GENERIC SETS IN SPACES OF MEASURES AND GENERIC SINGULAR CONTINUOUS SPECTRUM FOR DELONE HAMILTONIANS

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To J. Weidmann on the occasion of his 65th birthday

Abstract
We show that geometric disorder leads to some purely singular continuous spectrum generically. The main input is a result of Simon known as the Wonderland theorem *in [17, Section 2]. Here we provide an alternative approach and actually a slight strengthening by showing that various sets of measures defined by regularity properties are generic in the set of all measures on a locally compact metric space. As a byproduct, we obtain the fact that a generic measure on euclidean space is singular continuous.*

1. Introduction

In this article, we study the spectral type of continuum Hamiltonians describing solids with a specific form of disorder that is sometimes called *geometric disorder*. The corresponding Hamiltonians are defined using Delone sets (uniformly discrete and uniformly dense subsets of euclidean space) in the following way. The ions of a solid are assumed to be distributed in space according to points of a Delone set. We fix an effective potential v for each of the ions and consider the effective Hamiltonian

$$H(\omega) := -\Delta + \sum_{x\in\omega} v(\cdot - x)$$

for every Delone set $\omega \in \mathbb{D}_{r,R}$ (see Section 3 for the precise definition). We can now show that under some mild assumptions concerning v, r, R, there exists a dense G_δ-set of ω's for which $H(\omega)$ exhibits a purely singular continuous spectral component.

Let us put this result into perspective. Two extremal forms of geometrically disordered sets are periodic sets and highly random sets with points distributed according to, say, a Poisson law. In the periodic case, it is known that the spectrum is

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purely absolutely continuous. For the highly random case, some pure point spectrum is expected. The region between these two extremes contains the case of *aperiodic order*. This form of disorder has attracted much attention recently due to the discovery of solids exhibiting this kind of order (see, e.g., [15], [1] for reviews and further references). These solids are called *quasi crystals*. For them, singularly continuous spectrum is assumed to occur.

This picture is well substantiated by one-dimensional investigations (see [5]). However, in the higher-dimensional case, essentially nothing is known (except, of course, absolute continuity of the spectrum for the periodic case).

Thus, our result provides a first modest step in the investigation of geometric disorder in the higher-dimensional case.

The main tool we use goes back to Simon's Wonderland theorem in [17, Section 2]. It basically tells us that certain sets of operators defined by spectral types are in fact regular in the sense that they are G_δ-sets. If, by chance, one can prove denseness of these sets, then their intersection is dense as well. For instance, in the situation indicated in the first paragraph of the introduction, we can prove that a dense set of ω's leads to continuous spectral measures (actually, even absolutely continuous spectral measures) and that a dense set of ω's leads to singular spectral measures (in some energy interval). Since both of these sets turn out to be G_δ, we get a dense G_δ-set of ω's such that the spectrum of $H(\omega)$ is singular continuous.

We need a slight generalization of Simon's results. More importantly, we provide a new proof of his result which gives some new insights. Namely, we consider the relevant sets directly in spaces of measures and show that the singular measures (with respect to some comparison measure) form a G_δ-set in the space of Borel measures on a nice metric space. The same holds for the continuous measures. By standard continuity arguments, this regularity can be pulled back to spaces of operators.

Since we work directly in spaces of measures, we obtain as a byproduct of our investigation the fact that the set of singular continuous measures on euclidean space is a dense G_δ-set in the space of measures. Although this could certainly be expected in view of all the strange things that typically happen as a consequence of Baire's theorem, we are not aware of a proof of this fact.

2. Back to Wonderland

The following result is the main abstract input to our proof of generic appearance of a purely singular continuous component in the spectrum of certain Delone Hamiltonians.

It is a soft result that basically follows from Baire's theorem and only a minor strengthening of Simon's Wonderland theorem from [17]. In order to formulate it efficiently, let us introduce the following notation.

For a fixed separable Hilbert space $\mathfrak{H}$, consider the space $\mathfrak{S} = \mathfrak{S}(\mathfrak{H})$ of self-adjoint operators in $\mathfrak{H}$. We endow $\mathfrak{S}$ with the *strong resolvent topology* τ_{srs}, the weakest topology for which all the mappings

$$\mathfrak{S} \to \mathbb{C}, \qquad A \mapsto (A+i)^{-1}\xi \quad (\xi \in \mathfrak{H}),$$

are continuous. Therefore, a sequence (A_n) converges to A with respect to τ_{srs} if and only if

$$(A_n + i)^{-1}\xi \to (A+i)^{-1}\xi$$

for all $\xi \in \mathfrak{H}$.

We denote by $\sigma_*(A)$ for $* = $ c, s, ac, sc, pp the continuous, singular, absolutely continuous, singular continuous, and pure point spectrum, respectively.

THEOREM 2.1
Let (X, ρ) be a complete metric space, and let $H : (X, \rho) \to (\mathfrak{S}, \tau_{\mathrm{srs}})$ be a continuous mapping. Assume that for an open set $U \subset \mathbb{R}$,
(1) *the set $\{x \in X \mid \sigma_{\mathrm{pp}}(H(x)) \cap U = \emptyset\}$ is dense in X,*
(2) *the set $\{x \in X \mid \sigma_{\mathrm{ac}}(H(x)) \cap U = \emptyset\}$ is dense in X, and*
(3) *the set $\{x \in X \mid U \subset \sigma(H(x))\}$ is dense in X.*
Then the set

$$\Big\{x \in X \,\Big|\, U \subset \sigma\big(H(x)\big), \sigma_{\mathrm{ac}}\big(H(x)\big) \cap U = \emptyset, \sigma_{\mathrm{pp}}\big(H(x)\big) \cap U = \emptyset\Big\}$$

is a dense G_δ-set in X.

Remark. If the mapping H in Theorem 2.1 happens to be injective, then its range $H(X)$ can be endowed with the metric ρ given on X. In this case, $H(X)$ is a regular metric space of operators in the sense of [17], and [17, Theorem 2.1] gives the result.

As can be seen from this remark, Theorem 2.1 is only slightly stronger than Simon's result. (A referee kindly pointed out related work by Choksi and Nadkarni; see [6], [7].) Our proof, however, is somewhat different. It is based on the fact that certain sets in spaces of measures are G_δ's. In that sense, we get some extra information in comparison with [17]. The basic idea is that once one has established the G_δ-property for the set of all measures that are purely continuous and purely singular, respectively, this regularity can be pulled back to obtain the G_δ-property for the sets appearing in assumptions (1) and (2) of Theorem 2.1. Clearly, this gives the asserted denseness of the intersection of these sets. Since the set from Theorem 2.1(3) is G_δ as well, the set appearing in the assertion is the intersection of three dense G_δ-sets and as such is dense.

We now start our program of proof by studying certain subsets of the set of positive, regular Borel measures $\mathscr{M}_+(S)$ on some locally compact, σ-compact, separable metric space S. Of course, $\mathcal{M}_+(S)$ is endowed with the weak topology from $C_c(S)$, also called the *vague topology*. We refer the reader to [3] for standard results concerning the space of measures. In particular, we note that the vague topology is metrizable such that $\mathcal{M}_+(S)$ becomes a complete metric space as we consider second countable spaces. For the application we have in mind, S is just the open subset U of the real line that appears in Theorem 2.1.

We call a measure $\mu \in \mathcal{M}_+(S)$ *diffusive* or *continuous* if its *atomic* or *pure point part* vanishes, that is, if $\mu(\{x\}) = 0$ for every $x \in S$. (We prefer the former terminology in the abstract framework and the latter for measures on the real line.) Two measures are said to be *mutually singular*, $\mu \perp \nu$, if there exists a set $C \subset S$ such that $\mu(C) = 0 = \nu(S \backslash C)$. We have the following theorem.

THEOREM 2.2
Let S be as in the preceding two paragraphs.
(1) *The set $\{\mu \in \mathcal{M}_+(S) \mid \mu$ is diffusive$\}$ is a G_δ-set in $\mathcal{M}_+(S)$.*
(2) *For any $\lambda \in \mathcal{M}_+(S)$, the set $\{\mu \in \mathcal{M}_+(S) \mid \mu \perp \lambda\}$ is a G_δ-set in $\mathcal{M}_+(S)$.*
(3) *For any closed $F \subset S$, the set $\{\mu \in \mathcal{M}_+(S) \mid F \subset \operatorname{supp}(\mu)\}$ is a G_δ-set in $\mathcal{M}_+(S)$.*

Proof of Theorem 2.2(3)
First, consider the case $F = \{x\}$. Choose a basis $(V_n)_{n\in\mathbb{N}}$ of open neighborhoods of x. Then

$$\left\{\mu \in \mathcal{M}_+(S) \,\middle|\, x \in \operatorname{supp}(\mu)\right\}^{c} = \bigcup_{n\in\mathbb{N}} \left\{\mu \in \mathcal{M}_+(S) \,\middle|\, \mu(V_n) = 0\right\}$$

is an F_σ since $\{\mu \in \mathcal{M}_+(S) \mid \mu(V) = 0\}$ is closed for any open set $V \subset S$. Therefore $\{\mu \in \mathcal{M}_+(S) | \; x \in \operatorname{supp}(\mu)\}$ is a G_δ.

To treat the general case, take a dense subset $\{x_n \mid n \in \mathbb{N}\}$ in F (which is possible since S is separable). Then

$$\left\{\mu \in \mathcal{M}_+(S) \,\middle|\, F \subset \operatorname{supp}(\mu)\right\} = \bigcap_{n\in\mathbb{N}} \left\{\mu \in \mathcal{M}_+(S) \,\middle|\, x_n \in \operatorname{supp}(\mu)\right\}$$

is a countable intersection of G_δ's and, hence, a G_δ. □

To prove Theorem 2.2(1) and (2), we use the following observations.

PROPOSITION 2.3
Let $\mathcal{K} \subset \mathcal{M}_+(S)$ be compact. Then

$$\mathcal{K}^\bullet := \left\{\mu \in \mathcal{M}_+(S) \,\middle|\, \exists \nu \in \mathcal{K} : \nu \leq \mu\right\}$$

is closed.

Proof
Let (μ_n) be a sequence in $\mathscr{K}^\bullet$ which converges to μ. Choose $\nu_n \in \mathscr{K}$ with $\nu_n \leq \mu_n$ for $n \in \mathbb{N}$. By compactness, we find a converging subsequence (ν_{n_k}) with limit $\nu \in \mathscr{K}$. For any $\varphi \in C_c(S)$, $\varphi \geq 0$, we get

$$\begin{aligned}\langle \mu, \varphi \rangle &= \lim_{k\to\infty} \langle \mu_{n_k}, \varphi \rangle \\ &\geq \lim_{k\to\infty} \langle \nu_{n_k}, \varphi \rangle \\ &= \langle \nu, \varphi \rangle,\end{aligned}$$

so that $\mu \in \mathscr{K}^\bullet$. □

PROPOSITION 2.4
Let $K \subset S$ be compact, and let $a > 0$. Then

$$\left\{ a \cdot \delta_x \,\middle|\, x \in K \right\}$$

is compact in $\mathscr{M}_+(S)$.

Proof
The proof is evident from the fact that $S \to \mathscr{M}_+(S)$, $x \mapsto \delta_x$ is continuous. □

PROPOSITION 2.5
Let $\lambda \in \mathscr{M}_+(S)$, let $K \subset S$ be compact, and let $\gamma > 0$ be given. Then

$$\mathscr{K} := \left\{ f \cdot \lambda \,\middle|\, f \in L^2(\lambda),\ \|f\|_{L^2(\lambda)} \leq 1, 0 \leq f, \operatorname{supp}(f) \subset K, \int f\, d\lambda \geq \gamma \right\}$$

is compact.

Proof
The densities considered in $\mathscr{K}$ form a closed subset of the unit ball of $L^2(K, \lambda)$. Since the latter is weakly compact and the mapping $L^2(K, \lambda)_+ \to \mathscr{M}_+(S)$, $f \mapsto f\lambda$ is w-w*-continuous, we get the desired compactness. □

Proofs of Theorem 2.2(1) and Theorem 2.2(2)
For the proof of Theorem 2.2(1), consider

$$\mathscr{M}_1 := \left\{ \mu \in \mathscr{M}_+(S) \,\middle|\, \mu \text{ is diffusive} \right\}^{c}.$$

We want to show that $\mathscr{M}_1$ is an F_σ. By assumption on S, we find a sequence of compacts $K_n \nearrow S$ and get that

$$\mathscr{K}_{1,n} = \left\{ \frac{1}{n} \cdot \delta_x \,\middle|\, x \in K_n \right\}$$

is compact by Proposition 2.4. Proposition 2.3 yields that $\mathscr{F}_{1,n} = \mathscr{K}^{\bullet}_{1,n}$ is closed. Since

$$\mathscr{M}_1 = \bigcup_{n \in \mathbb{N}} \mathscr{F}_{1,n},$$

we arrive at the desired conclusion. □

We show Theorem 2.2(2) with a similar argument. For K_n defined as in the proof of Theorem 2.2(1), let

$$\mathscr{K}_{2,n} = \left\{ f\lambda \,\middle|\, f \in L^2(\lambda),\, \|f\| \le 1,\, 0 \le f,\, \operatorname{supp}(f) \subset K_n,\, \int f\, d\lambda \ge \frac{1}{n} \right\},$$

which is compact by Proposition 2.5. Again by Proposition 2.3, we get that $\mathscr{F}_{2,n} = \mathscr{K}^{\bullet}_{2,n}$ is closed. Since

$$\mathscr{M}_2 = \bigcup_{n \in \mathbb{N}} \mathscr{F}_{2,n}$$

is an F_σ and

$$\mathscr{M}_2^{c} = \left\{ \mu \in \mathscr{M}_+(S) \,\middle|\, \mu \perp \lambda \right\},$$

Theorem 2.2(2) is proven. □

We now pull back the regularity properties derived in Theorem 2.2 to regularity properties in $\mathfrak{S}$. We denote the spectral measure of $A \in \mathfrak{S}$ for $\xi \in \mathfrak{H}$ by ρ_ξ^A. It is defined by

$$\langle \rho_\xi^A, \varphi \rangle = \left(\varphi(A)\xi \,\middle|\, \xi \right) \quad \text{for } \varphi \in C_c(\mathbb{R}).$$

It is easy to see that $A_n \overset{\text{srs}}{\to} A$ implies strong convergence $\varphi(A_n) \to \varphi(A)$ for every $\varphi \in C_c(\mathbb{R})$ (see, e.g., [19, Satz 9.20]), which in turn gives weak convergence $\rho_\xi^{A_n} \to \rho_\xi^A$ for every $\xi \in \mathfrak{H}$. Thus, for each fixed $\xi \in \mathfrak{H}$, the map $\mathfrak{S} \longrightarrow \mathscr{M}_+(\mathbb{R})$, $A \mapsto \rho_\xi^A$ is continuous.

The spectral subspaces of A are defined by

$$\mathfrak{H}_{ac}(A) = \left\{ \xi \in \mathfrak{H} \,\middle|\, \rho_\xi^A \text{ is absolutely continuous} \right\},$$

$$\mathfrak{H}_{sc}(A) = \left\{ \xi \in \mathfrak{H} \,\middle|\, \rho_\xi^A \text{ is singular continuous} \right\},$$

$$\mathfrak{H}_{\mathrm{c}}(A) = \left\{\xi \in \mathfrak{H} \,\middle|\, \rho_\xi^A \text{ is continuous}\right\},$$
$$\mathfrak{H}_{\mathrm{pp}}(A) = \mathfrak{H}_{\mathrm{c}}(A)^\perp, \qquad \mathfrak{H}_{\mathrm{s}}(A) = \mathfrak{H}_{\mathrm{ac}}(A)^\perp.$$

These subspaces are closed and invariant under A, and $\mathfrak{H}_{\mathrm{pp}}(A)$ is the closed linear hull of the eigenvectors of A. Recall that the spectra $\sigma_*(A)$ are just the spectra of A restricted to $\mathfrak{H}_*(A)$. Using Theorem 2.2, we get the following proposition.

PROPOSITION 2.6
Let $U \subset \mathbb{R}$ be open, and let $\mathfrak{G} \subset \mathfrak{H}$ be a closed subspace. Then
(1) $\{A \in \mathfrak{S} \mid \forall\xi \in \mathfrak{G} : \rho_\xi^A|_U$ *is continuous* $\}$ *is a* G_δ,
(2) $\{A \in \mathfrak{S} \mid \forall\xi \in \mathfrak{G} : \rho_\xi^A|_U$ *is singular* $\}$ *is a* G_δ.

Proof
First, fix $\xi \in \mathfrak{H}$. We use the fact that the mappings $\mathscr{M}_+(\mathbb{R}) \to \mathscr{M}_+(U)$, $\nu \mapsto \nu|_U$ and $\mathfrak{S} \to \mathscr{M}_+(\mathbb{R})$, $A \mapsto \rho_\xi^A$ are continuous. Therefore we get that

$$\left\{A \in \mathfrak{S} \,\middle|\, \rho_\xi^A|_U \text{ is continuous}\right\}$$

is a G_δ by Theorem 2.2(1) since continuous is synonymous with diffusive. In the same way,

$$\left\{A \in \mathfrak{S} \,\middle|\, \rho_\xi^A|_U \text{ is singular}\right\}$$

is a G_δ by Theorem 2.2(2) since singular means singular with respect to the Lebesgue measure. Using the fact that the spectral subspaces are closed, for any dense set $\{\xi_n \mid n \in \mathbb{N}\}$, we get that

$$\left\{A \in \mathfrak{S} \,\middle|\, \forall\xi \in \mathfrak{G} : \rho_\xi^A|_U \text{ is continuous}\right\} = \bigcap_{n\in\mathbb{N}} \left\{A \in \mathfrak{S} \,\middle|\, \rho_{\xi_n}^A|_U \text{ is continuous}\right\}$$

is a G_δ, as is

$$\left\{A \in \mathfrak{S} \,\middle|\, \forall\xi \in \mathfrak{G} \,\middle|\, \rho_\xi^A|_U \text{ is singular}\right\} = \bigcap_{n\in\mathbb{N}} \left\{A \in \mathfrak{S} \,\middle|\, \rho_{\xi_n}^A|_U \text{ is singular}\right\}. \qquad \square$$

For completeness' sake, let us reproduce the well-known fact of lower semicontinuity of spectra under strong continuity (see, e.g., [17, Lemma 1.6], [19, Satz 9.26(b)]).

PROPOSITION 2.7
Let $V \subset \mathbb{R}$ be open. Then

$$\left\{A \in \mathfrak{S} \,\middle|\, \sigma(A) \cap V = \emptyset\right\}$$

is closed.

Proof
We have that

$$\{A \in \mathfrak{S} \,|\, \sigma(A) \cap V = \emptyset\} = \bigcap_{\varphi \in C_c(V)} \{A \in \mathfrak{S} \,|\, \varphi(A) = 0\}$$

is closed by the above-mentioned continuity of $A \mapsto \varphi(A)$. □

We can now put all that together for the following.

Proof of Theorem 2.1
By continuity of H and Proposition 2.6 applied with $\mathfrak{S} = \mathfrak{H}$, we get that the sets appearing in assumptions (1) and (2) of Theorem 2.1 are G_δ's. Choosing a countable base V_n, $n \in \mathbb{N}$ of U, we find that

$$\{A \in \mathfrak{S} \,|\, U \subset \sigma(A)\}^c = \bigcup_{n \in \mathbb{N}} \{A \in \mathfrak{S} \,|\, \sigma(A) \cap V_n = \emptyset\}$$

is an F_σ. Thus, invoking again the continuity of H, we infer that the set appearing in assumption (3) of Theorem 2.1 is a G_δ. Therefore the asserted denseness follows by Baire's theorem since the set appearing in the assertion is just the intersection of the three dense G_δ's from Theorem 2.1(1)–(3). □

Let us end our excursion to Wonderland by emphasizing the special role of singular continuity witnessed so far in the article. Let U be an open subset of $\mathbb{R}^d$. Then both

$$\mathcal{M}_{ac}(U) := \{\mu \in \mathcal{M}_+(U) \,|\, \mu \text{ is absolutely continuous}\}$$

and

$$\mathcal{M}_{pp}(U) := \{\mu \in \mathcal{M}_+(U) \,|\, \mu \text{ is pure point}\}$$

are dense in $\mathcal{M}_+(U)$, as can be seen by standard arguments. Thus, by Theorem 2.2(1) and Theorem 2.2(2), the sets $\mathcal{M}_c(U) := \{\mu \in \mathcal{M}_+(U) \,|\, \mu \text{ is continuous}\}$ and $\mathcal{M}_s(U) := \{\mu \in \mathcal{M}_+(U) \,|\, \mu \text{ is singular}\}$ are dense G_δ's. Therefore

$$\mathcal{M}_{sc}(U) := \{\mu \in \mathcal{M}_+(U) \,|\, \mu \text{ is singular continuous}\} = \mathcal{M}_c(U) \cap \mathcal{M}_s(U)$$

is a dense G_δ by Baire's theorem. Then another application of Baire's theorem shows that neither $\mathcal{M}_{ac}(U)$ nor $\mathcal{M}_{pp}(U)$ can be a G_δ, as they do not intersect $\mathcal{M}_{sc}(U)$.

Let us state the consequences for the special case of euclidean space.

COROLLARY 2.8
Let U be an open subset of $\mathbb{R}^d$. Then the singular continuous measures $\mathscr{M}_{sc}(U)$ form a dense G_δ in the space $\mathscr{M}_+(U)$ of Borel measures.

Of course, much more general spaces and reference measures can be treated; the only important restriction is that the measures singular with respect to the reference measure and the diffusive measures form dense sets.

Once more, we find that the silent majority consists of rather strange individuals. In mathematical terms, we owe this fact to Baire and completeness. We refer to [20] and [21] for a similar result, saying that most continuous monotonic functions on the real line are not differentiable. Let us also mention the classical articles [2] and [14] dealing with the lack of differentiability for a typical (in the sense of dense G_δ's) continuous function.

3. Delone sets and Delone Hamiltonians

Let us now define the operators for which we want to prove generic singular continuous spectral components. We start by recalling what a *Delone set,* a notion named after B. N. Delone (Delaunay), is (see [8]). We write $U_r(x)$ and $B_r(x)$, respectively, for the open ball and closed ball in $\mathbb{R}^d$. The euclidean norm on $\mathbb{R}^d$ is denoted by $\|\cdot\|$.

Definition
A set $\omega \subset \mathbb{R}^d$ is called an (r, R)-*set* if
- $\forall x, y \in \omega, x \neq y : U_r(x) \cap U_r(y) = \emptyset$;
- $\bigcup_{x\in\omega} B_R(x) = \mathbb{R}^d$.

By $\mathbb{D}_{r,R}(\mathbb{R}^d) = \mathbb{D}_{r,R}$ we denote the set of all (r, R)-sets. We say that $\omega \subset \mathbb{R}^d$ is a *Delone set* if it is an (r, R)-set for some $0 < r \leq R$, so that $\mathbb{D}(\mathbb{R}^d) = \mathbb{D} = \bigcup_{0<r\leq R} \mathbb{D}_{r,R}(\mathbb{R}^d)$ is the set of all Delone sets.

Delone sets turn out to be quite useful in the description of quasi crystals and more general aperiodic solids (see also [4], where the relation to discrete operators is discussed). In fact, if we regard an infinitely extended solid whose ions are assumed to be fixed, then the positions are naturally distributed according to the points of a Delone set. Fixing an effective potential v for all the ions, this leads us to consider the Hamiltonian

$$H(\omega) := -\Delta + \sum_{x\in\omega} v(\cdot - x) \quad \text{in } \mathbb{R}^d,$$

where $\omega \in \mathbb{D}$. For simplicity, let us assume that v is bounded, measurable, and compactly supported.

In order to apply our analysis in Section 2, we need to introduce a suitable topology on $\mathbb{D}$. This can be done in several ways (see [4], [12]). We follow the strategy from [12] and proceed as follows. We need some notation. Whenever (X, e) is a metric space, the Hausdorff distance e_H on the compact subsets of X is defined by

$$e_H(K_1, K_2) := \inf\big(\{\epsilon > 0 : K_1 \subset U_\epsilon(K_2) \text{ and } K_2 \subset U_\epsilon(K_1)\} \cup \{1\}\big),$$

where K_1, K_2 are compact subsets of the metric space (X, e) and $U_\epsilon(K)$ denotes the open ϵ-neighborhood around K with respect to e. It is well known that the set of all compact subsets of X becomes a complete compact metric space in this way whenever (X, e) is complete and compact. Using the inverse of the stereographic projection

$$j : \mathbb{R}^d \cup \{\infty\} \longrightarrow \mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \|x\| = 1\},$$

we can identify $\mathbb{R}^d \cup \{\infty\}$ and $\mathbb{S}^d$. We denote the euclidean distance on $\mathbb{S}^d$ by ρ. Denote by $\mathscr{F} := \mathscr{F}(\mathbb{R}^d)$ the set of closed subsets of $\mathbb{R}^d$. We can then define a metric δ on $\mathscr{F}(\mathbb{R}^d)$ by

$$\delta(F, G) := \rho_H\big(j(F \cup \{\infty\}), j(G \cup \{\infty\})\big).$$

This makes sense since $j(F \cup \{\infty\})$ is compact in $(\mathbb{S}^d, \rho)$ whenever F is closed in $\mathbb{R}^d$. The emerging topology is called the *natural topology*. It is not hard to see that $\{j(F \cup \{\infty\}) : F \in \mathbb{D}_{r,R}\}$ is closed within the compact subsets of $\mathbb{S}^d$ and, hence, compact. Thus, the natural topology defines a compact, completely metrizable topology on the set of all closed subsets of $\mathbb{R}^d$ for which $\mathbb{D}_{r,R}(\mathbb{R}^d)$ is a compact, complete space. The following lemma describes convergence with respect to this topology.

LEMMA 3.1
A sequence (ω_n) of Delone sets converges to $\omega \in \mathbb{D}$ in the natural topology if and only if there exists for any $l > 0$ an $L > l$ such that the $\omega_n \cap U_L(0)$ converge to $\omega \cap U_L(0)$ with respect to the Hausdorff distance as $n \to \infty$.

Proof
We start by proving the "only if" part. Let $\omega_n, \omega \in \mathbb{D}$ be given with $\omega_n \to \omega$ in $(\mathscr{F}, \delta)$. Fix $l > 0$. Since ω is discrete, we find $L > l$ such that $d(\partial U_L(0), \omega) =: \gamma > 0$, where d refers to the euclidean distance. There clearly exists C_L with

$$\rho\big(j(x), j(y)\big) \leq 2\|x - y\| \leq C_L \rho\big(j(x), j(y)\big) \qquad (*)$$

for all $x, y \in U_L(0)$. Fix $\epsilon > 0$, and use convergence of ω_n to ω in $\mathscr{F}$ to find $n_0 \in \mathbb{N}$ such that $\delta(\omega_n, \omega) \leq \beta\epsilon$ for $n \geq n_0$ for β small enough, so that

$$C_L \beta \epsilon \leq \min\{\epsilon, \gamma\}.$$

By definition of δ, this means that for every $x \in \omega \cap U_L(0)$, there is $x_n \in \omega_n$ such that $\rho(j(x), j(x_n)) \leq \beta\epsilon$. From $(*)$, we get

$$\|x - x_n\| \leq \frac{C_L}{2}\beta\epsilon \leq \frac{1}{2}\min\{\epsilon, \gamma\},$$

so that $x_n \in U_L(0)$ and $\omega \cap U_L(0) \subset U_\epsilon(\omega_n \cap U_L(0))$ for all $n \geq n_0$.

Conversely, given $x_n \in \omega_n \cap U_L(0)$, we find $x \in \omega$ such that $\rho(j(x), j(x_n)) \leq \beta\epsilon$. Again, by our choice of γ, β, and $j(x) \in U_L(0)$, this implies $\omega_n \cap U_L(0) \subset U_\epsilon(\omega \cap U_L(0))$.

These considerations show that $\omega_n \to \omega$ in $(\mathscr{F}, \delta)$ implies that for all $l > 0$, there exists $L > l$ with $d_H(\omega_n \cap U_L(0), \omega \cap U_L(0)) \longrightarrow 0$. This proves the "only if" part of Lemma 3.1.

We are now going to prove the "if" part of Lemma 3.1. Assume that ω_n, ω are Delone sets satisfying the condition of the lemma. We use a standard compactness argument to show that $\omega_n \to \omega$ with respect to δ. Choose an arbitrary subsequence (ω_{n_k}) of (ω_n). By compactness, there is a subsequence $(\omega_{n_{k_l}})$ converging to some $\widetilde{\omega}$ with respect to δ. By the first part of the proof, for every $l > 0$, we then find $L > 0$ with $d_H(\omega_{n_{k_l}} \cap U_L(0), \widetilde{\omega} \cap U_L(0)) \longrightarrow 0$. This implies that $\omega = \widetilde{\omega}$. Therefore, every subsequence of (ω_n) has a subsequence converging to ω with respect to δ. Thus, the sequence (ω_n) itself converges to ω with respect to δ. This finishes the proof of Lemma 3.1. □

The proof of the lemma shows effectively that $\omega_n \to \omega$ in $(\mathscr{F}, \delta)$ if and only if the following two conditions hold.

(i) For every $x \in \omega$, there exists (x_n) with $x_n \in \omega_n$ for every $n \in \mathbb{N}$ and $x_n \to x$.

(ii) Whenever (x_n) is a sequence with $x_n \in \omega_n$ for every $n \in \mathbb{N}$ and $x_n \to x$, then $x \in \omega$.

Given the lemma, it is straightforward to show that the map

$$H : \mathbb{D}_{r,R}(\mathbb{R}^d) \longrightarrow \mathfrak{S}\big(L^2(\mathbb{R}^d)\big), \qquad \omega \mapsto H(\omega)$$

is continuous.

We say that a Delone set ρ is *crystallographic* if $\mathrm{Per}(\rho) := \{t \in \mathbb{R}^d : \rho = t + \rho\}$ is a lattice of full rank.

We are now in position to state the main application of this article.

THEOREM 3.2
Let $r, R > 0$ with $2r \leq R$ and v be given such that there exist crystallographic $\gamma, \tilde{\gamma} \in \mathbb{D}_{r,R}$ with $\sigma(H(\gamma)) \neq \sigma(H(\tilde{\gamma}))$. Set $U_1 := \sigma(H(\gamma))^\circ \setminus \sigma(H(\tilde{\gamma}))$, $U_2 := \sigma(H(\tilde{\gamma}))^\circ \setminus \sigma(H(\gamma))$, and $U := U_1 \cup U_2$. Then U is nonempty, and there exists a dense G_δ-set $\Omega_{sc} \subset \mathbb{D}_{r,R}$ such that for every $\omega \in \Omega_{sc}$, the spectrum of $H(\omega)$ contains U and is purely singular continuous in U.

To prove the theorem, we need two results on extension of Delone sets. To state the results, we use the following notation. For $S > 0$, we define $Q(S) := [-S, S]^d \subset \mathbb{R}^d$.

LEMMA 3.3
Let $r, R > 0$ with $2r \leq R$ be given. Let both $\omega \in \mathbb{D}_{r,R}$ and $S > 0$ be arbitrary. Then there exists a crystallographic $\rho \in \mathbb{D}_{r,R}$ with $\rho \cap Q(S) = \omega \cap Q(S)$.

Proof
Let $P : \mathbb{R}^d \longrightarrow \mathbb{R}^d/2(S + R + r)\mathbb{Z}^d =: \mathbb{T}$ be the canonical projection. Note that the euclidean norm on $\mathbb{R}^d$ induces a canonical metric e on $\mathbb{T}$ with $e(P(x), P(y)) = \|x - y\|$ whenever $x, y \in \mathbb{R}^d$ are close to each other.

Let $F_0 := P(\omega \cap Q(S + R))$. By assumption on ω, we have $e(p, q) \geq 2r$ for all $p, q \in F_0$ with $p \neq q$. Moreover, $\bigcup_{p \in F_0} B_e(p, R) \supset P(Q(S)) =: C$, where $B_e(p, R)$ denotes the ball around p with radius R in the metric e.

Adding successively points from $\mathbb{T} \setminus C$ to F_0, we obtain a finite set F which is maximal among the sets satisfying

$$e(p, q) \geq 2r \quad \text{for all } p, q \in F \text{ with } p \neq q.$$

As any larger set violates this condition and $R \geq 2r$, we infer

$$\bigcup_{p \in F} B_e(p, R) = \mathbb{T}.$$

Now, $\rho := P^{-1}(F)$ has the desired properties. □

LEMMA 3.4
Let $r, R > 0$ with $2r \leq R$ be given. Let $\gamma, \omega \in \mathbb{D}_{r,R}$ and $S > 0$ be arbitrary. Then there exists a $\rho \in \mathbb{D}_{r,R}$ with

$$\rho \cap Q(S) = \omega \cap Q(S) \quad \text{and} \quad \rho \cap \big(\mathbb{R}^d \setminus Q(S + 2R + r)\big) = \gamma \cap \big(\mathbb{R}^d \setminus Q(S + 2R + r)\big).$$

Proof
Define

$$\rho' := \big(\omega \cap Q(S+R)\big) \cup \big(\gamma \cap (\mathbb{R}^d \backslash Q(S+R+r))\big).$$

Then $\bigcup_{x\in\rho'} B(x,R) \supset Q(S) \cup (\mathbb{R}^d \backslash Q(S+2R+r))$ and $\|x-y\| \geq 2r$ for all $x, y \in \rho'$ with $x \neq y$. Adding successively points from $Q(S+2R+r)\backslash Q(S)$ to ρ, we arrive at the desired set ρ. □

Proof of Theorem 3.2
Since $\gamma, \tilde{\gamma}$ are crystallographic, the corresponding operators are periodic and their spectra are, consequently, purely absolutely continuous.

We first consider the case where U_1 is nonempty. We verify conditions (1)–(3) from Theorem 2.1.

(1) Fix $\omega \in \mathbb{D}_{r,R}$. For $n \in \mathbb{N}$, consider $\nu_n := \omega \cap Q(n)$. By Lemma 3.3, we can find a crystallographic ω_n in $\mathbb{D}_{r,R}$ with $\omega_n \cap Q(n) = \nu_n$. For given $L > 0$, we get $\omega_n \cap U_L(0) = \omega \cap U_L(0)$ if n is large enough. Therefore, by Lemma 3.1, we find that $\omega_n \to \omega$ with respect to the natural topology. On the other hand, $\sigma_{\text{pp}}(H(\omega_n)) = \emptyset$ since the potential of $H(\omega_n)$ is periodic. Consequently,

$$\big\{\omega \in \mathbb{D}_{r,R} \,\big|\, \sigma_{\text{pp}}\big(H(\omega)\big) \cap U_1 = \emptyset\big\}$$

is dense in $\mathbb{D}_{r,R}$.

(2) To get the denseness of ω for which $\sigma_{\text{ac}}(H(\omega)) \cap U_1 = \emptyset$, fix $\omega \in \mathbb{D}_{r,R}$. For $n \in \mathbb{N}$ large enough, we apply Lemma 3.4 to obtain $\omega_n \in \mathbb{D}_{r,R}$ such that

$$\omega_n \cap U_n(0) = \omega \cap U_n(0) \qquad \text{and} \qquad \omega_n \cap U_{2n}(0)^{\text{c}} = \widetilde{\gamma} \cap U_{2n}(0)^{\text{c}}.$$

By virtue of the last property, $H(\omega_n)$ and $H(\widetilde{\gamma})$ differ only by a compactly supported, bounded potential, so that $\sigma_{\text{ac}}(H(\omega_n)) = \sigma_{\text{ac}}(H(\widetilde{\gamma})) \subset U_1^{\text{c}}$. In fact, standard arguments of scattering theory give that $e^{-H(\omega_n)} - e^{-H(\widetilde{\gamma})}$ is a trace class operator. By the invariance principle, the wave operators for $H(\omega_n)$ and $H(\widetilde{\gamma})$ exist and are complete, which in turn implies equality of the absolutely continuous spectra (see, e.g., [16, Section XI.3], [9, Section 2], [18, Corollary of Theorem 2] for a much more general result). Again, $\omega_n \to \omega$ yields condition (2) of Theorem 2.1.

(3) This can be checked with a similar argument, this time with $\tilde{\gamma}$ instead of γ. More precisely, we proceed as follows. Fix $\omega \in \mathbb{D}_{r,R}$. For $n \in \mathbb{N}$ large enough, we apply Lemma 3.4 to obtain $\omega_n \in \mathbb{D}_{r,R}$ such that

$$\omega_n \cap U_n(0) = \omega \cap U_n(0) \qquad \text{and} \qquad \omega_n \cap U_{2n}(0)^{\text{c}} = \gamma \cap U_{2n}(0)^{\text{c}}.$$

By virtue of the last property, $H(\omega_n)$ and $H(\gamma)$ differ only by a compactly supported, bounded potential, so that $\sigma_{\text{ac}}(H(\omega_n)) = \sigma_{\text{ac}}(H(\gamma)) \supset U_1$. By $\omega_n \to \omega$, we obtain Theorem 2.1(3).

Summarizing what we have shown so far, an appeal to Theorem 2.1 gives that

$$\{\omega \in \mathbb{D}_{r,R} \mid \sigma_{\mathrm{pp}}(H(\omega)) \cap U_1 = \emptyset, \sigma_{\mathrm{ac}}(H(\omega)) \cap U_1 = \emptyset, U_1 \subset \sigma(H(\omega))\}$$

is a dense G_δ-set if U_1 is not empty.

An analogous argument shows the same statement with U_2 instead of U_1. This proves the assertion if exactly one of the U_i, $i = 1, 2$, is not empty. If both are not empty, the assertion follows after intersecting the two corresponding dense G_δ's.

It remains to show that U_i, $i = 1, 2$, are not both empty. This can be seen as follows. As γ, $\tilde{\gamma}$ are periodic, the spectra of the corresponding operators consist of a union of closed intervals with only finitely many gaps in every compact subset of the reals. Hence, by $\sigma(H(\gamma)) \neq \sigma(H(\tilde{\gamma}))$, both U_1 and U_2 cannot be empty. This finishes the proof of Theorem 3.2. □

Remark. The assumption of Theorem 3.2 combines the nontriviality of v and the existence of suitable crystallographic γ, $\tilde{\gamma}$.

A simple way of ensuring this condition for $v \neq 0$ of fixed sign is to choose $R > 2^d r$. By an argument as in the proof of Lemma 3.3, we can then find crystallographic γ, $\tilde{\gamma}$ with the same lattice of periods such that $0 \in \tilde{\gamma} \setminus \gamma$. The corresponding periodic operators differ by a periodic potential with the same periodicity lattice. In this case, the analysis of [10] can be applied, showing that the spectra of $H(\tilde{\gamma})$ and $H(\gamma)$ differ.

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References

[1] M. BAAKE and R. V. MOODY, eds., *Directions in Mathematical Quasicrystals,* CRM Monogr. Ser. **13**, Amer. Math. Soc., Providence, 2000. MR 1798986 204

[2] S. BANACH, *Über die Bairesche Kategorie gewisser Funktionenmengen,* Studia Math. **3** (1931), 174–179. 211

[3] H. BAUER, *Mass- und Integrationstheorie*, 2nd ed., de Gruyter Lehrbuch, de Gruyter, Berlin, 1992. MR 1181881 206

[4] J. BELLISSARD, D. J. L. HERMANN, and M. ZARROUATI, "Hulls of aperiodic solids and gap labeling theorems" in *Directions in Mathematical Quasicrystals,* CRM Monogr. Ser. **13**, Amer. Math. Soc., Providence, 2000, 207–258. MR 1798994 211, 212

[5] D. BUSCHMANN and G. STOLZ, *Two-parameter spectral averaging and localization for non-monotonic random Schrödinger operators*, Trans. Amer. Math. Soc. **353** (2001), 635–653. MR 1804511 204

[6] J. R. CHOKSI and M. G. NADKARNI, "Baire category in spaces of measures, unitary operators and transformations" in *Invariant Subspaces and Allied Topics (Delhi, 1986)*, Narosa, New Delhi, 1990, 147 – 163. 205

[7] ———, *Genericity of certain classes of unitary and self-adjoint operators,* Canad. Math. Bull. **41** (1998), 137 – 139. MR 1624153 205

[8] B. N. DELAUNAY [B. N. DELONE], *Sur la sphère vide,* Bull. Acad. Sci. URSS, VII. Ser. **1934**, no. 6, 793 – 800. 211

[9] A. JENSEN and T. KATO, *Asymptotic behavior of the scattering phase for exterior domains*, Comm. Partial Differential Equations **3** (1978), 1165 – 1195. MR 0512084 215

[10] W. KIRSCH, P. STOLLMANN, and G. STOLZ, *Localization for random perturbations of periodic Schrödinger operators*, Random Oper. Stochastic Equations **6** (1998), 241 – 268. MR 1630995 216

[11] D. LENZ and P. STOLLMANN, *Quasicrystals, aperiodic order, and groupoid von Neuman algebras,* C. R. Math. Acad. Sci. Paris **334** (2002), 1131 – 1136. MR 1911659

[12] ———, "Delone dynamical systems and associated random operators" in *Operator Algebras and Mathematical Physics (Constanţa, Romania, 2001),* Theta, Bucharest, 2003, 267 – 285. MR 2018235 212

[13] ———, *Aperiodic order and quasicrystals: Spectral properties,* Ann. Henri Poincaré **4**, suppl. 2 (2004), S933 – S942. MR 2038109 216

[14] S. MAZURKIEWICZ, *Sur les fonctions non dérivables*, Studia Math. **3** (1931), 92 – 94. 211

[15] R. V. MOODY, ed., *The Mathematics of Long-Range Aperiodic Order (Waterloo, Canada, 1995),* NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **489**, Kluwer, Dordrecht, 1997. MR 1460016 204

[16] M. REED and B. SIMON, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press, New York, 1979. MR 0529429 215

[17] B. SIMON, *Operators with singular continuous spectrum, I: General operators*, Ann. of Math. (2) **141** (1995), 131 – 145. MR 1314033 203, 204, 205, 209

[18] P. STOLLMANN, *Scattering by obstacles of finite capacity,* J. Funct. Anal. **121** (1994), 416 – 425. MR 1272133 215

[19] J. WEIDMANN, *Lineare Operatoren in Hilberträumen, Teil I: Grundlagen*, Math. Leitfäden, Teubner, Stuttgart, 2000. MR 1887367 208, 209

[20] T. ZAMFIRESCU, *Most monotone functions are singular*, Amer. Math. Monthly **88** (1981), 47 – 49. MR 0619420 211

[21] ———, *Typical monotone continuous functions*, Arch. Math. (Basel) **42** (1984), 151 – 156. MR 0751487 211

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