,loc} := \{\mu \text{ measure on } \mathfrak{B};\ 1_K \mu \in \mathbf{S}_K\ (K \in \mathscr{K})\}$$
$$(= \{\mu;\ \varphi\mu \in \mathbf{S}_K\ (\varphi \in C_c(X))\}).$$

5.5. THEOREM. *Assume that* U^* *satisfies* (SF), *and let* $\mu_\pm \in \mathbf{S}_{K,\,loc}$, $\mu_- \in \hat{\mathbf{S}}_K$, $c(\mu_-) < 1$.

(a) *Then* U^*_μ *satisfies* (F).

(b) *If additionally* U^* *satisfies* (L) *then* U^*_μ *satisfies* (SF).

5.6. LEMMA. *Let* $\mu = \mu_+ - \mu_-$ *where* $\mu_+ \in \mathbf{M}_0$, $\mu_- \in \hat{\mathbf{S}}_K$, $c(\mu_-) < 1$. *Let* $\mathscr{F} \subset \{\psi \in \mathscr{L}_\infty(m);\ 0 \leqslant \psi \leqslant 1\}$ *be directed under "$\leqslant$ quasi-everywhere," and* $q-\sup \mathscr{F} = 1$ (*i.e.*, $1 \geqslant \tilde{\psi} \geqslant \psi$ *q.e. for all* $\psi \in \mathscr{F}$ *implies* $\tilde{\psi} = 1$ *q.e.*). *Then*

$$U_\mu(t) = s - \lim_{\psi \in \mathscr{F}} U_{\psi\mu}(t) \qquad (t \geqslant 0).$$

Proof. It is clear that

$$\mathfrak{h} + \mu = \lim_{\psi \in \mathscr{F}} (\mathfrak{h} + \mu_+ - \psi\mu_-) = \lim_{\psi \in \mathscr{F}} (\mathfrak{h} + \mu_+ - \psi\mu_-),$$

$$\mathfrak{h} + \mu = \sup_{\psi \in \mathscr{F}} (\mathfrak{h} + \psi\mu_+ - \mu_-) = \lim_{\psi \in \mathscr{F}} (\mathfrak{h} + \psi\mu_+ - \mu_-),$$

where the nets are monotone decreasing and increasing, respectively. The monotone convergence theorems for forms imply $H_{\mu_+ - \psi\mu_-} \to H_\mu$, $H_{\psi\mu_+ - \mu_-} \to H_\mu$ in strong resolvent sense. (See [RS; Theorems S.14 and S.16, p. 373] for densely defined forms and [We] for the general case. In both of these references the results are formulated for sequences, but the proofs hold equally for nets.) We denote by $U_{\mu_+-\mu_-}(\cdot)$ etc. the associated L_1-semigroups. The inequalities

$$\psi\mu_+ - \mu_- \leqslant \psi(\mu_+ - \mu_-) \leqslant \mu_+ - \psi\mu_-$$

imply then

$$U_{\psi\mu_+-\mu_-}(t) \geqslant U_{\psi(\mu_+-\mu_-)}(t) \geqslant U_{\mu_+-\psi\mu_-}(t) \tag{5.1}$$

($t \geqslant 0$) in the sense of Banach lattice order, by [SV2; Remark 4.4] (see also [Ou]). By the same reference, the nets $(U_{\psi\mu_+-\mu_-}(t))_{\psi\in\mathscr{F}}$, and $(U_{\mu_+-\psi\mu_-}(t))_{\psi\in\mathscr{F}}$ are monotone decreasing and increasing, respectively; therefore they are strongly convergent. The limit of both of these nets is $U_{\mu_+-\mu_-}(t)$, by the strong resolvent convergence of the L_2-generators shown before. Therefore inequality (5.1) implies the assertion. ∎

The proof of the first part of Theorem 5.5 will mainly consist in a paraphrasis of the proof of Theorem 3.3. In order to carry this out we single out an inequality which corresponds to the estimate obtained from Equation (3.1).

5.7. Lemma. *Let $\mu_\pm \in \hat{\mathbf{S}}_K \cap \mathbf{S}_0$, $\gamma < 1$, $\beta > 0$, such that $c_\beta(\mu_-) \leqslant \gamma$, and denote $\mu = \mu_+ - \mu_-$. Then there exist constants $\alpha > 0$, $C \geqslant 0$, only depending on γ, β, with the following property: If $K \in \mathscr{K}$ and $\psi_1, \psi_2 \in C_c(X) \cap D(\mathfrak{h})$ are functions such that $1_K \leqslant \psi_1 \leqslant \psi_2 \leqslant 1$ then for all $\varphi \in C_0(X)$, $0 \leqslant t \leqslant \alpha$*

$$\| U^*_{\psi_1\mu}(t)\varphi - U^*_{\psi_2\mu}(t)\varphi \|_\infty \leqslant C \sup_{0\leqslant s\leqslant\alpha} \|1_{X\setminus K} U^*(s)\varphi\|_\infty .$$

Proof. By [SV2; Theorem 2.1] there exist sequences $(V_{\pm,n})$ in $L_2 \cap L_\infty(m)_+$ such that

$$c_\beta(V_{\pm,n}) \leqslant \gamma_\pm \qquad (n\in\mathbb{N})$$

(where $\gamma_+ := c_\beta(\mu_+)$, $\gamma_- := \gamma$),

$$\int V_\pm\, |u|^2\, dm \leqslant \gamma_\pm(\mathfrak{h}[u] + \beta\, \|u\|^2) \qquad (n\in\mathbb{N},\, u\in D(\mathfrak{h})),$$

and $V_{\pm,n} \to \mu_\pm$ strongly in $L(D(\mathfrak{h}), D(\mathfrak{h})^*)$. The first two properties carry over immediately to $\psi_j V_{\pm,n}$ (instead of $V_{\pm,n}$). Moreover, the proof of Theorem 2.1 in [SV2] yields that

$$\psi_j V_{\pm,n} \to \psi_j \mu_\pm \qquad (n\to\infty,\, j=1,2)$$

strongly in $L(D(\mathfrak{h}), D(\mathfrak{h})^*)$, and [SV2; Theorem A.1] implies

$$H + \psi_j V_n \to H_{\psi_j \mu} \qquad (n \to \infty, j = 1, 2)$$

in strong resolvent sense (with $V_n := V_{+,n} - V_{-,n}$). Fixing $\gamma' \in (\gamma, 1)$ one can find $\alpha > 0$ only dependent on γ, β, γ' such that $c'_\alpha(\psi_1 V_{-,n}) \leqslant \gamma'$ $(n \in \mathbb{N})$; cf. [Vo2; Proposition 4.7]. Let $\varphi \in C_c(X)$. For $n \in \mathbb{N}$, $0 \leqslant t \leqslant \beta$ we have the Duhamel formula

$$e^{-t(H+\psi_1 V_n)}\varphi - e^{-t(H+\psi_2 V_n)}\varphi$$

$$= \int_0^t e^{-(t-s)(H+\psi_2 V_n)}(\psi_2 - \psi_1)\, V_n e^{-s(H+\psi_1 V_n)}\varphi \, ds,$$

and the considerations as in the proof of Theorem 3.3 show

$$\| U_{\psi_1 V_n}(t)\varphi - U_{\psi_2 V_n}(t)\varphi \|_\infty \leqslant \frac{\gamma' + 1}{\gamma' - 1} \sup_{0 \leqslant s \leqslant t} \| 1_{X \setminus K} U_{\psi_1 V_n}(s)\varphi \|_\infty .$$

Further, Lemma 3.6 implies that there exists $C' \geqslant 0$, only dependent on α, γ', such that

$$\| 1_{X \setminus K} U_{\psi_1 V_n}(t)\varphi \|_\infty \leqslant C' \, \| 1_{X \setminus K} U(t)\varphi \|_\infty$$

$(0 \leqslant t \leqslant \alpha)$. Choosing $C = ((\gamma' + 1)/(\gamma' - 1))\, C'$ we obtain

$$\| U_{\psi_1 V_n}(t)\varphi - U_{\psi_2 V_n}(t)\varphi \|_\infty \leqslant C \sup_{0 \leqslant s \leqslant \alpha} \| 1_{X \setminus K} U(s)\varphi \|_\infty .$$

For $n \to \infty$ the asserted inequality follows. By continuity, the inequality carries over to all $\varphi \in C_0(X)$. ▮

Proof of Theorem 5.5. (a) The properties of Dirichlet forms imply that the set $\mathscr{F} := \{\psi \in C_c(X) \cap D(\mathfrak{h});\ 0 \leqslant \psi \leqslant 1\}$ is directed by $\leqslant$. The regularity of $\mathfrak{h}$ implies $\sup \mathscr{F} = 1$.

There exist $\beta > 0$, $\gamma < 1$ such that $c_\beta(\mu_-) < \gamma$. Choose $\alpha > 0$, $C \geqslant 0$ corresponding to γ, β according to Lemma 5.7. Let $\varphi \in C_0(X)$. We show that the net $(t \mapsto U^*_{\psi\mu}(t)\varphi)_{\psi \in \mathscr{F}}$ is a Cauchy net in $C([0, \alpha]; C_0(X))$. (Note that $U^*_{\psi\mu}$ satisfies (F), by Corollary 5.4.)

Let $\varepsilon > 0$. From (F) for U^* we obtain $K_0 \in \mathscr{K}$ such that

$$\| 1_{X \setminus K_0} U^*(s)\varphi \|_\infty \leqslant \varepsilon \qquad (0 \leqslant s \leqslant \alpha).$$

Let ψ_0, $\psi \in \mathscr{F}$ be such that $1_{K_0} \leqslant \psi_0 \leqslant \psi$. With $K := \operatorname{supp} \psi$ we have

$$\psi\mu = \psi(1_K \mu), \qquad \psi_0 \mu = \psi_0(1_K \mu),$$

and therefore we can apply Lemma 5.7 with μ replaced by $1_K\mu$ and obtain

$$\| U^*_{\psi_0\mu}(t)\varphi - U^*_{\psi\mu}(t)\varphi\|_\infty \leqslant C \sup_{0\leqslant s\leqslant\alpha} \|1_{X\setminus K_0} U^*(s)\varphi\| \leqslant C\varepsilon.$$

Lemma 5.6 implies

$$U_\mu(t) = s\text{-}\lim_{\psi\in\mathscr{F}} U_{\psi\mu}(t),$$

and therefore $U^*_{\psi\mu}(t)\varphi \to U^*_\mu(t)\varphi$ in the w^*-sense. This implies that $t\mapsto U^*_\mu(t)\varphi$ is continuous on $[0,\alpha]$, and thus U^*_μ satisfies (F).

(b) For the proof of this part we refer to the proof of Theorem 4.6 and note that the required inequalities have to be proved by approximation procedures similarly as in the proof of Lemma 5.7. ∎

6. Necessary Conditions for the Feller Property for Schrödinger Semigroups

In this section we consider the case where $X=\mathbb{R}^d$, m Lebesgue–Borel measure, and U the C_0-semigroup associated with the heat equation $\partial_t u = \frac{1}{2}\Delta u$. Since U has (SF) and (L) we know from Theorem 5.5 that U_μ satisfies (SF) whenever $\mu_\pm \in \mathbf{S}_{K,\,loc}$, $\mu_-\in\hat{\mathbf{S}}_K$, $c(\mu_-)<1$. In this section we are going to show that these conditions are not far from necessary. Before we do this we want to make it clear by a simple argument that continuity cannot be expected even for rather nice perturbations.

6.1. Example. Let $d\geqslant 2$. Then there exists a regular $V:\mathbb{R}^d\to[0,\infty]$ such that $V\notin L_1(U)$ for any nonempty open $U\subset\mathbb{R}^d$; cf. [SV1]. (Recall that "regular" means that U_V is strongly continuous, and $s\text{-}\lim_{\alpha\to 0+} U_{\alpha V}(t) = U(t)$ $(t\geqslant 0)$.) For this V the following property holds: If $f\in L_1(\mathbb{R}^d)$ and $t>0$ are such that $U_V(t)f$ is a continuous function then $U_V(t)f=0$. (Hence $f=0$ since $U_V(\cdot)f$ is analytic on $(0,\infty)$, by [AB; Theorem 6.1], and continuous on $[0,\infty)$, as an L_1-valued function.) Indeed, by the holomorphy of the semigroup U_V, the function $U_V(t)f$ belongs to $D(T_V)$, and the latter is equal to $D(T)\cap D(V)$, by [Vo2; Corollary 4.3]. The properties of V imply that the only continuous function contained in $D(V)$ is the zero function.

In order to show the characterization stated at the beginning we recall, as a preparation, the following property.

6.2. LEMMA. *Let* $\mu \in \mathbf{M}_0$.

(a) ([Stu; Korollar 4.7]) *Then* $\mu \in \mathbf{S}_K$ *if and only if* $(H+\alpha)^{-1}\mu$ *is bounded and uniformly continuous for some (all)* $\alpha > 0$.

(b) ([BHH; Proposition 7.1]) *Moreover*, $\mu \in \mathbf{S}_{K,\,loc}$ *if and only if* $(H+\alpha)^{-1}(1_K\mu)$ *is bounded and continuous for each compact set* $K \subset \mathbb{R}^d$ *and some (all)* $\alpha > 0$.

("$(H+\alpha)^{-1}\mu$ bounded" means: The functional $C_c(\mathbb{R}^d) \ni \varphi \mapsto \int ((H+\alpha)^{-1}\varphi)^{\sim}\, d\mu$ is continuous with respect to the L_1-norm, and $(H+\alpha)^{-1}\mu$ is the L_∞-function generating this functional. This implies that μ is a Radon measure and that $(H+\alpha)^{-1}\mu$ is obtained as the convolution of the corresponding resolvent kernel with μ.)

Proof. (a) Let $\mu \in S_K$, $\alpha > 0$. Then $(H+\alpha)^{-1}\mu$ is a bounded function, and therefore $(H+\alpha')^{-1}(H+\alpha)^{-1}\mu$ is uniformly continuous for all $\alpha' > 0$. In the resolvent equation

$$(H+\alpha)^{-1}\mu = (\alpha'-\alpha)(H+\alpha')^{-1}(H+\alpha)^{-1}\mu + (H+\alpha')^{-1}\mu$$

the term $(H+\alpha')^{-1}\mu$ tends to zero uniformly (this is by the definition of the Kato class $\mathbf{S}_K$ of measures), and therefore $(H+\alpha)^{-1}\mu$ is uniformly approximated by uniformly continuous functions.

On the other hand, assume that $(H+\alpha')^{-1}\mu$ is uniformly continuous for some $\alpha > 0$. Then $(\alpha'-\alpha)(H+\alpha')^{-1}(H+\alpha)^{-1}\mu \to (H+\alpha)^{-1}\mu$ uniformly for $\alpha' \to \infty$, and therefore the resolvent equation implies $(H+\alpha')^{-1}\mu \to 0$ uniformly.

(b) The necessity of the condition is a trivial consequence of (a).

For the converse let $K \subset \mathbb{R}^d$ be compact. The boundedness of $(H+\alpha)^{-1}(1_K\mu)$ implies that $1_K\mu$ has finite total mass, and therefore $(H+\alpha)^{-1}(1_K\mu)$ tends to zero at ∞ in $\mathbb{R}^d$. Now the continuity implies uniform continuity, and $1_K\mu \in \mathbf{S}_K$ follows from (a). ∎

6.3. THEOREM. (a) *Let* $\mu \in \mathbf{M}_0$ *and let* U^*_μ *satisfy* (F). *Then* $\mu \in \mathbf{S}_{K,\,loc}$.

(b) *Let* $\mu \in \hat{\mathbf{S}}_K$, $c(\mu) < 1$, *and let* $U^*_{-\mu}$ *satisfy* (F). *Then* $\mu \in \mathbf{S}_{K,\,loc}$.

Proof. (a) Let $\varphi \in C_c(\mathbb{R}^d)$, $\alpha > 0$. We use the following version of the second resolvent equation which we shall prove subsequently:

$$(H+\alpha)^{-1}\mu(H+\mu+\alpha)^{-1}\varphi = (H+\alpha)^{-1}\varphi - (H+\mu+\alpha)^{-1}\varphi,$$

where "$H+\mu$" stands for H_μ, and the left hand side is a more intuitive way to write $(H+\alpha)^{-1}(((H+\mu+\alpha)^{-1}\varphi)\mu)$. By the hypothesis and the fact that U satisfies (F) we have that the right hand side is a continuous function, and Lemma 6.2 shows $((H+\mu+\alpha)^{-1}\varphi)\mu \in \mathbf{S}_K$. Choosing φ and α properly we obtain $\mu \in \mathbf{S}_{K,\,loc}$.

The equation used at the beginning follows from the equality

$$
\begin{aligned}
\int (H+\alpha)^{-1}\,\psi(H+\mu+\alpha)^{-1}\varphi\,d\mu
&= \mu[(H+\alpha)^{-1}\psi, (H+\mu+\alpha)^{-1}\varphi] \\
&= (\mathfrak{h}+\mu+\alpha-(\mathfrak{h}+\alpha))[(H+\alpha)^{-1}\psi, (H+\mu+\alpha)^{-1}\varphi] \\
&= ((H+\alpha)^{-1}\,\psi\,|\,\varphi) - (\psi\,|\,(H+\mu+\alpha)^{-1}\varphi) \\
&= (\psi\,|\,(H+\alpha)^{-1}\,\varphi - (H+\mu+\alpha)^{-1}\varphi).
\end{aligned}
$$

(b) As in (a), with μ replaced by $-\mu$. ∎

6.4. *Remark*. A result corresponding to Theorem 6.3 for general U but for perturbations given by V is proved in [Vo4].

REFERENCES

[A] W. Arendt, Gaussian estimates and interpolation of the spectrum in L^p, *Differential Integral Equations* **7** (1994), 1153–1168.

[AB] W. Arendt and C. J. K. Batty, Absorption semigroups and Dirichlet boundary conditions, *Math. Ann.* **295** (1993), 427–448.

[AM] S. Albeverio and Z. Ma, Perturbation of Dirichlet forms—lower semiboundedness, closability, and form cores, *J. Funct. Anal.* **99** (1991), 332–356.

[AS] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators, *Comm. Pure. Appl. Math.* **35** (1982), 209–273.

[ArSe] D. G. Aronson and J. Serrin, Local behaviour of solutions of quasilinear parabolic equations, *Arch. Rational Mech. Anal.* **25** (1967), 81–122.

[BHH] A. Boukricha, W. Hansen, and H. Hueber, Continuous solutions of the generalized Schrödinger equation and perturbation of harmonic spaces, *Exposition Math.* **5** (1987), 97–135.

[Bou] N. Bourbaki, "Topologie générale," Chaps. 5–10, Hermann, Paris, 1974.

[D] E. B. Davies, L^p spectral independence and L^1 analyticity, *J. London Math. Soc.*, to appear.

[FOT] M. Fukushima, Y. Oshima, and M. Takeda, "Dirichlet Forms and Symmetric Markov Processes," De Gruyter, Berlin, 1994.

[HP] E. Hille and R. S. Phillips, "Functional Analysis and Semigroups," Amer. Math. Soc. Coll. Publ., Providence, RI, 1957.

[HV1] R. Hempel and J. Voigt, The spectrum of a Schrödinger operator in $L_p(\mathbb{R}^\nu)$ is p-independent, *Comm. Math. Phys.* **104** (1986), 243–250.

[HV2] R. Hempel and J. Voigt, The spectrum of Schrödinger operators in $L_p(\mathbb{R}^d)$ and in $C_0(\mathbb{R}^d)$, *in* "Mathematical Results in Quantum Mechanics" (M. Demuth, P. Exner, H. Neidhardt, and V. Zagrebnov, Eds.), Birkhäuser, Basel, 1994.

[Mi1] I. Miyadera, On perturbation theory for semi-groups of operators, *Tôhoku Math. J.* **18** (1966), 299–310.

[Mi2] I. Miyadera, On perturbation for semigroups of linear operators, *Sci. Res. School of Ed. Waseda Univ.* **21** (1972), 21–24 [in Japanese].

[Ou] E. M. Ouhabaz, Invariance of closed convex sets and domination criteria for semi-groups, *Potential Anal.*, to appear.

[Pa] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.

[RS] M. Reed and B. Simon, "Methods of Modern Mathematical Physics. I. Functional Analysis," rev. and enlarged ed., Academic Press, New York, 1980.

[ScV] G. Schreieck and J. Voigt, Stability of the L_p-spectrum of Schrödinger operators with form small negative part of the potential, *in* "Functional Analysis, Proc. Essen Conference 1991" (Bierstedt, Pietsch, Ruess, and Vogt, Eds.), pp. 95–105, Dekker, New York, 1994.

[Si] B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 447–526.

[Sto] P. Stollmann, Processes corresponding to semigroups on L_1, *Semesterber. Funktionalanal. (Tübingen)* **19** (1990/1991), 205–218.

[SV1] P. Stollmann and J. Voigt, A regular potential which is nowhere in L_1, *Lett. Math. Phys.* **9** (1985), 227–230.

[SV2] P. Stollmann and J. Voigt, Perturbation of Dirichlet forms by measures, *Potential Anal.* **5** (1996), 109–138.

[Stu] K.-Th. Sturm, "Störung von Hunt-Prozessen durch signierte additive Funktionale," thesis, Erlangen, 1989.

[vN] J. van Neerven, "The Adjoint of a Semigroup of Linear Operators," Lecture Notes in Math., Vol. 1529, Springer, Berlin, 1992.

[Vo1] J. Voigt, On the perturbation theory for strongly continuous semigroups, *Math. Ann.* **229** (1977), 163–171.

[Vo2] J. Voigt, Absorption semigroups, their generators, and Schrödinger semigroups, *J. Funct. Anal.* **67** (1986), 167–205.

[Vo3] J. Voigt, Absorption semigroups, *J. Operator Theory* **20** (1988), 117–131.

[Vo4] J. Voigt, Absorption semigroups, Feller property, and Kato class, *in* "Partial Differential Operators and Mathematical Physics" (Demuth and Schulze, Eds.), Operator Theory: Advances and Applications, Vol. 78, pp. 389–396, Birkhäuser, Basel, 1995.

[We] J. Weidmann, Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet, *Math. Scand.* **54** (1984), 51–69.

[Zh] Z. Zhao, A probabilistic principle and generalized Schrödinger perturbation, *J. Funct. Anal.* **101** (1991), 162–176.