MULTI-SCALE ANALYSIS IMPLIES STRONG DYNAMICAL LOCALIZATION

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Abstract. We prove that a strong form of dynamical localization follows from a variable energy multi-scale analysis. This abstract result is applied to a number of models for wave propagation in disordered media.

1. Introduction

In the present paper we prove that a variable energy multi-scale analysis implies dynamical localization in a strong (expectation) form. Thus we accomplish a goal of a long line of research. Ever since Anderson's paper [5], the dynamics of waves in random media has been a subject of intensive research in mathematical physics. The breakthrough as far as mathematically rigorous results are concerned came with the paper [19] by Fröhlich and Spencer in which absence of diffusion is proven. They also introduced a technique of central importance to the topic: multi-scale analysis.

The next step was a proof of exponential localization, by which one understands pure point spectrum with exponentially decaying eigenfunctions; see the bibliography for a list of results in different generality. However, from the point of view of transport properties, exponential localization does not yield too much information. We refer to [13, 14] where a strengthening of exponential decay is introduced, property (SULE), which in fact allows one to prove dynamical localization. In [20] it was shown that a variable energy multi-scale analysis implies (SULE) almost surely, so that

$$\sup_{t>0} \| |X|^p e^{-itH(\omega)} P_I(H(\omega))\chi_K \| < \infty \quad \mathbb{P}\text{-a.s.},$$

where $H(\omega)$ is a random Hamiltonian which admits multi-scale analysis in the interval I, P_I is the spectral projector onto that interval, and K is compact.

We will strengthen the last statement to

$$\mathbb{E}\Big\{ \sup_{t>0} \| |X|^p e^{-itH(\omega)} P_I(H(\omega))\chi_K \| \Big\} < \infty.$$

Here, as in [20], the p which is admissible depends on the characteristic parameters of multi-scale analysis. In order to explain this we will sketch in the next section an abstract form of multi-scale analysis and introduce the necessary set-up. In

Section 3, we show that multi-scale analysis implies dynamical localization in the expectation. We do so by showing that (more or less) for $\eta \in L^\infty$, $\operatorname{supp} \eta \subset I$ (the localized region),

$$\mathbb{E}\Big\{ \|\chi_{\Lambda_1} \eta(H(\omega)) \chi_{\Lambda_2}\| \Big\} \le \|\eta\|_\infty \cdot \operatorname{dist}(\Lambda_1, \Lambda_2)^{-2\xi},$$

where ξ is one of the characteristic exponents of multi-scale analysis. We should note here that the main progress concerns continuum models, since for discrete models the Aizenman technique [1, 3] is available, which gives even exponential decay of the expectation above (see [2] for an exposition in which a number of applications is presented and the very recent [4] which shows that the Aizenman technique is applicable in the energy region in which multi-scale analysis works). However, our results clearly apply to discrete models with singular single-site distribution, most notably the one-dimensional Bernoulli-Anderson model. Moreover, we refer to [7] where a study of time means instead of the sup is undertaken. However, the latter paper does not contain too much about continuum models, and the results we present contain the estimates given there. In Section 4 we present our applications to a number of models for wave propagation in disordered media, including band edge dynamical localization for Schrödinger and divergence form operators as well as Landau Hamiltonians.

2. The multi-scale scenario

In this section we present the abstract framework for multi-scale analysis developed in [29]. We start with a number of properties which are easily verified for the applications we shall discuss, where $H(\omega)$ is a random operator in $L^2(\mathbb{R}^d)$ and $H_\Lambda(\omega)$ denotes its restriction to an open cube $\Lambda \subset \mathbb{R}^d$ with suitable boundary conditions.

We call a cube $\Lambda = \Lambda_L(x)$ of sidelength L centered at x *suitable* if $x \in \mathbb{Z}^d$ and $L \in 3\mathbb{N} \setminus 6\mathbb{N}$. In this case $\overline{\Lambda}$ itself as well as $\overline{\Lambda}_{L/3}(x)$ are unions of closed unit cubes centered on the lattice. Denote

$$\Lambda^{\text{int}} := \Lambda_{L/3}(x),\ \Lambda^{\text{out}} := \Lambda_L(x) \setminus \Lambda_{L-2}(x),$$

and denote the respective characteristic functions by $\chi^{\text{int}} = \chi^{\text{int}}_\Lambda = \chi^{\text{int}}_{L,x} := \chi_{\Lambda^{\text{int}}}$ and $\chi^{\text{out}} = \chi^{\text{out}}_\Lambda = \chi^{\text{out}}_{L,x} := \chi_{\Lambda^{\text{out}}}$.

The first condition concerns measurability and independence:

(INDY) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space; for every cube Λ, $H_\Lambda(\omega)$ is a self-adjoint operator in $L^2(\Lambda)$, measurable in ω, such that $H_{\Lambda_L(x)}(\omega)$ is stationary in $x \in \mathbb{Z}^d$ and H_Λ and $H_{\Lambda'}$ are independent for disjoint cubes Λ and Λ'.

So far, H_Λ and $H_{\Lambda'}$ are not related if $\Lambda \subset \Lambda'$. The next condition supplies a relation. In concrete examples it is the so-called geometric resolvent inequality which follows from commutator estimates and the resolvent identity. For $E \in \rho(H_\Lambda(\omega))$, we denote

$$R_\Lambda(E) = R_\Lambda(\omega, E) = (H_\Lambda(\omega) - E)^{-1}.$$

(GRI) For given bounded $I_0 \subset \mathbb{R}$, there is a constant C_{geom} such that for all suitable cubes Λ, Λ' with $\Lambda \subset \Lambda'$, $A \subset \Lambda^{\text{int}}$, $B \subset \Lambda' \setminus \Lambda$, $E \in I_0$ and $\omega \in \Omega$, the following inequality holds:

$$\|\chi_B R_{\Lambda'}(E)\chi_A\| \leq C_{\text{geom}} \cdot \|\chi_B R_{\Lambda'}(E)\chi_\Lambda^{\text{out}}\| \cdot \|\chi_\Lambda^{\text{out}} R_\Lambda(E)\chi_A\|.$$

Finally, we need an upper bound for the trace of the local Hamiltonians H_Λ in a given bounded energy region I_0, which follows from Weyl's law in concrete cases at hand.

(WEYL) For each interval $J \subset I_0$, there is a constant C such that

$$\operatorname{tr}(P_J(H_\Lambda(\omega)) \leq C \cdot |\Lambda| \quad \text{for all } \omega \in \Omega.$$

Here $P_J(\cdot)$ denotes the spectral projection of the operator in question. Given this basic set-up, multi-scale analysis deals with an inductive proof of resolvent decay estimates. This resolvent decay is measured in terms of the following concept:

Definition 2.1. *Let $\Lambda = \Lambda_L(x)$, $x \in \mathbb{Z}^d$, $L \in 2\mathbb{N}+1$. Λ is called (γ, E)-*good *for $\omega \in \Omega$ if*

$$\|\chi^{\text{out}} R_\Lambda(E)\chi^{\text{int}}\| \leq \exp(-\gamma \cdot L).$$

*Λ is called (γ, E)-*bad *for $\omega \in \Omega$ if it is not (γ, E)-good for ω.*

We can now define the property on which we base our induction:

G(I, L, γ, ξ) $\forall x, y \in \mathbb{Z}^d$, $d(x,y) \geq L$ the following estimate holds:

$$\mathbb{P}\{\forall E \in I : \Lambda_L(x) \ \text{ or } \Lambda_L(y) \ \text{ is } (\gamma, E)\text{-good for } \omega\} \geq 1 - L^{-2\xi}.$$

The basic idea of the multi-scale induction is that we consider some larger cube Λ' with sidelength $L' = L^\alpha$. With high probability there are not too many disjoint bad cubes of sidelength L in Λ'. Of course, since the number of cubes in Λ' is governed by α, this will only hold if α is not too large, depending on ξ.

By virtue of the geometric resolvent inequality (GRI), each of the good cubes of sidelength L in Λ' will add to exponential decay on the big cube. In order to make this work, we will additionally need a "worst case estimate." This is given by the following weak form of a Wegner estimate:

W(I, L, Θ, q) For all $E \in I$ and $\Lambda = \Lambda_L(x)$, $x \in \mathbb{Z}^d$, the following estimate holds:

$$\mathbb{P}\{\operatorname{dist}(\sigma(H_\Lambda(\omega)), E) \leq \exp(-L^\Theta)\} \leq L^{-q}.$$

We have the following theorem:

Theorem 2.2. *Let $I_0 \subset \mathbb{R}$ be a bounded open set and assume that $H_\Lambda(\omega)$ satisfies* (INDY), (GRI) *and* (WEYL) *for I_0.*

Assume that there are $L_0 \in 2\mathbb{N}+1$, $q > d$, $\Theta \in (0, \frac{1}{2})$ such that for $L \geq L_0$, $L \in 2\mathbb{N}+1$, the Wegner estimate $W(I_0, L, \Theta, q)$ is valid.

Furthermore, fix $\xi_0 > 0$ and $\beta > 2\Theta$. Let $\alpha \in (1,2)$ be such that

$$4d\frac{\alpha - 1}{2 - \alpha} \leq \xi_0 \wedge \frac{1}{4}(q - d).$$

Then there exist $C_1 = C_1(d, C_{\text{geom}})$ and $L^* = L^*(q, d, \xi_0, \Theta, \beta, \alpha)$ such that the following implication holds:

If for $\overline{I} \subset I_0$, $L \geq L^*$, $L \in 3\mathbb{N} \setminus 6\mathbb{N}$, and $\gamma_L \geq L^{\beta-1}$, the estimate $G(I, L, \gamma_L, \xi_0)$ is satisfied, then $G(I, L', \gamma_{L'}, \xi)$ also holds, where

(i) *$L' \in 3\mathbb{N} \setminus 6\mathbb{N}$, $L^\alpha \leq L' \leq L^\alpha + 6$,*
(ii) *$\xi \geq \xi_0 \wedge [\frac{1}{4}(q - d)]$,*

(iii) $\gamma_{L'} \geq \gamma_L(1 - 8L^{1-\alpha}) - C_1 \cdot L^{-1} - 6L^{\alpha(\Theta-1)} \geq (L')^{1-\beta}$.

For a proof of the result in this form we refer to [29]. It is modelled after the variable multi-scale analysis by von Dreifus-Klein [15]. See also [17] and [20] for continuum versions.

Let us now formulate an immediate consequence of the preceding theorem.

Corollary 2.3. *Let $I_0, (H_\Lambda(\omega)), \xi_0, \beta, q, \Theta, \alpha \in (1,2)$ be as in Theorem 2.2. There exists $\overline{L} = \overline{L}(\xi_0, \beta, \Theta, q, C_{\mathrm{geom}}, \alpha)$ such that the following holds.*

If $I \subset I_0$ and $G(I, L, \gamma_L, \xi_0)$ is satisfied for some $\gamma_L \geq L^{\beta-1}$ and some $L \geq \overline{L}$, then there exist a sequence $(L_k)_{k\in\mathbb{N}} \subset 3\mathbb{N} \setminus 6\mathbb{N}$ and $\gamma_\infty > 0$ with the following properties:

(i) *For all $k \in \mathbb{N}$, the estimate $G(I, L_k, \gamma_\infty, \xi)$ is satisfied, where $\xi = \xi_0 \wedge \frac{1}{4}(q-d)$.*
(ii) $L_k^\alpha \leq L_{k+1} \leq L_k^\alpha + 6$.

3. Multi-scale estimates imply strong dynamical localization

We keep the framework introduced in the preceding section. Thus we start out with a family H_Λ of random local Hamiltonians where Λ runs through the suitable cubes. Now we introduce a link to a Hamiltonian on the whole space $\mathbb{R}^d$. Consider the statement

(EDI) Assume that $H(\omega)$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$, measurable with respect to ω, and suppose that there is a measurable set Ω_1 with $\mathbb{P}(\Omega_1) = 1$ and a constant C_{EDI} such that for every $\omega \in \Omega_1$, the spectrum of $H(\omega)$ in I_0 is pure point and every eigenfunction u of $H(\omega)$ corresponding to $E \in I_0$ satisfies

$$\|\chi_\Lambda^{\mathrm{int}} u\| \leq C_{\mathrm{EDI}} \cdot \|\chi_\Lambda^{\mathrm{out}} (H_\Lambda(\omega) - E)^{-1} \chi_\Lambda^{\mathrm{int}} u\| \cdot \|\chi_\Lambda^{\mathrm{out}} u\|. \qquad \text{(EDI)}$$

For the operators $H(\omega)$ we shall consider in Section 4 and $H_\Lambda(\omega)$ the restriction to Λ with respect to suitable boundary conditions, the eigenfunction decay inequality (EDI) readily follows. Moreover, in this case, we can use the multi-scale machinery to prove pure point spectrum almost surely. Therefore, the condition above seems to be a natural abstract condition. We can now state the main result of the present paper:

Theorem 3.1. *Assume that $H(\omega)$ and $H_\Lambda(\omega)$ satisfy* (INDY), (GRI), (WEYL) *and* (EDI) *above for a given bounded open set $I_0 \subset \mathbb{R}$. Moreover, assume*

(i) *$\chi_\Lambda P_{I_0}(H(\omega))\chi_\Lambda$ is trace class for every suitable cube Λ and*

$$\mathrm{tr}(\chi_\Lambda P_{I_0}(H(\omega)) \leq C_{\mathrm{tr}} \cdot |\Lambda|^\kappa$$

for some fixed κ.

(ii) *There exist $L_0 \in \mathbb{N}$, $q > d$ and $\Theta \in (0, \frac{1}{2})$ such that for $L \in 3\mathbb{N} \setminus 6\mathbb{N}$, $L \geq L_0$, the Wegner estimate $W(I_0, L, \Theta, q)$ is valid.*
(iii) *For $q > d$ from the Wegner estimate and $\xi_0 > 0$ we have that*

$$p < 2\xi_0 \wedge \tfrac{1}{4}(q-d).$$

Then there exists $\overline{L} = L(p, \xi_0, \beta, \Theta, q, C_{\mathrm{geom}}, d)$ such that if

(iv) *for some $L \in 3\mathbb{N} \setminus 6\mathbb{N}$, $L \geq \overline{L}$, there is an open interval $I \neq \emptyset$, $I \subset I_0$, such that $G(I, L, L^{\beta-1}, \xi_0)$ holds,* *then for every* $\eta \in L^\infty$ *with* $\operatorname{supp} \eta \subset I$, *it follows that*

$$\mathbb{E}\{\||X|^p \eta(H(\omega))\chi_K\|\} < \infty$$

for every compact set $K \subset \mathbb{R}^d$.

Let us first sketch the idea of the proof which is quite simple. Of course, by $|X|^p$ we denote the operator of multiplication with $|x|^p$.

We write

$$\|\chi_{\Lambda_1}\eta(H(\omega))\chi_{\Lambda_2}\| \leq \sum_{E_n \in I_0} \|\chi_{\Lambda_1}\phi_n(\omega)\| \cdot \|\chi_{\Lambda_2}\phi_n(\omega)\| \cdot |\eta(E_n(\omega))|, \tag{3.1}$$

where $E_n(\omega)$, $\phi_n(\omega)$ denote the eigenvalues and eigenfunctions of $H(\omega)$ in I. The probability that both Λ_1 and Λ_2 are bad for the same $E_n(\omega)$ is small, roughly polynomially in the distance between Λ_1, Λ_2. If one of them is good, the eigenfunction decay inequality (EDI) says that one of the norms appearing in the rhs of (3.1) is exponentially small. This leads to a polynomial decay of $\mathbb{E}\{\|\chi_{\Lambda_1}\eta(H(\omega))\chi_{\Lambda_2}\|\}$ once the interval I is suitably chosen to guarantee the necessary probabilistic estimates. The assumption in (iii) of the theorem ensures that the polynomial growth of $|X|^p$ is killed by this polynomial decay. To make all of this work we have to overcome the difficulty that in the sum in (3.1) we have infinitely many terms. This is taken care of by analyzing the centers of localization $x_n(\omega)$ of $\phi_n(\omega)$. All this will be done relatively to a certain length scale L_k.

We proceed in several steps. The first steps will be used to choose an appropriate α and set up a multi-scale scenario. Then we take care of those $\phi_n(\omega)$ whose centers are far away from K. To this end, we employ the Weyl-type trace condition (i).

Proof. Step 1. Choose $\alpha \in (1, 2)$ such that

$$4d\,\frac{\alpha - 1}{2 - \alpha} \leq \frac{1}{4}(q - d) \wedge \xi_0 =: \xi$$

and

$$3d(\alpha - 1) + \alpha p < 2\xi.$$

Note that the latter condition can be achieved for $\alpha > 1$ small enough, since $p < 2\xi$. For this choice of α, let $\overline{L}$ be the minimal length scale from Corollary 2.3.

We can now use Theorem 2.2 and Corollary 2.3 to find a sequence $(L_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and a constant $\gamma > 0$ such that for every k,

- $L_k \in 3\mathbb{N} \setminus 6\mathbb{N}$,
- $L_k^\alpha \leq L_{k+1} \leq 6L_k^\alpha$,
- $G(I, L_k, \gamma, \xi)$ is satisfied.

For $j \in \mathbb{N}$, denote $\Gamma_j = \left(\frac{L_j}{3}\mathbb{Z}\right)^d$ and

$$E_j = \{\omega \in \Omega;\ \text{for some } E \in I \text{ there exist } y, z \in \Gamma_j \cap \Lambda_{3L_{j+1}} \text{ such that } \Lambda_{L_j}(y) \text{ and } \Lambda_{L_j}(z) \text{ are disjoint and both not } (\gamma, E)\text{-good}\}.$$

Since $\Gamma_j \cap \Lambda_{3L_{j+1}} \leq \left(\frac{9L_{j+1}}{L_j}\right)^d \leq (54)^d L_j^{d(\alpha-1)}$ and $G(I, L_j, \gamma, \xi)$ holds, we have

$$\mathbb{P}(E_j) \leq c_d L_j^{2d(\alpha-1)-2\xi}.$$

For $k \in \mathbb{N}$, denote

$$\Omega_{2\text{bad}}^k = \bigcup_{j \geq k} E_j.$$

Claim. For every $k \in \mathbb{N}$,

$$\mathbb{P}(\Omega^k_{2\mathrm{bad}}) \leq c(\alpha, d, \xi) \cdot L_k^{2d(\alpha-1)-2\xi}. \tag{3.2}$$

Proof. We have

$$\begin{aligned}\mathbb{P}(\Omega^k_{2\mathrm{bad}}) &\leq c_d \cdot \sum_{j \geq k} L_j^{2d(\alpha-1)-2\xi} \\ &\leq c_d \cdot L_k^{2d(\alpha-1)-2\xi} \cdot \Big(1 + \sum_{j \geq k+1} \Big(\frac{L_j}{L_k}\Big)^{2d(\alpha-1)-2\xi}\Big).\end{aligned}$$

Now, for $j \geq k+1$,

$$\frac{L_j}{L_k} \geq \frac{L_k^{\alpha^{j-k}}}{L_k} = L_k^{\alpha^{j-k}-1} \geq 3^{\alpha^{j-k}},$$

which gives the assertion.

Step 2. Denote by $\phi_n(\omega)$ the normalized eigenfunctions of $H(\omega)$, $\omega \in \Omega_1$ with corresponding eigenvalues $E_n(\omega) \in I$. For each ω, n, define a *center of localization* $x_n(\omega) \in \mathbb{Z}^d$ by

$$\|\chi_{\Lambda_1(x_n(\omega))}\phi_n(\omega)\| = \max\{\|\chi_{\Lambda_1(y)}\phi_n(\omega)\|; y \in \mathbb{Z}^d\}.$$

Since $\phi_n(\omega) \in L^2$, such a center always exists.

Claim. There is $k_0 = k_0(\gamma, d, C_{\mathrm{EDI}})$ such that for $\omega \in \Omega_1$, $k \geq k_0$ and $x_n(\omega) \in \Lambda^{\mathrm{int}}_{L_k}(x)$, the cube $\Lambda_{L_k}(x)$ is $(\gamma, E_n(\omega))$-bad.

Proof. Assume otherwise. Then by (EDI) it follows that

$$\|\chi_{\Lambda_1(x_n(\omega))}\phi_n(\omega)\| \leq \|\chi^{\mathrm{int}}_{L_k,x}\phi_n(\omega)\| \leq C_{\mathrm{EDI}} \cdot e^{-\gamma L_k} \cdot \|\chi^{\mathrm{out}}_{L_k,x}\phi_n(\omega)\|.$$

Estimating the number of unit cubes in $\Lambda^{\mathrm{out}}_{L_k}(x)$ very roughly by L_k^d we find that

$$\cdots \leq C_{\mathrm{EDI}} \cdot e^{-\gamma L_k} \cdot L_k^d \cdot \max_{\tilde{x} \in \Lambda^{\mathrm{out}}_{L_k}(x)} \|\chi_{\Lambda_1(\tilde{x})}\phi_n(\omega)\|.$$

If k_0 is large enough to ensure

$$C_{\mathrm{EDI}} \cdot e^{-\gamma L_{k_0}} \cdot L_{k_0}^d < 1,$$

the inequality above contradicts the choice of $x_n(\omega)$.

Step 3: Let $\omega \in \Omega^k_{2\mathrm{good}} = (\Omega^k_{2\mathrm{bad}})^c \cap \Omega_1$ with $k \geq k_0$. Then there exists $j_0 = j_0(\gamma, \alpha, d, C_{\mathrm{EDI}})$ such that for $j \geq j_0$, $j \geq k$ and $x_n(\omega) \in \Lambda_{L_{j+1}}$,

$$\|(1 - \chi_{3L_{j+2}})\phi_n(\omega)\|^2 \leq \tfrac{1}{4},$$

where χ_L is shorthand for $\chi_{\Lambda_L(0)}$.

Proof. We divide $\Lambda^c_{3L_{j+2}}$ into angular regions M_i,

$$M_i = \Lambda_{3L_{i+1}} \setminus \overline{\Lambda}_{3L_i}, \ i \geq j+2.$$

We have

$$\begin{aligned}\|(1 - \chi_{3L_{j+2}})\phi_n(\omega)\|^2 &= \sum_{i \geq j+2} \|\chi_{M_i}\phi_n(\omega)\|^2 \\ &= \sum_{i \geq j+2} \sum_{\tilde{x} \in M_i \cap \Gamma_i} \|\chi^{\mathrm{int}}_{L_i,\tilde{x}}\phi_n(\omega)\|^2.\end{aligned}$$

By construction of M_i, for every $\tilde{x} \in M_i \cap \Gamma_i$, we find $\tilde{x}_n \in \Gamma_i \cap \Lambda_{L_{j+1}}$ such that $x_n(\omega) \in \Lambda^{\mathrm{int}}_{L_i}(\tilde{x}_n)$ and $d(\tilde{x}, \tilde{x}_n) \geq L_i$.

Since $\Lambda_{L_i}(\tilde{x}_n)$ is $(\gamma, E_n(\omega))$-bad and $\omega \in \Omega^k_{2\text{good}}$, it follows that $\Lambda_{L_i}(\tilde{x})$ is $(\gamma, E_n(\omega))$-good so that

$$\|\chi^{\text{int}}_{L_i,\tilde{x}}\phi_n\|^2 \le (C_{\text{EDI}})^2 \cdot e^{-2\gamma L_i}.$$

Since $\# M_i \cap \Gamma_i$ grows only polynomially in L_i, the assertion follows.

Step 4: There exists $C = C(\gamma, \alpha, d, \kappa, C_{\text{tr}})$ such that for $\omega \in \Omega^k_{2\text{good}}$, $j \ge k$,

$$\#\{n; x_n(\omega) \in \Lambda_{L_{j+1}}\} \le C \cdot L^{\alpha\kappa d}_{j+1}.$$

Proof. Since $\#\{\dots\}$ is non-decreasing in j, and since j_0 from Step 3 only depends on (γ, α, d), we can restrict ourselves to the case $j \ge j_0$ and adapt the constant C.

We start by observing

$$\sum_{x_n \in \Lambda_{L_{j+1}}} (\chi_{3L_{j+2}} P_I(H(\omega)) \chi_{3L_{j+2}} \phi_n(\omega) | \phi_n(\omega)) \le \operatorname{tr}(\chi_{3L_{j+2}} P_I(H(\omega))).$$

We want to show that each of the terms in the sum is at least $\frac{1}{2}$, thus giving an estimate on the number as asserted. Using Step 3 and suppressing ω, we have

$$\begin{aligned}(\chi_{3L_{j+2}} P_I \chi_{3L_{j+2}} \phi_n | \phi_n) &= (\chi_{3L_{j+2}} P_I \phi_n | \phi_n) - (\chi_{3L_{j+2}} P_I (1 - \chi_{3L_{j+2}}) \phi_n | \phi_n) \\ &\ge (\chi_{3L_{j+2}} \phi_n | \phi_n) - \tfrac{1}{4} \\ &= (\phi_n | \phi_n) - ((1 - \chi_{3L_{j+2}}) \phi_n | \phi_n) - \tfrac{1}{4} \\ &\ge \tfrac{1}{2}.\end{aligned}$$

Plugging this into the above estimate on the trace, we get the claimed bound for the number $\#\{n; \dots\}$.

Step 5: There is $k_1 = k_1(C_{\text{EDI}}, \alpha, C_{\text{tr}}, \kappa, L_0, \gamma, d)$ such that for $k \ge k_1$, $\omega \in \Omega^k_{2\text{good}}$ and $x \in \Gamma_k \cap \Lambda_{L_{k+1}} \setminus \Lambda_{L_k}$,

$$\|\chi^{\text{int}}_{L_k,x} \eta(H(\omega)) \chi^{\text{int}}_{L_k,0}\| \le \exp(-\tfrac{\gamma}{2} L_k) \cdot \|\eta\|_\infty.$$

Proof. We have

$$\|\chi^{\text{int}}_{L_k,x} \eta(H(\omega)) \chi^{\text{int}}_{L_k,0}\| \le \sum_{E_n \in I} |\eta(E_n(\omega))| \cdot \|\chi^{\text{int}}_{L_k,x} \phi_n(\omega)\| \cdot \|\chi^{\text{int}}_{L_k,0} \phi_n(\omega)\|. \tag{3.3}$$

We now divide the sum according to where the $x_n(\omega)$ are located:

$$\sum_{\substack{E_n \in I \\ x_n(\omega) \in \Lambda_{k+1}}} \|\chi^{\text{int}}_{L_k,x} \phi_n(\omega)\| \cdot \|\chi^{\text{int}}_{L_k,0} \phi_n(\omega)\| \le C \cdot L^{\alpha\kappa d}_{k+1} \cdot C_{\text{EDI}} \cdot e^{-\gamma L_k},$$

since one of the cubes $\Lambda_{L_k}(x), \Lambda_{L_k}(0)$ has to be $(\gamma, E_n(\omega))$-good and the number of $x_n(\omega)$ has been estimated in Step 4.

For k large enough, depending only on the indicated parameters, $k \ge k_0$,

$$\sum_{\substack{E_n \in I \\ x_n(\omega) \in \Lambda_{k+1}}} \|\chi^{\text{int}}_{L_k,x} \phi_n(\omega)\| \cdot \|\chi^{\text{int}}_{L_k,0} \phi_n(\omega)\| \le \tfrac{1}{2} \exp(-\tfrac{\gamma}{2} L_k). \tag{3.4}$$

We now treat the remaining terms. Note that for $j \ge k+1$ and $x_n(\omega) \in \Lambda_{L_{j+1}} \setminus \Lambda_{L_j}$, we find an $\tilde{x}_n(\omega) \in \Lambda_{L_{j+1}} \cap \Gamma_j$ such that $x_n(\omega) \in \Lambda^{\text{int}}_{L_j}(\tilde{x}_n(\omega))$. From Step 2 we know that $\Lambda_{L_j}(\tilde{x}_n(\omega))$ must be $(\gamma, E_n(\omega))$-bad so that $\Lambda_{L_j}(0)$ has to be $(\gamma, E_n(\omega))$-good since $\omega \in \Omega^k_{2\text{good}}$. Therefore

$$\|\chi^{\text{int}}_{L_k,0} \phi_n(\omega)\| \le \|\chi^{\text{int}}_{L_j,0} \phi_n(\omega)\| \le C_{\text{EDI}} \cdot \exp(-\gamma L_j).$$

Using Step 4 again, we see that

$$\begin{aligned}\sum_{j=k+1}^{\infty}\Big(\sum_{x_n\in\Lambda_{L_{j+1}}\setminus\Lambda_{L_j}}\|\chi^{\rm int}_{L_k,x}\phi_n(\omega)\|\cdot\|\chi^{\rm int}_{L_k,0}\phi_n(\omega)\|\Big)&\le C\cdot C_{\rm EDI}\cdot\sum_{j=k+1}^{\infty}e^{-\gamma L_j}L_{j+1}^{\alpha\kappa d}\\&\le\tfrac12\exp(-\tfrac{\gamma}{2}L_k)\end{aligned}$$

if $k\ge k_1(C_{\rm EDI},\alpha,C_{\rm tr},\kappa,L_0,\gamma,d)$. The latter estimate, together with (3.3) and (3.4), gives the assertion.

Step 6: For $k\ge k_1$ from Step 5 and $x\in\Gamma_k\cap\Lambda_{L_{k+1}}\setminus\Lambda_{L_k}$, we have

$$\mathbb{E}\{\|\chi^{\rm int}_{L_k,x}\eta(H(\omega))\chi^{\rm int}_{L_k,0}\|\}\le\|\eta\|_\infty\cdot\Big(c(\alpha,d,\xi)\cdot L_k^{2d(\alpha-1)-2\xi}+\exp(-\tfrac{\gamma}{2}L_k)\Big).$$

Proof. For $\omega\in\Omega^k_{\rm 2bad}$, we can estimate the norm by $\|\eta\|_\infty$ and use Step 1, while for $\omega\in\Omega^k_{\rm 2good}$, we can use Step 5.

Put together, we have

$$\begin{aligned}\mathbb{E}\{\dots\}&\le\|\eta\|_\infty\cdot\Big(\mathbb{P}(\Omega^k_{\rm 2bad})+\exp(-\tfrac{\gamma}{2}L_k)\mathbb{P}(\Omega^k_{\rm 2good})\Big)\\&\le\|\eta\|_\infty\cdot\Big(c(\alpha,d,\xi)L_k^{2d(\alpha-1)-2\xi}+\exp(-\tfrac{\gamma}{2}L_k)\Big).\end{aligned}$$

Step 7: End of the proof.
For compact K, we find $k\ge k_1$ such that $K\subset\Lambda^{\rm int}_{L_k}(0)$. Then with $D=D(d,k,p,\|\eta\|_\infty)$, we have

$$\begin{aligned}\mathbb{E}\Big\{\||X|^p\eta(H(\omega))\chi_K\|\Big\}&\le c_dL_k^p\|\eta\|_\infty+\mathbb{E}\Big\{\sum_{j\ge k}\||X|^p\chi_{\Lambda_{L_{j+1}}\setminus\Lambda_{L_j}}\eta(H(\omega))\chi_K\|\Big\}\\&\le D+\sum_{j\ge k}c_dL_{j+1}^p\sum_{\substack{\tilde x\in\Lambda_{L_{j+1}}\setminus\Lambda_{L_j}\\ \tilde x\in\Gamma_j}}\mathbb{E}\{\|\chi^{\rm int}_{L_j,\tilde x}\eta(H(\omega))\chi^{\rm int}_{L_j,0}\|\}\\&\le D\Big[1+\sum_{j\ge1}L_j^{\alpha p}L_j^{d(\alpha-1)}\Big(L_j^{2d(\alpha-1)-2\xi}+\exp(-\tfrac{\gamma}{2}L_j)\Big)\Big]\\&<\infty,\end{aligned}$$

since $\alpha p+3d(\alpha-1)-2\xi<0$ and the L_j grow fast enough. □

Although we cannot apply the theorem directly, a look at the proof, particularly at Steps 5 to 7, shows that we have the following:

Corollary 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Then we have*

$$\mathbb{E}\Big\{\sup_{t>0}\||X|^pe^{-itH(\omega)}P_I(H(\omega))\chi_K\|\Big\}<\infty.$$

4. Applications

In this section we present a list of models for which the variable energy multiscale analysis has been established and which therefore exhibit strong dynamical localization by the results of the preceding section.

4.1. **Periodic plus Anderson.** Here we discuss band edge localization for alloy-type models which consist of a periodic background operator with impurities sitting on the periodicity lattice. We take $\mathbb{Z}^d$ as this lattice simply for notational convenience; a reformulation for more general lattices presents no difficulties whatsoever. Note that compared with most results available in the literature, we assume minimal conditions on the single-site measure:

1. Let $p = 2$ if $d \le 3$ and $p > d/2$ if $d > 3$.
2. Let $V_0 \in L^p_{\text{loc}}(\mathbb{R}^d)$, V_0 periodic w.r.t. $\mathbb{Z}^d$ and $H_0 = -\Delta + V_0$.
3. Let $f \in L^p(\Lambda_1(0))$, $f \ge 0$ and $f \ge \sigma$ on $\Lambda_s(0)$ for some $\sigma > 0$, $s > 0$; f is called the *single-site potential.*
4. Let μ be a probability measure on $\mathbb{R}$, with $\operatorname{supp} \mu = [q_-, q_+]$, where $q_- < q_+ \in \mathbb{R}$; μ is called the *single-site measure.*
5. Let
$$\Omega = [q_-, q_+]^{\mathbb{Z}^d}, \mathbb{P} = \bigotimes_{\mathbb{Z}^d} \mu \text{ on } \Omega$$
and $q_k : \Omega \to \mathbb{R}$, $q_k(\omega) = \omega_k$.
6. Let
$$V_\omega(x) := \sum_{k \in \mathbb{Z}^d} q_k(\omega) f(x - k)$$
and
$$H^{\mathrm{A}}(\omega) = -\Delta + V_0 + V_\omega.$$

For an elementary discussion of this model and all the ingredients necessary to prove localization, we refer to [29]; see also [22, 23]. Note that to conform with standard notation, we denote by p both the power of the moment operator in the dynamical bounds and the power defining the appropriate L^p space the potentials have to belong to. This, however, should not lead to any real confusion.

Theorem 4.1. *Let $H^{\mathrm{A}}(\omega)$ be as above. Assume that the single-site measure μ is Hölder continuous, that is, there exists $a > 0$ such that for every interval J of length small enough, $\mu(J) \le |J|^a$. Denote $\Sigma = \sigma(H^{\mathrm{A}}(\omega))$ a.e. and $E_0 = \inf \Sigma$. Let $p > 0$. Then there exists $\varepsilon_0 > 0$ such that for $\eta \in L^\infty(\mathbb{R})$ with $\operatorname{supp} \eta \subset [E_0, E_0 + \varepsilon_0]$ and compact K, we have*
$$\mathbb{E}\{\| |X|^p \eta(H^{\mathrm{A}}(\omega)) \chi_K \|\} < \infty.$$
Moreover, for $I \subset [E_0, E_0 + \varepsilon_0]$ and K compact:
$$\mathbb{E}\Big\{ \sup_t \| |X|^p e^{-iH^{\mathrm{A}}(\omega)t} P_I(H^{\mathrm{A}}(\omega)) \chi_K \| \Big\} < \infty.$$

Proof. It is well known that (INDY), (WEYL), (GRI) and (EDI) are satisfied if we take for H^{A}_Λ the operator H^{A} restricted to Λ with periodic boundary conditions. Due to [22, 28] we have a Wegner estimate of the form
$$\mathbb{P}\{\operatorname{dist}(\sigma(H^{\mathrm{A}}_\Lambda(\omega)), E_0) \le \exp(-L^\Theta)\} \le C \cdot L^2 \cdot d \cdot \exp(-aL^\Theta),$$
where L denotes the sidelength of the cube Λ. In particular, $W(I_0, L, \Theta, q)$ is satisfied for a neighborhood I_0 of E_0, arbitrarily given Θ and q, and L large enough. For given $p > 0$, we can start the multi-scale induction with $2\xi > p$ by Lifshitz asymptotics. □

Note that the above theorem includes the case of single-site potentials with small support. Moreover, using Klopp's analysis of internal Lifshitz tails [26], Veselic establishes the necessary initial length scale estimates at lower band edges in the case where H_0 exhibits a non-degenerate behavior at the corresponding edge [30], so the result above extends to this case. If one does not know that H_0 has a non-degenerate band edge, one can still derive an initial length scale estimate by requiring a disorder assumption. This, however, might put some restriction on the power p.

Theorem 4.2. *Let $H^{\mathrm{A}}(\omega)$ be as above. Assume*

(i) *The single-site measure μ is Hölder continuous.*
(ii) *There exists $\tau > d$ such that for small $h > 0$,*

$$\mu([q_-, q_- + h]) \leq h^\tau \text{ and } \mu([q_+ - h, q_+]) \leq h^\tau.$$

Denote $\Sigma = \sigma(H(\omega))$ a.e. and let $E_0 \in \partial\Sigma$. Let $p < 2(2\tau - d)$. Then there exists $\varepsilon_0 > 0$ such that for $\eta \in L^\infty(\mathbb{R})$ with $\operatorname{supp} \eta \subset [E_0 - \varepsilon_0, E_0 + \varepsilon_0]$ and compact K, we have

$$\mathbb{E}\{\| |X|^p \eta(H^{\mathrm{A}}(\omega))\chi_K \|\} < \infty.$$

Moreover, for $I \subset [E_0 - \varepsilon_0, E_0 + \varepsilon_0]$ and K compact:

$$\mathbb{E}\Big\{ \sup_t \| |X|^p e^{-iH^{\mathrm{A}}(\omega)t} P_I(H^{\mathrm{A}}(\omega))\chi_K \| \Big\} < \infty.$$

Proof. We have already checked everything except for the initial length scale estimate $G(I, L, \gamma, \xi)$, and in particular how large ξ can be taken. By an elementary argument, we can take ξ subject to the condition $\xi < 2\tau - d$ (see [22]), which gives the claimed result. $\square$

With a modification of independent multi-scale analysis given in [23] we can also treat the correlated or long-range case, by which we understand that the single-site potential f is no longer assumed to have support in the unit cube; see [23].

Theorem 4.3. *Let $H^{\mathrm{A}}(\omega)$ be as above, with condition 3 replaced by*

3. *Let $f \in L^p_{\mathrm{loc}}$, $f \geq 0$ and $f \geq \sigma$ on $\Lambda_s(0)$ for some $\sigma > 0$, $s > 0$;*

$$f \leq C|x|^{-m} \text{ for } |x| \text{ large.}$$

Then the conclusions of Theorems 4.1 *and* 4.2 *hold true with*

$$p < 2\left(\frac{m}{4} - d\right) \text{ and } p < 2\left(\frac{m}{4} - d\right) \wedge 2(2\tau - d),$$

respectively.

Remark. Although discrete models are not explicitly included in the above framework, our principal strategy pursued in Section 3 is clearly able to treat random operators in $\ell^2(\mathbb{Z}^d)$ for which a multi-scale analysis has been established. In particular, building on results from [8] one may establish strong dynamical localization for the discrete Anderson model, where for $d = 1$, even pure point single-site measures (e.g., the Bernoulli case) are within the scope of this result. See [8] for explicit requirements to make the multi-scale machinery work. We thus obtain new results on strong dynamical localization also in the discrete case since the Aizenman method does not cover single-site distributions which are too singular (e.g., the Bernoulli case).

4.2. **Random divergence form operators.** The following type of model has been introduced in [17, 27] in order to study classical waves (see [17] for a motivation). These models are also intensively studied in [29].

1. Let $\mathbf{a}_0 : \mathbb{R}^d \to M(d \times d)$ be measurable, $\mathbb{Z}^d$-periodic and such that for some $\eta > 0$, $M > 0$,
$$\eta \leq \mathbf{a}_0(x) \leq M \text{ for all } x \in \mathbb{R}^d$$
as matrices, that is, $\eta||\zeta||^2 \leq (\mathbf{a}_0(x)\zeta|\zeta) \leq M||\zeta||^2$ for every $\zeta \in \mathbb{C}^d$.
2. Let $S = [0, \lambda_{\max}]^d \times \mathcal{O}(d)$, where $\lambda_{\max} > 0$ and $\mathcal{O}(d)$ denotes the orthogonal matrices.
3. Let ν be a probability measure on $\mathcal{O}(d)$ and let γ_i, $i = 1, ..., d$ be probability measures on $\mathbb{R}$ with $\operatorname{supp} \gamma_i = [0, \lambda_{\max}]$.
4. S is called the *single-site space* and $\mu = \gamma_1 \otimes \cdots \otimes \gamma_d \otimes \nu$ is called the *single-site measure*.
5. Let
$$\Omega = S^{\mathbb{Z}^d},\ \mathbb{P} = \mu^{\mathbb{Z}^d},$$
and for $\omega(k) = (\lambda_1(k), ..., \lambda_d(k), u(k))$, define
$$\mathbf{a}_k(\omega) = u(k)^* \operatorname{diag}(\lambda_1(k), ..., \lambda_d(k)) u(k),$$
where $\operatorname{diag}(\lambda_1(k), ..., \lambda_d(k))$ denotes the diagonal matrix with the indicated diagonal elements.
6. Define
$$\mathbf{a}_\omega(x) := \sum_{k \in \mathbb{Z}^d} \chi_{\Lambda_1(k)}(x) \mathbf{a}_k(\omega)$$
and
$$H^{\mathrm{DIV}}(\omega) = -\nabla(\mathbf{a}_0 + \mathbf{a}_\omega)\nabla.$$

Although the formulas may seem intricate, it is easy to see what is happening. For site k, we choose a non-negative matrix $\mathbf{a}_k(\omega)$ at random by choosing its d eigenvalues and a unitary conjugation matrix. This is done independently at different sites and we get an Anderson-like random matrix function $\mathbf{a}_\omega$ which is used as a perturbation to the perfectly periodic medium $\mathbf{a}_0$. Note that $\mathbf{a}_0 + \mathbf{a}_\omega$ have uniform upper and lower bounds (η and $M + \lambda_{\max}$) so that the operators can be defined via quadratic forms with the Sobolev space $W^{1,2}(\mathbb{R}^d)$ as common form domain. The initial value problem we are now interested in is governed by the wave equation
$$\frac{\partial^2 v}{\partial t^2} = -H^{\mathrm{DIV}}(\omega) v, \quad v(0) = v_0, \quad \frac{\partial v}{\partial t}|_{t=0} = v_1 \tag{WE}$$
rather than the Schrödinger equation. Solutions are given by
$$v(t) = \cos\left(t\sqrt{H^{\mathrm{DIV}}(\omega)}\right) v_0 + \sin\left(t\sqrt{H^{\mathrm{DIV}}(\omega)}\right) w_1,$$
where $v_1 = \sqrt{H^{\mathrm{DIV}}(\omega)} w_1$, and v_0, w_1 have to belong to the appropriate operator domains. The following result yields a strong form of dynamical localization in this case:

Theorem 4.4. *Let $H^{\mathrm{DIV}}(\omega)$ be as above. Assume*

(i) *The measures γ_i, $i = 1, ..., d$ are Hölder continuous.*
(ii) *There exists $\tau > d$ such that for small $h > 0$,*
$$\gamma_i([0, h]) \leq h^\tau \text{ and } \gamma_i([\lambda_{\max} - h, \lambda_{\max}]) \leq h^\tau$$
for all $i = 1, ..., d$.

Denote $\Sigma = \sigma(H^{\mathrm{DIV}}(\omega))$ *a.e. and let* $E_0 \in \partial\Sigma \setminus \{0\}$.

Then there exists $\varepsilon_0 > 0$ *such that for* $\eta \in L^\infty(\mathbb{R})$ *with* $\operatorname{supp}\eta \subset [E_0 - \varepsilon_0, E_0 + \varepsilon_0]$ *and compact* K*, we have*

$$\mathbb{E}\{\| |X|^p \, \eta(H^{\mathrm{DIV}}(\omega))\chi_K \|\} < \infty.$$

Moreover, for $I \subset [E_0 - \varepsilon_0, E_0 + \varepsilon_0]$ *and* K *compact:*

$$\mathbb{E}\Big\{ \sup_t \| |X|^p \cos\left(t\sqrt{H^{\mathrm{DIV}}(\omega)}\right) P_I(H^{\mathrm{DIV}}(\omega))\chi_K \| \Big\} < \infty$$

and

$$\mathbb{E}\Big\{ \sup_t \| |X|^p \sin\left(t\sqrt{H^{\mathrm{DIV}}(\omega)}\right) P_I(H^{\mathrm{DIV}}(\omega))\chi_K \| \Big\} < \infty.$$

By the results from [17, 27] the conditions for multi-scale analysis are satisfied.

4.3. **Random quantum waveguides.** Quantum waveguides have been introduced for the investigation of two- or three-dimensional motion of electrons in small channels, tubes or layers of crystalline matter of high purity. Mathematically speaking, one considers the free Laplacian in a domain which should be thought of as a perturbation of a strip. The following random model is taken from [24], where all the necessary conditions for multi-scale analysis are verified:

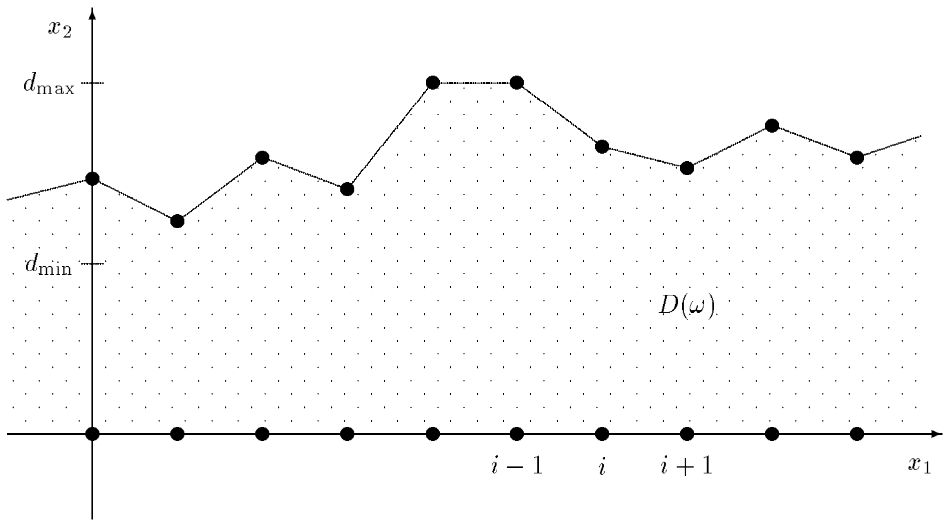
It consists of a collection of randomly dented versions of a parallel strip $\mathbb{R} \times (0, d_{\max}) = D_{\max}$. More precisely, let $d_{\max} > 0$, $0 < d < d_{\max}$, and consider $\Omega = [0, d]^{\mathbb{Z}}$. The i-th coordinate $\omega(i)$ of $\omega \in \Omega$ gives the deviation of the width of the random strip from $d_{\max}$, that is,

$$d_i(\omega) := d_{\max} - \omega(i),$$

which lies between $d_{\min} = d_{\max} - d$ and $d_{\max}$. Define $\gamma(\omega) : \mathbb{R} \to [d_{\min}, d_{\max}]$ as the polygon in $\mathbb{R}^2$ joining the points $\{(i, d_i(\omega))\}_{i \in \mathbb{Z}}$ and

$$D(\omega) = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, 0 < x_2 < \gamma(\omega)(x_1)\}.$$

The following picture will help in visualizing this domain:



We fix a probability measure μ on $[0, d]$ with $0 \in \operatorname{supp}\mu \neq \{0\}$ and introduce $\mathbb{P} = \mu^{\mathbb{Z}}$, a probability measure on Ω. Consider $H^{\mathrm{W}}(\omega) = -\Delta_{D(\omega)}$, the Laplacian on $D(\omega)$ with Dirichlet boundary conditions, which is a self-adjoint operator in $L^2(D(\omega))$. Note that

$$\inf \sigma(H^{\mathrm{W}}(\omega)) = E_0 := \frac{\pi^2}{d_{\max}^2} \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

In [24], exponential localization in a neighborhood of E_0 is proven with the help of a variable energy multi-scale analysis similar to the one presented in Section 2 of the present paper. In particluar, suitably modified versions of the assumptions of Theorem 3.1 are established which enable one to prove strong dynamical localization along the lines of Section 3. Thus, we have

Theorem 4.5. *Let $H^{\mathrm{W}}(\omega)$ and E_0 be as above. Assume that the single-site measure μ is Hölder continuous. Let $p > 0$. Then there exists $\varepsilon_0 > 0$ such that for $\eta \in L^\infty(\mathbb{R})$ with* $\operatorname{supp} \eta \subset [E_0, E_0 + \varepsilon_0]$ *and compact K, we have*

$$\mathbb{E}\{\| |X|^p \eta(H^{\mathrm{W}}(\omega))\chi_K \|\} < \infty.$$

Moreover, for $I \subset [E_0, E_0 + \varepsilon_0]$ and K compact:

$$\mathbb{E}\Big\{ \sup_t \| |X|^p e^{-iH^{\mathrm{W}}(\omega)t} P_I(H^{\mathrm{W}}(\omega))\chi_K \| \Big\} < \infty.$$

4.4. **Landau Hamiltonians.** The models we discuss now are particularly interesting due to their importance for the quantum Hall effect and hence have been studied intensively [10, 11, 12, 20, 31]. We rely here on the set-up from [10], also considered in [20], as the latter authors provide a proof of the basic assumptions needed for our approach. In particular, the trace condition (i) from Theorem 3.1 is proven there and the validity of (GRI) is discussed.

We consider electrons confined to the plane $\mathbb{R}^2$ subject to a perpendicular constant B-field.

Assume

1. $H_0 = \left(\partial_1 + \frac{B}{2}x_2\right)^2 + \left(\partial_2 - \frac{B}{2}x_1\right)^2$, where $B > 0$ is constant.
2. Let $\operatorname{supp} f \in L^\infty(\Lambda_1(0))$, $f \geq 0$ and $f \geq \sigma$ on $\Lambda_s(0)$ for some $\sigma > 0$, $s > 0$; f is called the *single-site potential.*
3. Let μ be a probability measure on $\mathbb{R}$, with density g, $g \in C_0^2(\mathbb{R})$ even and strictly positive a.e. on its support $[-q, q]$; μ is called the *single-site measure.*
4. Let
$$\Omega = [-q, q]^{\mathbb{Z}^2},\ \mathbb{P} = \bigotimes_{\mathbb{Z}^2} \mu \text{ on } \Omega$$
and $q_k : \Omega \to \mathbb{R}$, $q_k(\omega) = \omega_k$.
5. Let
$$V_\omega(x) := \sum_{k \in \mathbb{Z}^2} q_k(\omega) f(x - k)$$
and
$$H^{\mathrm{L}}(\omega) = H_0 + V_\omega.$$

Recall that the spectrum of H_0 in this case consists of the sequence of Landau levels $E_n(B) = (2n+1)B$. We have the following:

Theorem 4.6. *Let $H^{\mathrm{L}}(\omega)$ be as above with B large enough. Let $p > 0$. Then for every $n \in \mathbb{N}$, there exists $\varepsilon_n(B) = O(B^{-1}) > 0$ such that for $\eta \in L^\infty(\mathbb{R})$ with* $\operatorname{supp} \eta \subset [E_n(B) + \varepsilon_n(B), E_{n+1}(B) - \varepsilon_n(B)]$ *and compact K, we have*

$$\mathbb{E}\{\| |X|^p \eta(H^{\mathrm{L}}(\omega))\chi_K \|\} < \infty.$$

Moreover, for $I \subset [E_n(B) + \varepsilon_n(B), E_{n+1}(B) - \varepsilon_n(B)]$ *and* K *compact:*

$$\mathbb{E}\Big\{ \sup_t \| |X|^p e^{-iH^{\mathrm{L}}(\omega)t} P_I(H^{\mathrm{L}}(\omega))\chi_K \| \Big\} < \infty.$$

In [31], a proof of exponential localization is given for a case which includes single-site potentials of changing sign. However, the use of microlocal techniques requires smoothness of the potential.

Acknowledgements. The results presented here were obtained during a visit of P. S. to Caltech. Partial financial support by the DFG and the hospitality at Caltech and Santa Monica are gratefully acknowledged. D. D. received financial support from the German Academic Exchange Service through Hochschulsonderprogramm III (Postdoktoranden). Moreover, we would like to thank F. Germinet, S. Jitomirskaya and A. Klein for helpful discussions.

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