Localization on a quantum graph with a random potential on the edges

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Dedicated to Jean-Michel Combes on the occasion of his 65th birthday

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Abstract

We prove spectral and dynamical localization on a cubic-lattice quantum graph with a random potential. We use multiscale analysis and show how to obtain the necessary estimates in analogy to the well-studied case of random Schrödinger operators.

1 Introduction

Since the middle of the 1980's the mathematical approach to the phenomenon of localization in random solids witnessed a rapid development. One of the techniques used to prove localization is multiscale analysis. Introduced by Fröhlich and Spencer in [FS83] and further developed by von Dreifus and Klein in [DK89] for the original Anderson model on the lattice, it had been extended to the continuum by Combes and Hislop in [CH94]. By now there is a large number of discrete and continuum models for which localization has been established this way, see [Sto01] and for more recent advances [GK01].

On the other hand in recent years the interest also turned to the shape of structures made of semiconductor and other materials. In particular, quantum graph models became popular as models of various superlattice structures. Therefore it seems natural to ask how one can extend the multiscale proof of localization to such graph models. In this paper we want to give an answer for a particular case of a special cubic lattice graph that can be embedded in $\mathbb{R}^d$, so that the known techniques work similarly as in the “continuum” case. Recall that rectangular lattice graphs also exhibit other interesting spectral properties [Ex95].

The embedding into $\mathbb{R}^d$ provides an easy way to describe our graph Γ. Let $V(\Gamma) = \mathbb{Z}^d$ be the vertex set and let the set of edges $E(\Gamma)$ consist of all line segments of length one between two neighbouring vertices in directions of the coordinate axes. As usual we identify each edge with the interval $[0,1]$ with orientation in the sense of the increasing coordinate in $\mathbb{R}^d$. The initial and endpoint of an edge e are labeled by $\iota(e)$ and $\tau(e)$.

The embedding of Γ into $\mathbb{R}^d$ allows us to define subgraphs of Γ in terms of suitable domains in $\mathbb{R}^d$. To make this precise, we will call a bounded domain $\Lambda \subset \mathbb{R}^d$ with piecewise smooth boundary $\Gamma-$*edge bounded (*Γ*-ebdd.)* if $\partial\Lambda \subset E(\Gamma)$ and for each edge $e \in E(\Gamma)$ either $e \subset \partial\Lambda$, or e intersects $\partial\Lambda$ at most in its endpoints. The graph $\Gamma \cap \Lambda$ arises from Γ by deleting all the edges outside Λ (including those on the boundary). For its sets of edges and vertices we write $E(\Gamma \cap \Lambda)$ and $V(\Gamma \cap \Lambda)$, respectively.

The Hilbert space underlying our model is $L_2(\Gamma) := \bigoplus_{e\in E(\Gamma)} L_2(0,1)$; in a similar way we associate $L_2(\Gamma\cap\Lambda) := \bigoplus_{e\in E(\Gamma\cap\Lambda)} L_2(0,1)$ with $\Gamma\cap\Lambda$. Further we need the Sobolev space of order one,

$$\begin{aligned} W_2^1(\Gamma) \quad &:= \quad \{f \in \bigoplus_{e\in E} W_2^1(0,1) \,|\, f \text{ continuous at all vertices } v \in V, \\ &\qquad \|f\|^2_{W_2^1(\Gamma)} := \sum_{e\in E(\Gamma)} \|f_e\|^2_{W_2^1(0,1)} < \infty\} \end{aligned}$$

with the obvious notation and terminology for edge components of f, and its analogue $W_2^1(\Gamma\cap\Lambda)$.

We can now define the random Schrödinger operator $H(\omega)$ for $\omega \in \Omega := [q_-, q_+]^E$ via their associated forms,

$$\begin{aligned} D(\mathfrak{h}_\omega) \quad &= \quad W_2^1(\Gamma), \\ \mathfrak{h}_\omega(f,g) \quad &= \quad \sum_{e\in E(\Gamma)} [(f_e' \,|\, g_e')_{L_2(0,1)} + (\omega_e \cdot f_e \,|\, g_e)_{L_2(0,1)}] \end{aligned}$$

These self-adjoint operators correspond to the differential expression $-f_e'' + \omega_e \cdot f_e$ on the edges, together with the free (often called Kirchhoff) boundary conditions at the inner vertices, i.e.

$$\sum_{\iota(e)=v} f_e'(0) - \sum_{\tau(e)=v} f_e'(0) = 0 \qquad (\forall v \in V \cap \Lambda),$$

The coupling constants ω_e carry the random structure. They are picked independently for different edges with a probability measure μ on $\mathbb{R}$ with $\operatorname{supp}\mu =$

$[q_-, q_+]$. For technical reasons we have to assume that μ is Hölder continuous with Hölder exponent α and further that μ satisfies the following assumption: there exists $\tau > \frac{d}{2}$ such that for h small

$$\mu([q_-, q_- + h]) \leq h^\tau. \tag{1}$$

This *single site measure* μ defines a probability $\mathbb{P} := \bigotimes_{e \in E} \mu$ on Ω.

We will also need restrictions $H_\Lambda^N(\omega)$ for an ebbd open Λ defined via the form

$$\begin{aligned} D(\mathfrak{h}_{\Lambda,\omega}^N) &= W_2^1(\Gamma \cap \Lambda) \\ \mathfrak{h}_{\Lambda,\omega}^N(f,g) &= \sum_{e \in E(\Gamma \cap \Lambda)} [(f_e' \,|\, g_e')_{L_2(0,1)} + (\omega_e \cdot f_e \,|\, g_e)], \end{aligned}$$

which corresponds to Neumann boundary conditions at the boundary vertices $v \in V \cap \partial\Lambda$ – cf. [Ku04].

2 The main results and the idea of their proof

Our family of random Schrödinger operators exhibits deterministic spectrum, i.e. there exists a closed subset $\Sigma \subset \mathbb{R}$ s.t. $\sigma(H(\omega)) = \Sigma$ almost surely. This is a standard result from the theory of random operators – see, e.g., [CL90] – and comes from fundamental properties of our construction, especially the ergodicity w.r.t. lattice translations. To locate the deterministic spectrum we can consider the free operator H_0 (the one with $V = 0$) and use some results that relate the spectrum of H_0 to the spectrum of its transition operator, the Laplacian on $\mathbb{Z}^d$ – see, e.g., [Ex97, Cat97]. In this way we get $\sigma(H_0) = [0, \infty)$ and hence again by standard theory $\Sigma = [q_-, \infty)$.

Our first claim is that in some neighborhood of $\inf \Sigma = q_-$ the operators exhibit pure point spectrum with exponentially decaying eigenfunctions almost surely:

2.1 Theorem (Spectral/Anderson localization) *There is an $\varepsilon > 0$ such that the spectrum of $H(\omega)$ in $[q_-, q_- + \varepsilon_0]$ is pure point for a.e. $\omega \in \Omega$. Furthermore, there exists a $\gamma > 0$ and for each eigenfunction u associated to an energy in this interval a constant C_u such that*

$$\|\chi_{\Lambda_1(x)} u\| \leq C_u \cdot \exp[-\gamma\, d(x, 0)] \qquad (x \in \Gamma),$$

where $\Lambda_1(x)$ is the intersection of Γ with the unit cube centered at $x \in \mathbb{Z}^d$.

The assertion of the preceding theorem is sometimes called Anderson localization or spectral localization (see [RJLS95] for a discussion of different concepts of localization). An alternative and stronger concept is dynamical localization, see [GdB98, DSt01] and [GK01] for more recent developments. In the context of our model the following result is valid.

2.2 Theorem (Strong dynamical localization) *Let $p > 2(2\tau - d)$ where τ refers to (1). Then there exists an $\varepsilon > 0$ such that for $K \subset \Gamma$ compact, each interval $I \subset [E_0, E_0 + \varepsilon]$ and $\eta \in L_\infty(\mathbb{R})$ with supp $\eta \subset I$ we have*

$$\mathbb{E}\{\| |X|^p \eta(H(\omega))\chi_K \|\} < \infty,$$

which in particular means that

$$\mathbb{E}\{\sup_{t>0} \| |X|^p e^{-itH(\omega)} P_I(H(\omega))\chi_K \|\} < \infty.$$

Both results will be proved by a multiscale induction as presented in detail in [Sto01]. As the framework introduced there is general enough to include our case it will be sufficient to establish the necessary model-dependent estimates that are to be plugged into the multiscale machinery.

For the readers convenience we will now briefly describe the idea behind the multiscale induction. The basic property one proves by induction is an exponential decay estimate for the kernel of the resolvent of $H^N_{\Lambda(L)}(\omega)$. More precisely, it is shown that with high probability (depending on L) the resolvent of $H^N_{\Lambda(L)}(\omega)$ shows exponential off-diagonal decay.

Note that, outside the spectrum of a Schrödinger operator, such an exponential decay estimate is just the content of the celebrated *Combes-Thomas estimate*. We will make clear that an analogue holds for quantum graphs as well. Actually, this kind of argument will give the starting point of our induction procedure, the *initial length scale estimate*. More precisely, the assumption (1) on the tail of the single site measure implies that energies near $\inf \Sigma$ are in the resolvent set of $H^N_{\Lambda(L)}(\omega)$ with high probability for any given L. However, keeping an interval near $\inf \Sigma$ fixed and letting L tend to infinity, the interval will be filled with eigenvalues of the box Hamiltonian. Therefore the sought property, the exponential decay, must be deduced by a more clever argument. One important ingredient is the relation between resolvents of different nested boxes, cast in the form of a *geometric resolvent identity*. This will allow to conclude exponential decay on a large box, knowing exponential decay on smaller sub-boxes. In this induction step, from length L one proceeds to L^α with suitable $\alpha > 1$. A very important a priori information is necessary, the so-called *Wegner estimate*. Putting these estimates together as in [Sto01] one arrives at the desired exponential decay estimates for larger and larger boxes. To conclude, finally, that the operators $H(\omega)$ exhibit pure point spectrum almost surely, we need to know that the spectrum is indeed determined by *generalized eigenfunctions*. In the next section we show how to obtain these steps.

2.3 Remarks (a) Our results can easily be extended to certain other cases, for instance, to a “rhombic” lattice, where the present method would work after adjusting constants appearing in the equivalence between the Euclidean and the

intrinsic metric.
(b) The results could be also extended to potentials, which are only relatively bounded, for instance, one can consider suitable $L_p(0,1)-$functions with a positive lower bound as "single edge" potentials, following [Sto01] and numerous other papers; we did not take this path and treated characteristic functions as random potentials here exclusively for the sake of simplicity.
(c) In a different direction, results are available for certain random quantum graphs, namely for random trees with random edge lengths; see the recent work in [ASW06, HP06].

3 The proofs

3.1 A Combes-Thomas estimate

The statements of this section will show how to obtain "exponential decay of the local resolvent" outside the spectrum. The results go back to the celebrated paper [CT73] and its improvement in [BCH97].

3.1 Theorem (Combes-Thomas estimate) *Let $R > 0$. There exist constants $c_1 = c_1(q_-, q_+, R)$, $c_2 = c_2(q_-, q_+, R)$, s.t. from the assumptions*

(i) $\Lambda \subset \mathbb{R}^d$ $\Gamma-$*ebdd. box,* $A, B \subset \Lambda$ $\Gamma-$*ebdd.,* $\mathit{dist}(A, B) =: \delta \geq 1$,

(ii) $(r, s) \subset \varrho(H_\Lambda^N) \cap (-R, R)$, $E \in (r, s)$, $\eta := \mathit{dist}(E, (r, s)^c) > 0$

it follows that

$$\|\chi_A(H_\Lambda^N - E)^{-1}\chi_B\| \leq c_1 \cdot \eta^{-1} \cdot e^{-c_2\sqrt{\eta(s-r)}\,\delta}.$$

Proof: Let $w : \Lambda \to \mathbb{R}$ be defined as $w(x) := \operatorname{dist}(x, B)$. By triangle inequality

$$|w(y) - w(x)| \leq |x - y|,$$

so that $\|\nabla w\|_\infty \leq 1$, and this in turn implies $\|w'\|_\infty \leq 1$ for the restriction to the graph. Furthermore, the functions $\psi(x) = e^{-w(x)}$ and $\varphi(x) = e^{w(x)}$ are uniformly Lipschitz continuous on all edges because

$$\begin{aligned} |e^{w(y)} - e^{w(x)}| &\leq \sup_{\xi\in\Gamma\cap\Lambda} |(\exp\circ w)'(\xi)| \cdot |y - x| \\ &\leq \sup_{\xi\in\Gamma\cap\Lambda} |\exp(w(\xi))||w'(\xi)| \cdot |y - x|. \end{aligned}$$

Hence for each $u \in D(\mathfrak{h})$ also the functions $\psi u, \varphi u$ belong to $D(\mathfrak{h})$, which means that

$$\mathfrak{h}_\beta(u, v) := \mathfrak{h}(e^{-\beta w}u, e^{\beta w}v)$$

is well defined for all $u, v \in D(\mathfrak{h})$. By the product rule we have the relation

$$\begin{aligned}
\mathfrak{h}_\beta(u,v) &= (e^{-\beta w}u' \,|\, e^{\beta w}v') - \beta((e^{-\beta w}uw' \,|\, e^{\beta w}v') \\
&\quad -\beta^2((e^{-\beta w}uw' \,|\, e^{\beta w}vw') + \beta((e^{-\beta w}u' \,|\, e^{\beta w}vw') + (Vu \,|\, v) \\
&= \mathfrak{h}(u,v) - \beta \underbrace{[(uw' \,|\, v') - (u \,|\, vw')]}_{(*)} - \beta^2(w'^2 u \,|\, v).
\end{aligned}$$

Referring to the term $(*)$ above we define the symmetric form

$$\mathfrak{k}(u,v) := i[(uw' \,|\, v') - (u \,|\, vw')].$$

Using $1 \geq m := w'^2 \geq 0$ one can write

$$\begin{aligned}
\mathfrak{h}_\beta(u,v) &= \tilde{\mathfrak{h}}(u,v) + i\beta\mathfrak{k}(u,v), \quad \text{where} \\
\tilde{\mathfrak{h}}(u,v) &= \mathfrak{h}(u,v) - \beta^2(mu \,|\, v).
\end{aligned}$$

Next we are going to show that $\mathfrak{h}_\beta$ is sectorial. From $\|w'\|_\infty \leq 1$ one gets

$$\mathfrak{k}(u) \leq 2\|u\|\,\|u'\| \leq \|u'\|^2 + \|u\|^2. \tag{2}$$

On the other hand, consider the operator $\tilde{H}$ associated with $\tilde{\mathfrak{h}}$ and $C = C(R)$, $C \geq \beta^2 + 1$, $C \geq 1 - r\mathrm{m}$ for which we have

$$\begin{aligned}
\|(\tilde{H} + C)^{\frac{1}{2}}u\|^2 &\geq \|(\tilde{H} + \beta^2 + 1)^{\frac{1}{2}}u\|^2 \\
&= \|u'\|^2 + ([V + \beta^2 \underbrace{(1-m)}_{\geq 0} + 1]u \,|\, u) \\
&\geq \|u'\|^2 + \|u\|^2.
\end{aligned} \tag{3}$$

It follows from (2) and (3) that

$$|\mathfrak{k}(u)| \leq \|(\tilde{H} + C)^{\frac{1}{2}}u\|^2 = (\tilde{\mathfrak{h}} + C)(u), \tag{4}$$

hence $\mathfrak{h}_\beta = \tilde{\mathfrak{h}} + i\beta\mathfrak{k}$ is sectorial and there exists an associated sectorial operator H_β – see, e.g., [Kato76].

In the next step we are going to show the existence of a bounded operator S on $L_2(\Gamma \cap \Lambda)$, $\|S\| \leq 1$, s.t.

$$\mathfrak{k}(u,v) = (S(\tilde{H} + C)^{\frac{1}{2}}u \,|\, (\tilde{H} + C)^{\frac{1}{2}}v) \qquad (\forall u, v \in D(\mathfrak{h})).$$

Let thus $D(\mathfrak{h})$ be equipped with the scalar product $(\tilde{h} + C)(\cdot,\cdot)$. By the Riesz representation theorem there exists a bounded operator K on $D(\mathfrak{h})$ with

$$\mathfrak{k}(u,v) = (\tilde{\mathfrak{h}} + C)(Ku, v)$$

and by (4) we have $\|K\| \leq 1$. Put

$$S := (\tilde{H} + C)^{\frac{1}{2}} K (\tilde{H} + C)^{-\frac{1}{2}} : L_2(\Gamma \cap \Lambda) \to L_2(\Gamma \cap \Lambda).$$

As $(\tilde{H} + C)^{\frac{1}{2}} : D(\mathfrak{h}) \to L_2(\Gamma \cap \Lambda)$ and $(\tilde{H} + C)^{-\frac{1}{2}} : L_2(\Gamma \cap \Lambda) \to D(\mathfrak{h})$ are unitary, we have $\|S\| = \|K\| \leq 1$, and for $u, v \in D(\mathfrak{h})$ we get the desired relation

$$\begin{aligned} ((\tilde{H} + C)^{\frac{1}{2}} u \,|\, (\tilde{H} + C)^{\frac{1}{2}} v) &= ((\tilde{H} + C)^{\frac{1}{2}} K u \,|\, (\tilde{H} + C)^{\frac{1}{2}} v) \\ &= \mathfrak{k}(u, v). \end{aligned}$$

Now we have to investigate invertibility of $H_\beta - E$ for $E \in (r, s)$ in dependence on β. Here we can use the proof of [Sto01] (which in turn uses Lemma 3.1. from [BCH97]) word by word, so we present just the result: let

$$\beta_1 := \min \left\{ \beta_0, \, \frac{1}{R + C} \sqrt{\frac{1}{32} \eta \, (s - r)} \right\},$$

then for $|\beta| \leq \beta_1$ the operator $T + i\beta S$ is invertible with

$$\|(T + i\beta S)^{-1}\| \leq 4 \, \frac{R + C}{\eta}. \tag{5}$$

Next we will find a connection between $T + i\beta S$ and $H_\beta - E$ which shows that for $|\beta| \leq \beta_1$ the operator $H_\beta - E$ is invertible too, namely

$$(H_\beta - E)^{-1} = (\tilde{H} + C)^{-\frac{1}{2}} (T + i\beta S)^{-1} (\tilde{H} + C)^{-\frac{1}{2}}. \tag{6}$$

Let $f \in L_2(\Gamma \cap \Lambda)$, then

$$u := (\tilde{H} + C)^{-\frac{1}{2}} (T + i\beta S)^{-1} (\tilde{H} + C)^{-\frac{1}{2}} f \in D(\mathfrak{h})$$

holds, since $(\tilde{H} + C)^{-\frac{1}{2}}$ maps $L_2(\Lambda)$ to $D(\mathfrak{h})$. Using the definitions of T, S and u we can calculate for $v \in D(\mathfrak{h})$ the expression

$$\begin{aligned} (\mathfrak{h}_\beta - E)(u, v) &= (\tilde{\mathfrak{h}} - E)(u, v) + i\beta \, \mathfrak{k}(u, v) \\ &= (T(\widetilde{H} + C)^{\frac{1}{2}} u | (\widetilde{H} + C)^{\frac{1}{2}} v) + i\beta \, (S(\widetilde{H} + C)^{\frac{1}{2}} u | (\widetilde{H} + C)^{\frac{1}{2}} v) \\ &= ((T + i\beta S)(\widetilde{H} + C)^{\frac{1}{2}} u | (\widetilde{H} + C)^{\frac{1}{2}} v) \\ &= ((\widetilde{H} + C)^{-\frac{1}{2}} f | (\widetilde{H} + C)^{\frac{1}{2}} v) \\ &= (f | v). \end{aligned}$$

Consequently, we have $u \in D(H_\beta - E)$ and $(H_\beta - E)u = f$, so (6) follows, and by (5) we get

$$\|(H_\beta - E)^{-1}\| \leq 4 \, \frac{R + C}{\eta}. \tag{7}$$

A straightforward calculation now shows that

$$(H_\beta - E)^{-1} f = e^{\beta w}(H-E)^{-1} e^{-\beta w} f,$$

and therefore

$$\|\chi_A (H-E)^{-1} \chi_B\| \leq \|\chi_A e^{-\beta w}\|_\infty \cdot \|(H_\beta - E)^{-1}\| \cdot \|e^{\beta w} \chi_B\|_\infty. \tag{8}$$

Putting $\beta := \frac{1}{2}\beta_1$ we analyze the factors on the right-hand side. By $w|_B = 0$ one has $\|e^{\beta w}\chi_B\|_\infty \leq 1$. The second factor is controlled by (7), and furthermore, by definition of β_1 there is a constant $c_2 = c_2(R)$ s.t.

$$\beta \geq c_2(R) \cdot \sqrt{\eta(s-r)}.$$

By assumption, $w(x) = \mathrm{dist}(x, B) \geq \delta$ for all $x \in A$, i.e.

$$\|\chi_A e^{-\beta w}\|_\infty \leq e^{-\beta \cdot \delta} \leq \exp(-c_2(R) \cdot \sqrt{\eta(s-r)} \cdot \delta).$$

Combining this argument with (8) we get finally the result,

$$\|\chi_A (H-E)^{-1} \chi_B\| \leq c_1(R) \cdot \eta^{-1} \cdot \exp(-c_2(R) \cdot \sqrt{\eta(s-r)} \cdot \delta).$$

3.2 The initial length scale estimate

The initial length scale estimate tells us something about the probability that an eigenvalue of the box hamiltonian is found inside a suitable interval. Specifically, we take an interval centered at the lower bound q_- of the deterministic spectrum and we suppose that its length depends on the size l of the box. The estimate we are interested in will only hold for lengths larger than some initial value l^*.

3.2 Theorem (Initial length scale estimate) *For each $\xi \in (0, 2\tau - d)$ there exist $\beta = \beta(\tau, \xi) \in (0,2)$ and $l^* = l^*(\tau, \xi)$ such that*

$$\mathbb{P}\{\mathit{dist}(\sigma(H_\Lambda^N(\omega)), q_-) \leq l^{\beta-2}\} \;\leq\; l^{-\xi} \tag{9}$$

holds for all $\Gamma-$ebdd. boxes $\Lambda = \Lambda_l(0)$ with $l \geq l^*$.

Proof: Let

$$\Omega_{l,h} := \{\omega \in \Omega \mid q_e(\omega) \geq q_- + h \text{ for all } e \in E(\Gamma \cap \Lambda\}.$$

By the min-max principle we infer that for $\omega \in \Omega_{l,h}$

$$E_0(H_\Lambda^N) \geq E_0((-\Delta + q_- + h)_\Lambda^N) = q_- + h,$$

where E_0 is the lowest eigenvalue of the respective operator. Using assumption (1) the probability of $\Omega_{l,h}$ can be estimated by

$$\begin{aligned}\mathbb{P}(\Omega_{l,h}) &\geq 1 - \sharp E(\Gamma \cap \Lambda) \cdot \mu([q_-, q_- + h]) \\ &\geq 1 - d \cdot |\Lambda| \cdot h^\tau.\end{aligned}$$

Let $\xi \in (0, 2\tau - d)$. Then it is always possible to choose $\beta \in (0,2)$ such that

$$\xi < \tau(2-\beta) - d,$$

and inserting $h := l^{\beta-2}$ we get for l large

$$\begin{aligned}\mathbb{P}(\Omega_{l,h}) &= 1 - d\,|\Lambda|\, l^{\tau(\beta-2)} \\ &= 1 - \underbrace{d\, l^{\xi-\tau(2-\beta)+d}}_{\leq 1 \text{ for } l \text{ large}}\, l^{-\xi} \\ &\geq 1 - l^{-\xi}.\end{aligned}$$

3.3 The geometric resolvent inequality

As we mentioned above, in the multiscale induction step one has to deal with restrictions of a Schrödinger operator to nested cubes on different length scales. Consequently, we need a tool that relates the resolvents of such restrictions. The first step on this way is the following lemma, called geometric resolvent equality.

3.3 Lemma (Geometric resolvent equality) *Let $\Lambda \subset \Lambda' \subset \mathbb{R}^d$ be some open $\Gamma-$ebdd. boxes, H_Λ and $H_{\Lambda'}$ the respective realizations of our model operator with Neumann b.c. Let $\psi \in \{f|_{\Gamma\cap\Lambda} \mid f \in C_c^1(\Lambda)\}$ be real-valued. Then we have for each $z \in \varrho(H_\Lambda) \cap \varrho(H_{\Lambda'})$ the relation*

$$R_\Lambda \psi = \psi\, R_{\Lambda'} + R_\Lambda\, [\psi' \cdot D + D\, \psi']\, R_{\Lambda'},$$

where we have denoted $R_\Lambda := (H_\Lambda - z)^{-1}$, $R_{\Lambda'} := (H_{\Lambda'} - z)^{-1}$, D is the first derivative, and all the terms are interpreted as operators on $L_2(\Gamma \cap \Lambda')$.

Proof: We regard $L_2(\Gamma \cap \Lambda)$ as a subspace of $L_2(\Gamma \cap \Lambda')$. In terms of the associated forms the assertion then reads as follows:

$$\begin{aligned}(\mathfrak{h}_\Lambda - z)(\psi\, R_{\Lambda'} + R_\Lambda\, [\psi' \cdot D + D\, \psi']\, R_{\Lambda'})g,\, w) &= (\psi g \mid w) \\ &\quad (\forall g \in L_2(\Gamma \cap \Lambda), w \in D(\mathfrak{h}));\end{aligned}$$

notice that in this case the first argument at the left-hand side, which we denote as u, belongs to $D(H_\Lambda)$ and $(H - z)u = \psi \cdot g$.

In the first step we have to show that $u \in D(\mathfrak{h})$ holds. By the product rule, $\psi_e\,(R_{\Lambda'}g)_e \in W_2^1(0,1)$ for all $e \in E(\Gamma \cap \Lambda)$. The continuity of $\psi\, R_{\Lambda'}g$ at the inner vertices of Λ' is clear, so the first term is controlled. Further we find $\psi' D R_{\Lambda'} : L_2(\Gamma\cap\Lambda) \to L_2(\Gamma\cap\Lambda)$, i.e.

$$R_\Lambda \psi' D R_{\Lambda'} g \in D(\mathfrak{h}).$$

For the analysis of the third term one has

$$\psi' R_{\Lambda'} g \in L_2(\Gamma\cap\Lambda).$$

Now $R_\Lambda D : W_2^1(\Gamma\cap\Lambda) \to D(\mathfrak{h}_\Lambda)$ extends to a bounded operator from $L_2(\Gamma\cap\Lambda)$ to $D(\mathfrak{h}_\Lambda)$. Indeed, we can always choose z small enough, in which case

$$R((H_\Lambda - z)^{-\frac{1}{2}}) = D((H_\Lambda - z)^{\frac{1}{2}}) = D(\mathfrak{h}_\Lambda) \subset W_2^1(\Gamma\cap\Lambda).$$

For $v \in D(\mathfrak{h}_\Lambda)$ we have

$$\begin{aligned} \|v'\|^2_{L_2(\Gamma\cap\Lambda)} &= \mathfrak{h}(v) - \sum_{e\in E(\Gamma\cap\Lambda)} V_e \int_0^1 v_e^2(x)dx \\ &\leq \|v\|_{D(\mathfrak{h}_\Lambda)}, \end{aligned}$$

i.e. $D\,(H_\Lambda - z)^{-\frac{1}{2}}$ is bounded on $L_2(\Gamma\cap\Lambda)$. Thus for $\varphi \in W_2^1(\Gamma\cap\Lambda)$ and $f \in L_2(\Gamma\cap\Lambda)$ we get

$$\begin{aligned} |((H_\Lambda - z)^{\frac{1}{2}}\varphi' \,|\, f)| &= |(\varphi|\; D\,(H_\Lambda - z)^{-\frac{1}{2}} f)| \\ &\leq c\cdot \|\varphi\|_{L_2(\Gamma\cap\Lambda)} \cdot \|f\|_{L_2(\Gamma\cap\Lambda)}, \end{aligned}$$

and from here finally the boundedness of the map

$$R_\Lambda D = (H_\Lambda - z)^{-\frac{1}{2}}(H_\Lambda - z)^{-\frac{1}{2}} D : \; L_2(\Gamma\cap\Lambda) \to L_2(\Gamma\cap\Lambda) \to D(\mathfrak{h}_\Lambda).$$

The next step is to control the behavior of some functions at the inner vertices. For a fixed inner vertex of $\Gamma\cap\Lambda$ let $e_{k,\mathrm{in}}$ and $e_{k,\mathrm{out}}$ be the in- and outcoming edges, respectively, parallel to the $k-$th coordinate axis, and let $\partial_k\psi(v)$ be the $k-$th partial derivative of the $C_c^1(\Lambda)-$continuation of ψ. Then

$$\begin{aligned} (D\psi' R_{\Lambda'} g \,|\, w)_{L_2(\Gamma\cap\Lambda)} &= \sum_{e\in E(\Gamma\cap\Lambda)} (D\psi_e' R_{\Lambda'} g_e \,|\, w_e)_{L_2(0,1)} \\ &= \sum_{e\in E(\Gamma\cap\Lambda)} \{(-\psi_e' R_{\Lambda'} g_e \,|\, w_e')_{L_2(0,1)} + \psi_e' R_{\Lambda'} g_e w_e \,|_0^1\} \\ &= -(\psi' R_{\Lambda'} g \,|\, w')_{L_2(\Gamma\cap\Lambda)} \\ &\quad + \sum_{v \text{ inn. vertex}} \sum_{k=1}^{d} \partial_k\psi(v) \{\underbrace{(R_{\Lambda'} g w)_{e_{k,\mathrm{in}}}(1) - (R_{\Lambda'} g w)_{e_{k,\mathrm{out}}}(0)}_{=0 \text{ by continuity at inner vertices}}\} \\ &= -(\psi' R_{\Lambda'} g \,|\, w')_{L_2(\Gamma\cap\Lambda)}. \end{aligned} \tag{10}$$

The following calculation now finishes the proof:

$$
\begin{aligned}
(\mathfrak{h}_\Lambda - z)(u,w) &= (\mathfrak{h}_\Lambda - z)(\psi\, R_{\Lambda'}g, w) + ((\psi' \cdot DR_{\Lambda'} + D\,\psi' R_{\Lambda'})g \,|\, w) \\
&\overset{(10)}{=} ((\psi\, R_{\Lambda'}g)' \,|\, w') + ((V-z)\psi\, R_{\Lambda'}g \,|\, w) \\
&\quad +(\psi'(R_{\Lambda'}g)' \,|\, w) - (\psi' R_{\Lambda'}g \,|\, w') \\
&= (\psi'\, R_{\Lambda'}g \,|\, w') + (\psi(R_{\Lambda'}g)' \,|\, w') + (\psi'(R_{\Lambda'}g)' \,|\, w) \\
&\quad +((V-z)\psi\, R_{\Lambda'}g \,|\, w) - (\psi' R_{\Lambda'}g \,|\, w') \\
&\overset{\psi \text{ real val.}}{=} ((R_{\Lambda'}g)' \,|\, (\psi w)') + ((V-z)\, R_{\Lambda'}g \,|\, \psi w) \\
&= (\mathfrak{h}_{\Lambda'} - z)(R_{\Lambda'}g, \psi w) \\
&= (g \,|\, \psi w) \\
&= (\psi g \,|\, w).
\end{aligned}
$$

We will next prove another preparatory lemma after which we will be ready to state the main theorem of this section.

3.4 Lemma *Let $\tilde{\Omega} \subset \Omega \subset \mathbb{R}^d$ be a $\Gamma-$ebdd. domains, dist$(\partial\tilde{\Omega}, \partial\Omega) > 0$, $E \in \mathbb{R}$ and $g \in L_2(\Gamma \cap \Omega)$. Then there exists $C = C(q_-, q_+, E)$ s.t. for all $u \in W_2^1(\Gamma \cap \Omega)$ with*

$$
(u' \,|\, \varphi')_{L_2(\Gamma\cap\Omega)} + (Vu \,|\, \varphi)_{L_2(\Gamma\cap\Omega)} = (g \,|\, \varphi)_{L_2(\Gamma\cap\Omega)} \qquad (\forall \varphi \in W_{2,0}^1(\Gamma \cap \Omega))
$$

it holds that

$$
\|u'\|_{L_2(\Gamma\cap\tilde{\Omega})} \le C(\|u\|_{L_2(\Gamma\cap\Omega)} + \|g\|_{L_2(\Gamma\cap\Omega)}).
$$

Proof: By construction, $\text{dist}(\partial\Omega, \partial\tilde{\Omega}) \ge 1$, hence there exists a vector

$$
\psi \in \{f|_{\Gamma\cap\Omega} \,|\, f \in C_c(\Omega), \text{supp}\, f' \subset \{x \in \Omega \,|\, \text{dist}(x, \tilde{\Omega}) < 1\}\}
$$

with $0 \le \psi \le 1$, $\psi|_{\Gamma\cap\tilde{\Omega}} = 1$ and $\|\psi'\|_\infty \le \tilde{C}(d)$. Let $w := u\psi^2$, then $w \in W_{2,0}^1(\Gamma \cap \Omega)$, and by product rule we find

$$
(u' \,|\, w')_{L_2(\Gamma\cap\Omega)} = (\psi u' \,|\, \psi u') + 2(\psi u' \,|\, u\psi').
$$

Using $\tilde{V} := V - E$ and support properties of the functions involved we get

$$
\begin{aligned}
\|\psi u'\|^2 &= (u' \,|\, w') - 2(\psi u' \,|\, u\psi') \\
&= (g \,|\, w) - (\tilde{V}u \,|\, w) - 2(\psi u' \,|\, u\psi') \\
&\le \|g\|\,\|u\| + |(\tilde{V}\psi u \,|\, \psi u)| + 2\,\|\psi u'\|\,\|u\|\,\|\psi'\|_\infty \\
&\le \|g\|\,\|u\| + \hat{C}(q_-, q_+, E)\|u\|^2 + 2\tilde{C}\,\|\psi u'\|\,\|u\|.
\end{aligned}
$$

We consider the latter as a quadratic inequality in $\|\psi u'\|$, and find after some simple manipulations, that it can only be fulfilled for

$$\begin{aligned}\|\psi u'\| &\leq \sqrt{\tilde{C}^2+\hat{C}}\,\|u\| + \frac{1}{2\sqrt{\tilde{C}^2+\hat{C}}}\,\|g\| \\ &= C(q_-,q_+,E)(\|u\|+\|g\|)\end{aligned}$$

By $\psi|_{\Gamma\cap\tilde{\Omega}}=1$ the assertion follows. $\square$

Before we come to the main point we introduce some notation. A $\Gamma-$ebdd. box $\Lambda=\Lambda_L(x)$ is called *suitable*, if $x\in\mathbb{Z}^d$, $L\in 6\mathbb{N}\setminus 12\mathbb{N}$ and $L\geq 42$. For such boxes we define

$$\begin{aligned}\Lambda_{\text{int}}(x)=\Lambda_{L,\text{int}(x)} &:= \Lambda_{L/3}(x), \\ \Lambda_{\text{out}}(x)=\Lambda_{L,\text{out}(x)} &:= \Lambda_L(x)\setminus\Lambda_{L-12}(x)\end{aligned}$$

and write for the respective characteristic functions on the graph:

$$\chi_\Lambda^{\text{int}}=\chi_{\Lambda_L(x)}^{\text{int}}:=\chi_{\Gamma\cap\Lambda^{\text{int}}(x)}^{\text{int}},\quad \chi_\Lambda^{\text{out}}=\chi_{\Lambda_L(x)}^{\text{out}}:=\chi_{\Gamma\cap\Lambda^{\text{int}}(x)}^{\text{out}}.$$

In general the symbol χ_A for a $\Gamma-$ebdd. domain is to be understood as $\chi_{\Gamma\cap A}$.

3.5 Theorem (Geometric resolvent inequality) *Let $\Lambda\subset\Lambda'\subset\mathbb{R}^d$ be suitable $\Gamma-$ebdd. boxes. Let further $A\subset\Lambda^{int}$ and $B\subset\Lambda'\setminus\Lambda$ be $\Gamma-$ebdd. domains, $I_0\subset\mathbb{R}$ bounded and $E\in I_0$. Then there exists $C_{geom}=C_{geom}(q_-,q_+,E)$ s.t.*

$$\|\chi_B R_{\Lambda'}(E)\chi_A\|\leq C_{geom}\cdot\|\chi_B R_{\Lambda'}(E)\chi_\Lambda^{out}\|\,\|\chi_\Lambda^{out}R_\Lambda(E)\chi_A\|.$$

Proof: Let $x\in\mathbb{Z}^d$ be the center of Λ. We choose $\varphi\in\{f|_{\Gamma\cap\Lambda}\,|\,f\in C_c^\infty(\Lambda)\}$ real-valued with $\operatorname{supp} f\subset\Lambda_{L-4}(x)$ s.t. $\varphi=1$ on $\Lambda_{L-8}(x)$. This can be certainly achieved, with $\|\varphi'\|_\infty$ bounded independent on Λ.

Let $\Omega:=\operatorname{int}\Lambda^{\text{out}}$, i.e. $\operatorname{dist}(\partial\Omega,\operatorname{supp}\varphi')\geq 2$. By the geometric resolvent equality (Lemma 3.3) we have

$$\begin{aligned}\|\chi_B R_{\Lambda'}\chi_A\| &= \|\chi_A R_{\Lambda'}\chi_B\| \\ &= \|\chi_A(\varphi R_{\Lambda'}-R_\Lambda\varphi)\chi_B\| \qquad (\varphi|_A=1,\varphi|_B=0) \\ &\overset{\text{Lemma 3.3}}{=} \|\chi_A(\varphi R_\Lambda(D\varphi'+\varphi' D)R_{\Lambda'}\chi_B\| \\ &\leq \underbrace{\|\chi_A\varphi R_\Lambda D\varphi' R_{\Lambda'}\chi_B\|}_{(*)}+\underbrace{\|\chi_A\varphi R_\Lambda\varphi' D R_{\Lambda'}\chi_B\|}_{(**)}.\end{aligned}$$

We start with the analysis of $(*)$. If $\tilde{\Omega}:=\operatorname{int}(\Lambda_{L-2}(x)\setminus\Lambda_{L-10}(x))$ it holds that

$$\begin{aligned}(*) &= \|\chi_A\varphi R_\Lambda D\chi_{\tilde{\Omega}}\,\chi_\Omega\varphi' R_{\Lambda'}\chi_B\| \\ &\leq \|\varphi'\|_\infty\,\underbrace{\|\chi_A\varphi R_\Lambda D\chi_{\tilde{\Omega}}\|}_{(***)}\,\|\chi_\Omega R_{\Lambda'}\chi_B\|.\end{aligned}$$

The term $(***)$ can be now controlled with the help of Lemma 3.4. We put

$$f \in L_2(\Gamma \cap \Lambda), \qquad g := \chi_A f, \qquad u := R_\Lambda g.$$

Then $u \in D(\mathfrak{h})$ and

$$(\mathfrak{h}_\Lambda - E)(u, w) = (g \,|\, w)$$

for all $w \in D(\mathfrak{h}_\Lambda)$. Furthermore, we have $g|_\Omega = 0$ as well as $\operatorname{dist}(\partial\Omega, \partial\tilde{\Omega}) = 1$. Consequently, Lemma 3.4 is applicable and it gives

$$\begin{aligned} \|\chi_{\tilde{\Omega}} u'\| &\leq C_1(q_-, q_+, I)\, \|u\|_{L_2(\Gamma\cap\Omega)} \\ &= C_1(q_-, q_+, I)\, \|\chi_\Omega R_\Lambda \chi_A f\|, \end{aligned}$$

i.e.

$$(***) \leq C_1\,(q_-, q_+, I)\, \|\chi_\Lambda^{\text{out}} R_\Lambda \chi_A\|.$$

The term $(**)$ can be treated in a similar way. $\square$

3.4 The Wegner estimate

The Wegner estimate represents a statement about the probability that the operator $H_\Lambda^N(\omega)$, restricted to a $\Gamma-$ebdd. box $\Lambda = \Lambda_l(x)$ centered at $x \in \mathbb{Z}^d$, will have an eigenvalue near some fixed energy. Typically – and sufficiently for our multiscale analysis – this probability is polynomially bounded in terms of the box volume.

3.6 Theorem (Wegner estimate) *For each $R > 0$ there exists a constant C_R such that for all $\Gamma-$ebdd. boxes $\Lambda = \Lambda_l(i)$, $i \in \mathbb{Z}^d$, and all intervals $I \subset (-R, R)$ of length $|I|$ the following estimate holds:*

$$\mathbb{P}\{\sigma(H_\Lambda^N(\omega)) \cap I \neq \emptyset\} \;\leq\; C_R \cdot |\Lambda|^2 \cdot |I|^\alpha.$$

Before we start with the proof let us recall the following elementary lemma from [Sto00].

3.7 Lemma *Let J be a finite index set, μ a Hölder continuous probability measure on $\mathbb{R}^d$ with Hölder exponent α, $\mu^J := \otimes_{i\in J}\mu$ the product measure on $\mathbb{R}^J$. Let $\Phi : \mathbb{R}^J \to \mathbb{R}$ a monotone function, for which there are constants δ and $a > 0$ s.t. for all $t \in [0, \delta]$, $q \in \mathbb{R}^J$ we have*

$$\Phi(q + t(1, \ldots, 1)) - \Phi(q) \;\geq\; t \cdot a. \tag{11}$$

Then for each interval I of length smaller than $\varepsilon \leq a\delta$ the following estimate holds:

$$\mu^J(\{q : \Phi(q) \in I\}) \leq |J| \cdot \left(\frac{\varepsilon}{a}\right)^\alpha.$$

Proof of Theorem 3.6: We start with an estimate for the number of eigenvalues smaller than a given energy R. To this aim we define the Neumann-decoupled operator $-\Delta_\Lambda^{\text{N,dec}}$ via its quadratic form

$$\begin{aligned} D(\mathfrak{h}_\Lambda^{\text{N, dec}}) &= \oplus_{e\in E(\Gamma\cap\Lambda)} W_2^1(0,1), \\ \mathfrak{h}_\Lambda^{\text{N, dec}}(f,g) &:= \sum_{e\in E(\Gamma\cap\Lambda)} (f' \mid g')_{L_2(\Gamma\cap\Lambda)}. \end{aligned}$$

By a direct calculation the eigenvalues of this operator are $\frac{\pi^2}{4}n^2$, $n \in \mathbb{N}_0$, with the multiplicity $\sharp\{E(\Gamma\cap\Lambda)\} \leq d\cdot l^d = d|\Lambda|$. Hence there exists a constant $\tilde{C}_R$ s.t. for the n-th eigenvalue, counting multiplicity, it holds that

$$E_n(-\Delta_\Lambda^{\text{N, dec}}) > R \qquad \text{for } n > \tilde{C}_R|\Lambda|.$$

Now we have

$$H_\Lambda^N(\omega) \geq (H_0 + q_-)_\Lambda^N \geq -\Delta_\Lambda^{\text{N, dec}}$$

since $q_- \geq 0$ by assumption, and thus by min-max principle the corresponding inequality for the n-th eigenvalues. Using the previous inequality we get

$$\mathbb{P}\{\sigma(H_\Lambda^N(\omega)) \cap I \neq \emptyset\} \leq \sum_{n\leq \tilde{C}_R\cdot|\Lambda|} \mathbb{P}\{E_n(H_\Lambda^N(\omega)) \in I\}. \tag{12}$$

Next we estimate the terms of the sum by means of Lemma 3.7. Because of the independence of $H_\Lambda(\omega)$ of coupling constants outside Λ we have

$$\begin{aligned} \mathbb{P}\{E_n(H_\Lambda(\omega)) \in I\} &= \mu^{E(\Gamma)}\{\omega \mid E_n(H_\Lambda(\omega)) \in I\} \\ &= \mu^{E(\Gamma\cap\Lambda)}\{\tilde{\omega} = (\omega_e)_{e\in E(\Gamma\cap\Lambda)} \mid E_n(H_\Lambda(\omega)) \in I\}. \end{aligned}$$

By $\Phi(\tilde{\omega}) := E_n(H_\Lambda(\tilde{\omega})) = E_n(H_\Lambda(\omega))$ a monotone function on $\mathbb{R}^E(\Gamma\cap\Lambda)$ is defined, and it fulfills condition (11) because

$$\begin{aligned} H_\Lambda(\tilde{\omega} + t(1,\ldots,1)) &= -\Delta + \sum_{e\in E(\Gamma\cap\Lambda)} (\omega_e + t)\chi_e \\ &= H_\Lambda(\tilde{\omega}) + t. \end{aligned}$$

Hence by Lemma 3.7 we have

$$\begin{aligned} \mathbb{P}\{E_n(H_\Lambda(\omega)) \in I\} &\leq \sharp E(\Gamma\cap\Lambda)\cdot|I|^\alpha \\ &\leq d|\,\Lambda|\,|I|^\alpha, \end{aligned}$$

which in combination with (12) yields the assertion. $\square$

3.5 Expansion in generalized eigenfunctions

Now we come to the last statement needed for the multiscale analysis, namely that polynomially bounded generalized eigenfunctions exist spectrally a.s.

We want to use the main result from [BMSt03], that gives the polynomial boundedness in terms of the intrinsic metric (see [Stu94]) generated by the free Laplacian H_0 on the graph. Using the embedding of our graph into $\mathbb{R}^d$ it can easily be seen that the intrinsic metric is equivalent to the Euclidean one on $\mathbb{R}^d$, and consequently, after adjusting some constants the statement can be written in terms of absolute values as well. We start by checking the assumptions of [BMSt03]. First of all one has to show that the form $\mathfrak{h}_0$ associated with the free Laplacian is a Dirichlet form. Note that $\|\cdot\|_{\mathfrak{h}_0}$ is equivalent to the norm $\|\cdot\|_{W_2^1(\Gamma)}$ so $\mathfrak{h}_0$ is closed. For $u \in D(\mathfrak{h}_0)$ which is real-valued we have $|u| \in D(\mathfrak{h}_0)$, and therefore

$$\mathfrak{h}_0(|u|) = \sum_{e\in E(\Gamma)} \int_0^1 (\operatorname{sgn} u_e(x) u_e'(x))^2 \, dx \;=\; \mathfrak{h}_0(u).$$

If u is in addition nonnegative, we have $u \wedge 1 \in D(\mathfrak{h}_0)$ and

$$\mathfrak{h}_0(u\wedge 1) = \sum_{e\in E(\Gamma)} \int_0^1 u_e'(x)^2 \cdot 1_{[u_e<1]}(x) \, dx \;\le\; \mathfrak{h}_0(u).$$

Obviously $\mathfrak{h}_0$ is strongly local and regular – see, e.g., [BMSt03] for definitions.

The next point is that the volume of balls with respect to the intrinsic metric ϱ does not grow too fast as $R\to\infty$. Because the graph is embedded into $\mathbb{R}^d$ and the intrinsic metric and the $\|\cdot\|_1$-metric are equivalent, the volume of the ball $B_R^\varrho(x)$ can be estimated by the number of edges contained inside a box $\Lambda_{2R}(x)$ and hence by $c_d R^d$ for large R.

Finally, the third assumption to be checked is that e^{-tH_0} is bounded as a map from $L_2(\Gamma)$ to $L_\infty(\Gamma)$ for some $t > 0$. To this aim we employ the following extension of the ultracontractivity result [KMS06], Lemma 3.2, demonstrated by using the same method as in the cited paper.

3.8 Lemma *For $t \in (0,1]$ it holds that*

$$\|e^{-tH_0}\|_{L_2(\Gamma)\to L_\infty(\Gamma)} \le ct^{-\frac{1}{4}}.$$

Proof: By [Ou05], Thm. 6.3 ff, see also [Na58, FSt86, Dav89], it is sufficient to show that

$$\|f\|_{L_2(\Gamma)} \le C \cdot \|f\|_{\mathfrak{h}}^{\frac{1}{3}} \cdot \|f\|_{L_1(\Gamma)}^{\frac{2}{3}}$$

for $f \in D(\mathfrak{h}) \cap L_1(\Gamma)$. Now, by [Ga59, Ni59], or by [Ma85, Sect. 1.4.8], we have the following Nash type inequality for $u \in W_2^1(0,1)$:

$$\begin{aligned} \|u\|_{L_2(0,1)} &\le c_1 \cdot \left(\|u'\|_{L_2(0,1)} + \|u\|_{L_1(0,1)}\right)^{\frac{1}{3}} \cdot \|u\|_{L_1(0,1)}^{\frac{2}{3}} \\ &\le c_1 \cdot \|u\|_{W_2^1(0,1)}^{\frac{1}{3}} \cdot \|u\|_{L_1(0,1)}^{\frac{2}{3}}, \end{aligned}$$

where in the second step the Hölder inequality has been applied to $u \cdot 1$. For $f \in D(\mathfrak{h}) \cap L_1(\Gamma)$ we have by another application of Hölder inequality

$$\begin{aligned}
\|f\|^2_{L_2(\Gamma)} &= \sum_{e\in E(\Gamma)} \|f_e\|^2_{L_2(0,1)} \\
&\leq c_1^2 \sum_{e\in E(\Gamma)} \|f_e\|^{\frac{2}{3}}_{W_2^1(0,1)} \cdot \|f_e\|^{\frac{4}{3}}_{L_1(0,1)} \\
&\leq c_1^2 \Big(\sum_{e\in E(\Gamma)} \|f_e\|^2_{W_2^1(0,1)}\Big)^{\frac{1}{3}} \cdot \Big(\sum_{e\in E(\Gamma)} \|f_e\|_{L_1(0,1)}\Big)^{\frac{4}{3}} \\
&= c_2 \cdot \|f\|^{\frac{2}{3}}_{\mathfrak{h}} \cdot \|f\|^{\frac{4}{3}}_{L_1(\Gamma)}.
\end{aligned}$$

With these assumptions, given using the arguments in the opening of the section, [BMSt03] yields the following result:

3.9 Theorem *For spectrally a.a. $E \in \sigma(H)$ there exists a generalized eigenfunction φ*

$$(1 + |\cdot|^2)^{-\frac{m}{2}} \varphi \in L_2(\Gamma).$$

satisfying for any $m > \frac{d+1}{2}$.

This completes the necessary input for the use of Theorem 3.2.2 from [Sto01] and thus the proof of Theorems 2.1 and 2.2.

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