AN ERGODIC THEOREM FOR DELONE DYNAMICAL SYSTEMS AND EXISTENCE OF THE INTEGRATED DENSITY OF STATES

By

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Abstract. We study strictly ergodic Delone dynamical systems and prove an ergodic theorem for Banach space valued functions on the associated set of pattern classes. As an application, we prove existence of the integrated density of states in the sense of uniform convergence in distribution for the associated random operators.

Dedicated to J. Voigt on the occasion of his 60th birthday

1 Introduction

This paper is concerned with Delone dynamical systems and the associated random operators.

Delone dynamical systems can be seen as the higher-dimensional analogues of subshifts over finite alphabets. They have attracted particular attention, as they can serve as models for so-called quasicrystals. These are substances, discovered in 1984 by Shechtman, Blech, Gratias and Cahn [37] (see the report [18] of Ishimasa et al. as well), which exhibit features similar to crystals but are non-periodic. Thus, they belong to the reign of disordered solids; and their distinctive feature is their special form of weak disorder.

This form of disorder and its effects have been extensively studied in recent years, both from the theoretical and the experimental point of view (see [2, 19, 34, 36] and references therein). On the theoretical side, there does not yet exist an axiomatic framework to describe quasicrystals. However, they are commonly modeled by either Delone dynamical systems or tiling dynamical systems [36] (see [25, 26] for recent study of Delone sets as well). In fact, these two descriptions are essentially equivalent (see, e.g., [31]). The main focus of the theoretical study

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lies then on diffraction properties, ergodic and combinatorial features and the associated random operators (see [2, 34, 36]).

Here, we deal with ergodic features of Delone dynamical systems and the associated random operators. The associated random operators (Hamiltonians) describe basic quantum mechanical features of the models (e.g., conductance properties). In the one-dimensional case, starting with [6, 39], various specific features of these Hamiltonians have been rigorously studied. They include purely singular continuous spectrum, Cantor spectrum and anomalous transport (see [8] for a recent review and an extended bibliography). In the higher-dimensional case, our understanding is much more restricted. In fact, information on spectral types is completely missing. However, there is K-theory, providing some overall type information on possible gaps in the spectra. This topic was initiated by Bellissard [3] for almost periodic operators and subsequently investigated for tilings, starting with the work of Kellendonk [20] (see [4, 21] for recent reviews).

Our aim in this paper is to study the integrated density of states. This is a key quantity in the study of random operators. It gives some average type of information on the involved operators.

We show uniform existence of the density of states in the sense of uniform convergence in distribution of the underlying measures. This result is considerably stronger than the corresponding earlier results of Kellendonk [20] and Hof [15], which gave only weak convergence. It fits well within the general point of view that quasicrystals should behave very uniformly due to their proximity to crystals.

These results are particularly relevant as the limiting distribution may well have points of discontinuity. In fact, points of discontinuity are an immediate consequence of the existence of locally supported eigenfunctions. Such eigenfunctions had already been observed in certain models [1, 13, 23, 24]. In fact, as discussed by the authors and Steffen Klassert in [22], they can easily be "introduced" without essentially changing the underlying Delone dynamical system. Moreover, based on the methods presented here, it is possible to show that points of discontinuity of the integrated density of states are exactly those energies for which locally supported eigenfunctions exist (see [22] again).

Let us emphasize that the limiting distribution is known to be continuous for models on lattices [10] (and, in fact, even stronger continuity properties hold [5]). In these cases, uniform convergence of the distributions is an immediate consequence of general measure theory.

To prove our result on uniform convergence (Theorem 3), we introduce a new method. It relies on studying convergence of averages in suitable Banach spaces. Namely, the integrated density of states turns out to be given by an almost additive function with values in a certain Banach space (Theorem 2). To apply our method, we prove an ergodic theorem (Theorem 1) for Banach space valued functions on the associated set of pattern classes.

This ergodic theorem may be of independent interest. It is an analogue of a result of Geerse/Hof [14] for tilings associated to primitive substitutions. For real valued almost additive functions on linearly repetitive Delone sets, related results have been obtained by Lagarias and Pleasants [26]. The one-dimensional case was studied by one of the authors in [29, 28].

The proof of our ergodic theorem uses ideas from the cited work of Geerse and Hof. Their work relies on suitable decompositions. These decompositions are naturally present in the framework of primitive substitutions. However, we need to construct them separately in the case we are dealing with. To do so, we use techniques of "partitioning according to return words," as introduced by Durand in [11, 12] for symbolic dynamics and later studied for tilings by Priebe [35]. However, this requires some extra effort, as we do not assume aperiodicity.

The paper is organized as follows. In Section 2, we introduce the notation and present our results. Section 3 is devoted to a discussion of the relevant decomposition. The ergodic theorem is proved in Section 4. Uniform convergence of the integrated density of states is proven in Section 5, after proving the necessary almost additivity.

2 Setting and results

The aim of this section is to introduce notation and to present our results, which cover part of what has been announced in [30]. In a companion paper [31], more emphasis is laid on the topological background and the basics of the groupoid construction and the noncommutative point of view.

For the remainder of the paper, an integer $d \geq 1$ is fixed and all Delone sets, patterns etc. are subsets of $\mathbb{R}^d$. The Euclidean norm on $\mathbb{R}^d$ is denoted by $\|\cdot\|$ as are the norms on various other normed spaces. For $s > 0$ and $p \in \mathbb{R}^d$, $B(p, s)$ is the closed ball in $\mathbb{R}^d$ around p with radius s. A subset ω of $\mathbb{R}^d$ is called Delone set if there exist $r > 0$ and $R > 0$ such that

- $2r \leq \|x - y\|$ whenever $x, y \in \omega$ with $x \neq y$,
- $B(x, R) \cap \omega \neq \emptyset$ for all $x \in \mathbb{R}^d$;

and the limiting values of r and R are called *packing radius* and *covering radius*, respectively. Such an ω is also called (r, R)-set. Of particular interest are the restrictions of ω to bounded subsets of $\mathbb{R}^d$. In order to treat these restrictions, we introduce the following definition.

Definition 2.1. (a) A pair (Λ, Q) consisting of a bounded subset Q of $\mathbb{R}^d$ and $\Lambda \subset Q$ finite is called a *pattern*. The set Q is called the *support of the pattern*.

(b) A pattern (Λ, Q) is called a *ball pattern* if $Q = B(x, s)$ with $x \in \Lambda$ for suitable $x \in \mathbb{R}^d$ and $s > 0$.

The diameter and the volume of a pattern are defined to be the diameter and the volume of its support, respectively. For patterns $X_1 = (\Lambda_1, Q_1)$ and $X_2 = (\Lambda_2, Q_2)$, we define $\sharp_{X_1} X_2$, the number of occurrences of X_1 in X_2, to be the number of elements in $\{t \in \mathbb{R}^d : \Lambda_1 + t = \Lambda_2 \cap (Q_1 + t), Q_1 + t \subset Q_2\}$. Moreover, for patterns $X_i = (\Lambda_i, Q_i)$, $i = 1, \dots, k$, and $X = (\Lambda, Q)$, we write $X = \bigoplus_{i=1}^k X_i$ if $\Lambda = \bigcup \Lambda_i$, $Q = \bigcup Q_i$ and the Q_i are disjoint up to their boundaries.

For further investigation, we have to identify patterns which are equal up to translation. Thus, on the set of patterns we introduce an equivalence relation by setting $(\Lambda_1, Q_1) \simeq (\Lambda_2, Q_2)$ if and only if there exists $t \in \mathbb{R}^d$ with $\Lambda_1 = \Lambda_2 + t$ and $Q_1 = Q_2 + t$. The equivalence class of a pattern (Λ, Q) is denoted by $[(\Lambda, Q)]$. The notions of diameter, volume occurrence, etc., can easily be carried over from patterns to pattern classes.

Every Delone set ω gives rise to a set of pattern classes, $\mathcal{P}(\omega)$, viz., $\mathcal{P}(\omega) = \{Q \wedge \omega : Q \subset \mathbb{R}^d \text{ bounded and measurable}\}$ and to a set of ball pattern classes $\mathcal{P}_B(\omega)) = \{[B(p, s) \wedge \omega] : p \in \omega, s \in \mathbb{R}\}$. Here we set

$$Q \wedge \omega = (\omega \cap Q, Q). \tag{2.1}$$

Furthermore, for arbitrary ball patterns P, we define $s(P)$ to be the radius of the underlying ball, i.e.,

$$s(P) = s \quad \text{for } P = [(\Lambda, B(p, s))]. \tag{2.2}$$

For $s \in (0, \infty)$, we denote by $\mathcal{P}_B^s(\omega)$ the set of ball patterns with radius s. A Delone set is said to be of finite type if for every radius s the set $\mathcal{P}_B^s(\omega)$ is finite.

The Hausdorff metric on the set of compact subsets of $\mathbb{R}^d$ induces the so-called *natural topology* on the set of closed subsets of $\mathbb{R}^d$. It is described in detail in [31] and enjoys certain nice properties: the set of all closed subsets of $\mathbb{R}^d$ is compact in the natural topology, and the natural action T of $\mathbb{R}^d$ on the closed sets given by $T_t C \equiv C + t$ is continuous.

Definition 2.2. (a) If Ω is a set of Delone sets which is invariant under the shift T and closed under the natural topology, then (Ω, T) is called a *Delone dynamical system*, abbreviated as *DDS*.

was shown in Theorem 1.6 in [31] (see [27] as well). It goes back to [38], Theorem 3.3, in the tiling setting.

Definition 2.3. Let Ω be a DDS and $\mathcal{B}$ be a vector space with seminorm $\|\cdot\|$. A function $F : \mathcal{P}(\Omega) :\longrightarrow \mathcal{B}$ is called *almost additive (with respect to* $\|\cdot\|$*)* if there exists a function $b : \mathcal{P}(\Omega) \longrightarrow [0,\infty)$ (called the *associated error function*) and a constant $D > 0$ such that

(A1) $\|F(\bigoplus_{i=1}^k P_i) - \sum_{i=1}^k F(P_i)\| \leq \sum_{i=1}^k b(P_i)$;
(A2) $\|F(P)\| \leq D|P| + b(P)$;
(A3) $b(P_1) \leq b(P) + b(P_2)$ whenever $P = P_1 \oplus P_2$;
(A4) $\lim_{n\to\infty} |P_n|^{-1} b(P_n) = 0$ for every van Hove sequence (P_n).

Our first result is:

Theorem 1. *For a minimal DDSF* (Ω, T)*, the following are equivalent.*
(i) (Ω, T) *is uniquely ergodic.*
(ii) *The limit* $\lim_{k\to\infty} |P_k|^{-1} F(P_k)$ *exists for every van Hove sequence* (P_k) *and every almost additive* F *on* (Ω, T) *with values in a Banach space.*

The proof of the theorem makes use of completeness of the Banach space in a crucial manner. However, it does not use the nondegeneracy of the norm. Thus, we get the following corollary (of its proof).

Corollary 2.4. *Let* (Ω, T) *be a strictly ergodic DDSF. Let the vector space* $\mathcal{B}$ *be complete with respect to the topology induced by the seminorms* $\|\cdot\|_\iota$, $\iota \in \mathcal{I}$. *If* $F : \mathcal{P} \longrightarrow \mathcal{B}$ *is almost additive with respect to every* $\|\cdot\|_\iota$, $\iota \in \mathcal{I}$, *then* $\lim_{k\to\infty} |P_k|^{-1} F(P_k)$ *exists for every van Hove sequence* (P_k) *in* $\mathcal{P}(\Omega)$.

Theorem 1 may also be rephrased as a result on additive functions on Borel sets. As this may also be of interest, we include a short discussion.

Definition 2.5. Let (Ω, T) be a DDS and $\mathcal{B}$ be a Banach space. Let $\mathcal{S}$ be the family of bounded measurable sets on $\mathbb{R}^d$. A function $F : \mathcal{S} \times \Omega \longrightarrow \mathcal{B}$ is called *almost additive* if there exists a function $b : \mathcal{S} \longrightarrow [0,\infty)$ and $D > 0$ such that

(A0) $b(Q) = b(Q + t)$ for arbitrary $Q \in \mathcal{S}$ and $t \in \mathbb{R}^d$ and $\|F_\omega(Q) - F_\omega(Q')\| \leq b(Q)$ whenever $\omega \wedge Q = \omega \wedge Q'$.

(A1) $\|F_\omega(\bigcup_{j=1}^n Q_j) - \sum_{j=1}^n F_\omega(Q_j)\| \leq \sum_{j=1}^n b(Q_j)$ for arbitrary $\omega \in \Omega$ and $Q_j \in \mathcal{S}$ which are disjoint up to their boundaries.

(A2) $\|F_\omega(Q)\| \leq D|Q| + b(Q)$.

(A3) $b(Q_1) \leq b(Q) + b(Q_2)$ whenever $Q = Q_1 \cup Q_2$ with Q_1 and Q_2 disjoint up to their boundaries.

(A4) $\lim_{k\to\infty} |Q_k|^{-1} b(Q_k) = 0$ for every van Hove sequence (Q_k).

Corollary 2.6. *Let (Ω, T) be a strictly ergodic DDS and $F : \mathcal{S} \times \Omega \to \mathcal{B}$ be almost additive. Then $\lim_{k\to\infty} |Q_k|^{-1} F_\omega(Q_k)$ exists for arbitrary $\omega \in \Omega$ and every van Hove sequence (Q_k) in $\mathbb{R}^d$, and the convergence is uniform on Ω.*

Our further results concern selfadjoint operators in a certain C^* algebra associated to (Ω, T). The construction of this C^* algebra was given in our earlier work [30, 31]. We recall the necessary details next.

Definition 2.7. Let (Ω, T) be a DDSF. A family (A_ω) of bounded operators $A_\omega : \ell^2(\omega) \longrightarrow \ell^2(\omega)$ is called a *random operator of finite range* if there exists a constant $s > 0$ with

- $A_\omega(x, y) = 0$ whenever $\|x - y\| \geq s$;
- $A_\omega(x, y)$ only depends on the pattern class of $((K(x, s) \cup K(y, s)) \wedge \omega)$.

The smallest such s is denoted by R^A.

The operators of finite range form a $*$-algebra under the obvious operations. There is a natural C^*-norm on this algebra and its completion is a C^*-algebra denoted as $\mathcal{A}(\Omega, T)$ (see [4, 30, 31] for details). It consists again of families $(A_\omega)_{\omega\in\Omega}$ of operators $A_\omega : \ell^2(\omega) \longrightarrow \ell^2(\omega)$.

Note that for selfadjoint $A \in \mathcal{A}(\Omega, T)$ and bounded $Q \subset \mathbb{R}^d$, the restriction $A_\omega|_Q$ defined on $\ell^2(Q \cap \omega)$ has finite rank. Therefore, the spectral counting function

$$n(A_\omega, Q)(E) := \#\{\text{eigenvalues of } A_\omega|_Q \text{ below } E\}$$

is finite; and $\frac{1}{|Q|} n(A_\omega, Q)$ is the distribution function of the measure $\rho(A_\omega, Q)$, defined by

$$\langle \rho(A_\omega, Q), \varphi \rangle := \frac{1}{|Q|} \operatorname{tr}(\varphi(A_\omega|_Q)) \quad \text{for } \varphi \in C_b(\mathbb{R}).$$

These spectral counting functions are obviously elements of the vector space $\mathcal{D}$ consisting of all bounded right continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x)$ exists. Equipped with the supremum norm $\|f\|_\infty \equiv \sup_{x\in\mathbb{R}} |f(x)|$, this vector space is a Banach space. It turns out that the spectral counting function is essentially an almost additive function. More precisely, the following holds.

Theorem 2. *Let* (Ω, T) *be a DDS. Let* A *be an operator of finite range. Then* $F^A : \mathcal{P}(\Omega) \longrightarrow \mathcal{D}$, *defined by* $F^A(P) \equiv n(A_\omega, Q_{R^A})$ *for* $P = [(\omega \wedge Q)]$, *is a well-defined almost additive function.*

Remark 1. This theorem seems to be new even in the one-dimensional case. (There, of course, it is very easy to prove.)

Based on the foregoing two theorems, it is rather clear how to show existence of the limit $\lim_{k\to\infty} |Q_k|^{-1} n(A_\omega, Q_n)$ for van Hove sequences (Q_k). This limit is called the *spectral density* of A. It is possible to express this limit in closed form using a certain trace on a von Neumann algebra [30, 31]. We do not discuss this trace here, but rather directly give a closed expression. To each selfadjoint element $A \in \mathcal{A}(\Omega, T)$, we associate the measure ρ^A defined on $\mathbb{R}$ by

$$\rho^A(F) \equiv \int_\Omega \mathrm{tr}_\omega(M_f(\omega)\pi_\omega(F(A)))d\mu(\omega).$$

Here, tr_ω is the standard trace on the bounded operators on $\ell^2(\omega)$, f is an arbitrary nonnegative continuous function with compact support on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} f(t)dt = 1$ and $M_f(\omega)$ denotes the operator of multiplication with f in $\ell^2(\omega)$ (see [30, 31] for details). It turns out that ρ^A is a spectral measure for A [31]. Our result on convergence of the integrated density of states is the following.

Theorem 3. *Let* (Ω, T) *be a strictly ergodic DDSF. Let* A *be a selfadjoint operator of finite range and* (Q_k) *an arbitrary van Hove sequence. Then the distributions* $E \mapsto \rho^{A_\omega}_{Q_k}((-\infty, E])$ *converge to the distribution* $E \mapsto \rho^A((-\infty, E])$ *with respect to* $\|\cdot\|_\infty$, *and this convergence is uniform in* $\omega \in \Omega$.

Remark 2. (a) The usual proofs of existence of the integrated density of states yield only weak convergence of the measures.

(b) The proof of the theorem uses the fact, already established in [31, 33], that the measures $\rho(A_\omega, Q_n)$ converge weakly towards the measure ρ^A for every $\omega \in \Omega$ and $A \in \mathcal{A}(\Omega, T)$.

As mentioned in the preceding remark, the usual proofs of existence of the integrated density of states give only weak convergence of the measures $\rho^{A_\omega}_{Q_n}$. Weak convergence of measures does not, in general, imply convergence in distribution. Convergence in distribution follows, however, from weak convergence if the limiting distribution is continuous. Thus, Theorem 3 is particularly interesting in view of the fact that the limiting distribution can have points of discontinuity.

Existence of such discontinuities is rather remarkable, as it is completely different from the behaviour of random operators associated to models with higher disorder. It turns out that a very precise understanding of this phenomenon can be obtained invoking the results presented above. Details of this will be given separately [22]. Here, we only mention the following theorem.

Theorem 4. *Let* (Ω, T) *be a strictly ergodic DDSF and* A *an operator of finite range on* (Ω, T). *Then* E *is a point of discontinuity of* ρ^A *if and only if there exists a locally supported eigenfunction of* $A_\omega - E$ *for one (every)* $\omega \in \Omega$.

3 Decomposing Delone sets

This section provides the main geometric ideas underlying the proof of our ergodic theorem, Theorem 1. We first discuss how to decompose a given Delone set into finite pieces, called *cells*, in a natural manner, Proposition 3.2. This is based on the Voronoi construction, as given in (3.1) and Lemma 3.1, together with a certain way to obtain Delone sets from a given Delone set and a pattern. This decomposition is performed on an increasing sequence of scales. As mentioned already, here we use ideas from [11, 35]. Having described these decompositions, our main concern is to study van Hove type properties of the induced sequences of cells. This study is undertaken in a series of lemmas, yielding as main results Proposition 3.12 and Proposition 3.14. Here, the proof of Proposition 3.14 requires considerable extra effort (compared with the proof of Proposition 3.12), as we have to cope with periods.

We start with a discussion of the well-known Voronoi construction. Let ω be an (r, R)-set. To an arbitrary $x \in \omega$, we associate the Voronoi cell $V(x, \omega) \subset \mathbb{R}^d$ defined by

$$V(x,\omega) \equiv \{p \in \mathbb{R}^d : \|p - x\| \leq \|p - y\| \text{ for all } y \in \omega \text{ with } y \neq x\} \tag{3.1}$$

$$= \bigcap_{y \in \omega, y \neq x} \{p \in \mathbb{R}^d : \|p - x\| \leq \|p - y\|\}. \tag{3.2}$$

Note that $\{p \in \mathbb{R}^d : \|p-x\| \leq \|p-y\|\}$ is a half-space. Thus, $V(x, \omega)$ is a convex set. Moreover, it is obviously closed and bounded and therefore compact. It turns out that $V(x, \omega)$ is already determined by the elements of ω close to x. More precisely, the following holds.

Lemma 3.1. *Let* ω *be an* (r, R)*-set. Then,* $V(x, \omega)$ *is determined by* $B(x, 2R) \wedge \omega$, *viz.,* $V(x,\omega) \equiv \bigcap_{y \in B(x,2R) \cap \omega} \{p \in \mathbb{R}^d : \|p - x\| \leq \|p - y\|\}$. *Moreover,* $V(x, \omega)$ *is contained in* $B(x, R)$.

Proof. The first statement follows from Corollary 5.2 in [36], and the second one is a consequence of Proposition 5.2 in [36]. □

Next we describe our notion of derived Delone sets. Let ω be an (r, R)-set and P be a ball pattern class with $P \in \mathcal{P}(\omega)$. We define the Delone set derived from ω by P, denoted ω_P, to be the set of all occurrences of P in $\mathbb{R}^d$, i.e.,

$$\omega_P \equiv \{t \in \mathbb{R}^d : [B(t, s(P)) \wedge \omega] = P\}.$$

Now let (Ω, T) be a minimal DDSF. Choose $\omega \in \Omega$ and $P \in \mathcal{P}_B(\Omega)$. Then the Voronoi construction applied to ω_P yields a decomposition of ω into cells

$$C(x, \omega, P) \equiv V(x, \omega_P) \wedge \omega, \quad x \in \omega_P.$$

More precisely,

$$\mathbb{R}^d = \bigcup_{x \in \omega_P} V(x, \omega_P), \text{ and } \operatorname{int}(V(x, \omega_P)) \cap \operatorname{int}(V(y, \omega_P)) = \emptyset,$$

whenever $x \neq y$. Here, $\operatorname{int}(V)$ denotes the interior of V. This way of decomposing ω is called the *P-decomposition* of ω. It is a crucial fact that each $C(x, \omega, P)$ is already determined by

$$B(x, 2R(P)) \wedge \omega,$$

as can be seen by Lemma 3.1, where $R(P)$ denotes the covering radius of ω_P. Thus, in particular, the following holds.

Proposition 3.2. *Let (Ω, T) be a minimal DDSF and let $P \in \mathcal{P}_B(\Omega)$ be fixed. Let $\omega \in \Omega$ with $0 \in \omega_P$ and set $Q = B(0, 2R(P)) \wedge \omega$. Then $C(Q) \equiv [V(0, \omega_P) \wedge \omega]$ depends only on $[Q]$ (and not on ω). Moreover, if $\widetilde{C}$ is a cell occurring in the P-decomposition of some $\omega_1 \in \Omega$, then $[\widetilde{C}] = C(Q)$ for a suitable $\omega \in \Omega$ with $0 \in \omega_P$.*

The proposition says that the occurrences of certain cells in the P-decompositions are determined by the occurrences of the larger

$$Q \in \{[B(x, 2R(P)) \wedge \omega] : x \in \omega_P, \omega \in \Omega\}.$$

The proposition does not say that different Q induce different $C(Q)$ (and this is not, in fact, true in general).

The main aim is now to study the decompositions associated to an increasing sequence of ball pattern classes (P_n). We begin by studying minimal and maximal distances between occurrences of a ball pattern class P. We need the following definition.

Definition 3.3. Let (Ω, T) be a minimal DDSF and $P \in \mathcal{P}_B$ be arbitrary. Define $r(P)$ as the packing radius of ω_P, i.e., by

$$r(P) \equiv \frac{1}{2}\inf\{\|x-y\| : x \neq y, x, y \in \omega_P, \omega \in \Omega\},$$

and the occurrence radius $R(P)$ by

$$R(P) \equiv \inf\{R > 0 : \sharp_P([B(p,R) \wedge \omega]) \geq 1 \text{ for every } p \in \mathbb{R}^d \text{ and } \omega \in \Omega\}.$$

Lemma 3.4. *Let (Ω, T) be minimal. Then*

$$R(P) \equiv \min\{R > 0 : \sharp_P([B(p,R) \wedge \omega]) \geq 1 \textit{ for every } p \in \mathbb{R}^d \textit{ and } \omega \in \Omega\}.$$

Moreover, ω_P is an $(r(P), R(P))$-set for every $\omega \in \Omega$.

Proof. We show that the infimum is a minimum. Assume the contrary and set $R' := R(P)$. Then there exist $p \in \mathbb{R}^d$ and $\omega \in \Omega$ such that $B(p, R') \wedge \omega$ does not contain a copy of P. However, by definition of R', $B(p, R' + \epsilon) \wedge \omega$ contains a copy of P for every $\epsilon > 0$. As ω is a Delone set, $B(p, R' + 1) \wedge \omega$ contains only finitely many copies of P, and a contradiction follows. The last statement of the lemma is immediate. □

We need to deal with Delone sets which are not aperiodic. To do so the following notions are useful. For a minimal DDS (Ω, T) let $\mathcal{L} \equiv \mathcal{L}(\Omega)$ be the periodicity lattice of (Ω, T), i.e.,

$$\mathcal{L} \equiv \mathcal{L}(\Omega) \equiv \{t \in \mathbb{R}^d : T_t\omega = \omega \text{ for all } \omega \in \Omega\}.$$

Clearly, $\mathcal{L}$ is a subgroup of $\mathbb{R}^d$; it is discrete, since every ω is discrete. Thus (see Proposition 2.3 in [36]), $\mathcal{L}$ is a lattice in $\mathbb{R}^d$, i.e., there exist $D(\mathcal{L}) \in \mathbb{N}$ and vectors $e_1, \ldots, e_{D(\mathcal{L})} \in \mathbb{R}^d$ which are linearly independent (in $\mathbb{R}^d$) such that

$$\mathcal{L} = \mathrm{Lin}_{\mathbb{Z}}\{e_j : j = 1, \ldots D(\mathcal{L})\} \equiv \{\sum_{j=1}^{D(\mathcal{L})} a_j e_j : a_j \in \mathbb{Z}, j = 1, \ldots, D(\mathcal{L})\}.$$

We define $r(\mathcal{L})$ by

$$r(\mathcal{L}) \equiv \begin{cases} \infty & : \ ; \text{if } \mathcal{L} = \{0\} \\ \frac{1}{2}\min\{\|t\| : t \in \mathcal{L} \setminus \{0\}\} & : \ ; \text{otherwise.} \end{cases}$$

Lemma 3.6 below provides a result on minimal distances. Variants of this result have been given in the literature on tilings [35] and on symbolic dynamics [11]. To prove it in our context, we require the following result from [32] concerning the natural topology.

Lemma 3.5. *A sequence (ω_n) of Delone sets converges to $\omega \in \mathcal{D}$ in the natural topology if and only if there exists for any $l > 0$ an $L > l$ such that $\omega_n \cap B(0, L)$ converges to $\omega \cap B(0, L)$ with respect to the Hausdorff distance as $n \to \infty$.*

Lemma 3.6. *Let (Ω, T) be a minimal DDSF. Let (P_n) be a sequence of ball pattern classes with $s(P_n) \longrightarrow \infty$, $n \longrightarrow \infty$. Then*

$$\liminf_{n\to\infty} r(P_n) \geq r(\mathcal{L}).$$

In particular, there exists $\rho > 0$ such that $r(P) \geq r(\mathcal{L})/2$ whenever P is a ball pattern with $s(P) \geq \rho$.

Proof. As (Ω, T) is minimal, it is an (r, R)-system. Assume that the claim is false. Thus, there exists a sequence (P_n) in $\mathcal{P}_B(\Omega)$ with $s(P_n) \to \infty$, $n \to \infty$, but $r(P_n) \leq C$ with a suitable constant $C > 0$ with $C < r(\mathcal{L})$. Then there exist $\omega_n \in \Omega$ and $t_n \in \mathbb{R}^d$ with $||t_n|| \leq 2C$ (and, of course, $||t_n|| \geq 2r$) with

$$B(0, s(P_n)) \wedge \omega_n = B(0, s(P_n)) \wedge (\omega_n - t_n). \tag{3.3}$$

By compactness of Ω and $B(0, 2C)$, we can assume without loss of generality that $\omega_n \to \omega$ and $t_n \to t$, with $t \in B(0, C)$, $n \to \infty$. Thus, (3.3) implies

$$\omega = \omega - t. \tag{3.4}$$

In fact, let $p \in \omega$. Fix $R > 0$ such that $p \in \omega \cap B(0, R)$. By Lemma 3.5, we find $p_n \in \omega_n$, for n sufficiently large, such that $p_n \to p$ for $n \to \infty$. Assuming $R < s(P_n)$ and utilizing (3.3), we find $q_n \in \omega_n$ such that $p_n = q_n - t_n$. Since $q_n \to p + t$ and $\omega_n \to \omega$, we see that $q = p + t \in \omega$, leaving us with

$$\omega \cap B(0, R) \subset (\omega - t) \cap B(0, R).$$

By symmetry and since R was arbitrary, this gives (3.4). Minimality yields that (3.4) extends to all $\omega \in \Omega$. Thus, t belongs to $\mathcal{L}$. As $0 < 2r \leq ||t|| \leq 2C < 2r(\mathcal{L})$, this gives a contradiction. □

Definition 3.7. For a compact convex set $C \subset \mathbb{R}^d$, denote by $s(C) > 0$ the *inradius* of C, i.e., the largest s such that C contains a ball of radius s.

In the sequel, we write $\omega_{n,P_n} := (\omega_n)_{P_n}$ to shorten notation.

Lemma 3.8. *Let (Ω_n, T), $n \in \mathbb{N}$, be a family of minimal DDS. Let a pattern class $P_n \in \mathcal{P}(\Omega_n)$, $\omega_n \in \Omega_n$ and $x_n \in \omega_{n,P_n}$ be given for any $n \in \mathbb{N}$. If $r(P_n) \longrightarrow \infty$, $n \longrightarrow \infty$, then $s(V(x_n, \omega_{n,P_n})) \to \infty$, for $n \to \infty$.*

Proof. Without loss of generality, we can assume that $x_n = 0$ for every $n \in \mathbb{N}$. By construction of $V_n \equiv V(x_n, \omega_{n,P_n})$, we have

$$s(V_n) \geq \text{dist}(0, \partial V_n) \geq r(P_n), \quad n \in \mathbb{N}.$$

This implies $s(V_n) \longrightarrow \infty$, $n \longrightarrow \infty$. □

Our next aim is to show that a sequence of convex sets with increasing inradii must be van Hove. We need the following two lemmas.

Lemma 3.9. *For every $d \in \mathbb{N}$, there exists a constant $c = c(d)$ with*

$$(1+s)^d - (1-s)^d \leq cs$$

for $|s| \leq 1$.

Proof. This follows by a direct computation. □

For $C \subset \mathbb{R}^d$ and $\lambda \geq 0$, we set

$$\lambda C \equiv \{\lambda x : x \in C\}.$$

Lemma 3.10. *Let C be a compact convex set in $\mathbb{R}^d$ with $B(0,s) \subset C$. Then the inclusion*

$$C^h \setminus C_h \subset \left(1 + \frac{h}{s}\right) C \setminus \left(1 - \frac{h}{s}\right) C$$

holds, where we set $(1 - hs^{-1})C = \emptyset$ if $h > s$. In particular,

$$|C^h \setminus C_h| \leq \kappa \max\left\{\frac{h}{s}, \frac{h^d}{s^d}\right\} |C|,$$

with a suitable constant $\kappa = \kappa(d)$.

Proof. The first statement follows by convexity of C. The second is then an immediate consequence of the change of variable formula combined with the foregoing lemma. □

Lemma 3.11. *Let (C_n) be a sequence of convex sets in $\mathbb{R}^d$ with $s(C_n) \longrightarrow \infty$, $n \longrightarrow \infty$. Then (C_n) is a van Hove sequence.*

Proof. Let $h > 0$ be given and assume without loss of generality that $B(0, s(C_n)) \subset C_n$. The result follows from the previous lemma. □

The following consequence of the foregoing results is a key ingredient of our proof of Theorem 1.

Proposition 3.12. *Let (Ω, T) be a minimal and aperiodic DDSF. Let (P_n) be a sequence in $\mathcal{P}_B(\Omega)$ with $s(P_n) \to \infty$, $n \to \infty$. Let $(\omega_n) \subset \Omega$ and $x_n \in \omega_{n,P_n}$ be arbitrary. Then $V(x_n, \omega_{n,P_n})$ is a van Hove sequence.*

Proof. By Lemma 3.6, aperiodicity of (Ω, T) together with $s(P_n) \to \infty$, $n \to \infty$ yields $r(P_n) \to \infty$, $n \to \infty$. Therefore, by Lemma 3.8, we have $s(V(x_n, \omega_{n,P_n})) \to \infty$, for $n \to \infty$. The statement is now immediate from Lemma 3.11. □

We also need an analogue of this proposition for arbitrary (i.e., not necessarily aperiodic) DDSF. To obtain this analogue requires some extra effort.

Let a minimal DDSF (Ω, T) with periodicity lattice $\mathcal{L}$ be given. Let $U = U(\mathcal{L})$ be the subspace of $\mathbb{R}^d$ spanned by the e_j, $j = 1, \ldots, D(\mathcal{L})$ and let $P_U : \mathbb{R}^d \longrightarrow U$ be the orthogonal projection onto U. The lattice $\mathcal{L}$ induces a grid on $\mathbb{R}^d$. Namely, we can set

$$G_0 \equiv \{x \in \mathbb{R}^d : P_U x = \sum_{j=1}^{D(\mathcal{L})} \lambda_j e_j; \text{ with } 0 \leq \lambda_j < 1, j = 1, \ldots, D(\mathcal{L})\}$$

and

$$G_{(n_1,\ldots,n_{D(\mathcal{L})})} \equiv n_1 e_1 + \cdots + n_{D(\mathcal{L})} e_{D(\mathcal{L})} + G_0,$$

for $(n_1, \ldots, n_{D(\mathcal{L})}) \in \mathbb{Z}^{D(\mathcal{L})}$.

We now use coloring of Delone sets to obtain new DDS from (Ω, T). These new systems are essentially the same sets but equipped with a coloring which "broadens" the periodicity lattice. Coloring has been discussed, e.g., in [31].

Let C be a finite set. A Delone set with colorings in C is a subset of $\mathbb{R}^d \times C$ such that $p_1(\omega)$ is a Delone set, where $p_1 : \mathbb{R}^d \times C$ is the canonical projection $p_1(x, c) = x$. When referring to an element (x, c) of a colored Delone set, we also say that x is colored with c. Notions such as patterns, pattern classes, occurrences, diameter, etc., can easily be carried over to colored Delone sets.

Fix $\omega \in \Omega$ with $0 \in \Omega$. For every $l \in \mathbb{N}$, we define a DDS as follows. Let $\omega^{(l)}$ be a Delone set with coloring in $\{0, 1\}$ introduced by the following rule: $x \in \omega$ is colored with 1 if and only if there exists $(n_1, \ldots, n_{D(\mathcal{L})}) \in \mathbb{Z}^{D(\mathcal{L})}$ with

$$x \in G_{(ln_1,\ldots,ln_{D(\mathcal{L})})};$$

in all other cases, $x \in \omega$ is colored with 0. Set $\Omega^{(l)} \equiv \Omega(\omega^{(l)}) \equiv \overline{\{T_t \omega^{(l)} : t \in \mathbb{R}^d\}}$, where the bar denotes the closure in the canonical topology associated to colored Delone sets [31]. The DDS $(\Omega^{(l)}, T)$ is minimal, as can easily be seen considering repetitions of patterns in $\omega^{(l)}$. Also, $(\Omega^{(l)}, T)$ is uniquely ergodic if (Ω, T) is uniquely ergodic, as follows by considering the existence of frequencies in $\omega^{(l)}$. The important point about $(\Omega^{(l)}, T)$ is the following fact.

Lemma 3.13. *Let (Ω, T) be a minimal DDS and $(\Omega^{(l)}, T)$ for $l \in \mathbb{N}$ be constructed as above. Then $r(\mathcal{L}(\Omega^{(l)})) = l \cdot r(\mathcal{L}(\Omega))$.*

Proof. This is immediate from the construction. □

We can now state the following analogue of Proposition 3.12.

Proposition 3.14. *Let (Ω, T) be a minimal DDSF and $(\Omega^{(n)}, T)$, $n \in \mathbb{N}$ constructed as above. Then, there exists a sequence ρ_n with $\rho_n \longrightarrow \infty$, $n \longrightarrow \infty$, such that $V(x_n, \omega_{n,P_n})$ is a van Hove sequence whenever (P_n) is a sequence of pattern classes with $P_n \in \mathcal{P}(\Omega^{(n)})$ and $s(P_n) \geq \rho_n$ for all $n \in \mathbb{N}$.*

Proof. By the foregoing lemma and Lemma 3.6, there exist for every $n \in \mathbb{N}$ a $\rho_n > 0$ such that $r(P_n) \geq nr(\mathcal{L}(\Omega))/2$ whenever $P_n \in \mathcal{P}(\Omega^{(n)})$ with $s(P_n) \geq \rho_n$. Thus,

$$r(P_n) \longrightarrow \infty, \quad n \longrightarrow \infty.$$

Now, the statement follows as in the proof of Proposition 3.12. □

4 The ergodic theorem

In this section, we prove Theorem 1. The main idea of the proof is to combine the geometric decompositions studied in the last section with the almost additivity of F to reduce the study of F on large patterns to the study of F on smaller patterns.

Proof of Theorem 1. (ii) $\Longrightarrow$(i). For every $Q \in \mathcal{P}$, the function $P \mapsto \sharp_Q(P)$ is almost additive on $\mathcal{P}$. Thus, its average $\lim_{n\to\infty} |P_n|^{-1} \sharp_Q(P_n)$ exists along arbitrary van Hove sequences (P_n) in $\mathcal{P}$. But this easily implies (2.3), which in turn implies unique ergodicity, as discussed in Section 2.

(i) $\Longrightarrow$ (ii). Let $F : \mathcal{P}(\Omega) \longrightarrow \mathcal{B}$ be almost additive with error function b. Let (P_n) be a van Hove sequence in $\mathcal{P}(\Omega)$. We have to show that $\lim_{n\to\infty} |P_n|^{-1} F(P_n)$ exists. As $\mathcal{B}$ is a Banach space, it is clearly sufficient to show that $(|P_n|^{-1} F(P_n))$ is a Cauchy sequence. To do so we construct $F^{(k)}$ in $\mathcal{B}$ such that

$$\| |P_n|^{-1} F(P_n) - F^{(k)} \| \text{ is arbitrarily small for } n \text{ large and } k \text{ large.}$$

To introduce $F^{(k)}$ we proceed as follows. Fix $\omega \in \Omega$ with $0 \in \omega$.

We first consider the case that (Ω, T) is aperiodic. The other case can be dealt with similarly. Let $B^{(k)}$ be the ball pattern class occurring in ω around zero with radius k, i.e.,

$$B^{(k)} \equiv [\omega \wedge B(0,k)]. \tag{4.1}$$

Thus, $(B^{(k)})$ is a sequence in $\mathcal{P}_B(\Omega)$ with $k = s(B^{(k)}) \longrightarrow \infty$ for $k \to \infty$, and the assumptions of Proposition 3.12 are satisfied.

As (Ω, T) is of finite local complexity, the set

$$\{[B(x, 2R(B^{(k)})) \wedge \omega] : x \in \omega, \omega \in \Omega \text{ with } [B(x,k) \wedge \omega] = B^{(k)}\}$$

is finite. We can thus enumerate its elements by $B_j^{(k)}, j = 1, \ldots, N(k)$ with suitable $N(k) \in \mathbb{N}$ and $B_j^{(k)} \in \mathcal{P}(\Omega)$. Let $C_j^{(k)} \equiv C(B_j^{(k)})$ be the cells associated to $B_j^{(k)}$ according to Proposition 3.2. By Proposition 3.12,

$$(*) \quad (C_{j_k}^{(l_k)}) \text{ is a van Hove sequence}$$

for arbitrary $(l_k) \subset \mathbb{N}$ with $l_k \to \infty$, $k \to \infty$, and $j_k \in \{1, \ldots, N(l_k)\}$. This is crucial. Denote the frequencies of the $B_j^{(k)}$ by $f(B_j^{(k)})$, i.e.,

$$f(B_j^{(k)}) = \lim_{n\to\infty} |P_n|^{-1} \sharp_{B_j^{(k)}} P_n. \tag{4.2}$$

Define

$$F^{(k)} \equiv \sum_{j=1}^{N(k)} f(B_j^{(k)}) F(C_j^{(k)}).$$

Choose $\epsilon > 0$. We have to show that

$$\| |P_n|^{-1} F(P_n) - F^{(k)}) \| < \epsilon, \quad \text{for } n \text{ and } k \text{ large}$$

(as this implies that $|P_n|^{-1}F(P_n)$ is a Cauchy sequence). By $(*)$, there exists $k(\epsilon) > 0$ with

$$|C_j^{(k)}|^{-1} b(C_j^{(k)}) < \epsilon/3 \quad \text{for every } j = 1, \ldots, N(k) \tag{4.3}$$

whenever $k \geq k(\epsilon)$. (Otherwise, we could find (l_k) in $\mathbb{N}$ and $j_k \in \{1, \ldots, N(l_k)\}$ with $l_k \to \infty, k \to \infty$ such that

$$|C_{j_k}^{(l_k)}|^{-1} b(C_{j_k}^{(l_k)}) \geq \epsilon/3.$$

Since $(C_{j_k}^{(k)})$ is a van Hoove sequence by $(*)$, this contradicts property (A4) from Definition 2.3.)

Let $P \in \mathcal{P}$ be an arbitrary pattern class. By minimality of (Ω, T), we can choose $Q = Q(P) \subset \mathbb{R}^d$ with $[Q \wedge \omega] = P$.

The idea is now to consider the decomposition of $\omega \wedge Q$ induced by the $B^{(k)}$-decomposition of ω. This decomposition of $\omega \wedge Q$ consists (up to a boundary term) of representatives of $C_j^{(k)}$, $j = 1, \ldots, N(k)$. For $P = P_n$ with $n \in \mathbb{N}$ large, the number of representatives of a $C_j^{(k)}$ for j fixed occurring in $Q \wedge \omega$ is essentially given by $f(C_j^{(k)})|P_n|$. Together with the almost additivity of F, this allows us to relate $F(P_n)$ to $F^{(k)}$ in the desired way. Here are the details.

Let $I(P,k) \equiv \{x \in \omega_{B^{(k)}} : B(x, 2R(B^{(k)})) \subset Q\}$. Then, by Lemma 3.1 and Proposition 3.2,

$$Q \wedge \omega = S \wedge \omega \oplus \bigoplus_{x \in I(P,k)} C(x, \omega, B^{(k)}) \tag{4.4}$$

with a suitable surface type set $S \subset \mathbb{R}^d$ with

$$S \subset Q \setminus Q_{4R(B^{(k)})}. \tag{4.5}$$

The triangle inequality implies

$$\begin{aligned} \left\| \frac{F(P)}{|P|} - F^{(k)} \right\| &\leq \left\| \frac{F(P) - F([S \wedge \omega]) - \sum_{x \in I(P,k)} F([C(x, \omega, B^{(k)})])}{|P|} \right\| \\ &\quad + \left\| \frac{F([S \wedge \omega]) + \sum_{x \in I(P,k)} F([C(x, \omega, B^{(k)})])}{|P|} - F^{(k)} \right\| \\ &\equiv D_1(P,k) + D_2(P,k). \end{aligned}$$

The terms $D_1(P,k)$ and $D_2(P,k)$ can be estimated as follows. By almost additivity of F, we have

$$\begin{aligned} D_1(P,k) &\leq \frac{b([S \wedge \omega])}{|P|} + \sum_{x \in I(P,k)} \frac{b([C(x, \omega, B^{(k)})])}{|C(x, \omega, B^{(k)})|} \frac{|C(x, \omega, B^{(k)})|}{|P|} \\ &\leq \frac{b(P) + b([\bigoplus_{x \in I(P,k)} C(x, \omega, B^{(k)})])}{|P|} \\ &\quad + \sup \left\{ \frac{b([C(x, \omega, B^{(k)})])}{|C(x, \omega, B^{(k)})|} : x \in I(P,k) \right\}. \end{aligned}$$

In the last inequality we have used (A3).

Fix $k = k(\epsilon)$ from (4.3) and consider the above estimate for $P = P_n$. Then

$$D_1(P_n, k) \leq \frac{b(P_n) + b([\bigoplus_{x \in I(P,k)} C(x, \omega, B^{(k)})])}{|P_n|} + \frac{\epsilon}{3}.$$

As (P_n) is a van Hove sequence, it is clear from (4.5) that $([\bigoplus_{x\in I(P,k)} C(x,\omega,B^{(k)})])$ is a van Hove sequence as well. Thus

$$\frac{b(P_n) + b([\bigoplus_{x\in I(P_n,k)} C(x,\omega,B^{(k)})]}{|P_n|} =$$

$$\frac{b(P_n)}{|P_n|} + \frac{b([\bigoplus_{x\in I(P_n,k)} C(x,\omega,B^{(k)})])}{|[\bigoplus_{x\in I(P_n,k)} C(x,\omega,B^{(k)})]|} \frac{|[\bigoplus_{x\in I(P_n,k)} C(x,\omega,B^{(k)})]|}{|P_n|}$$

tends to zero for n tending to infinity by the definition of b. Putting this together, we infer

$$D_1(P_n,k) < \epsilon/2$$

for large enough $n \in \mathbb{N}$.

Consider now D_2. Invoking the definition of $F^{(k)}$, we clearly have

$$D_2(P,k) \le \frac{\|F([S\wedge\omega])\|}{|P|}$$

$$+ \sum_{j=1}^{N(k)} \left| \frac{\sharp\{x \in I(P,k) : [B(x,2R(B^k))\wedge\omega] = B_j^{(k)}\}}{|P|} - f(B_j^{(k)}) \right| \|F(C_j^{(k)})\|.$$

Choose k as above and consider $P = P_n$. By (4.5) and the almost additivity of F (property (A2)), we infer that the first term tends to zero for n tending to infinity. Again by (4.5) and the definition of the frequency, we infer that the second term tends to zero as well. Thus,

$$D_2(P_n,k) < \epsilon/2$$

for n large. Putting these estimates together, we have

$$\||P_n|^{-1}F(P_n) - F^{(k)}\| \le D_1(n,k) + D_2(n,k) < \epsilon$$

for large n, and the proof is finished for aperiodic DDSF.

For arbitrary strictly ergodic DDSF, we replace the definition of $B^{(k)}$ in (4.1) by

$$B^{(k)} \equiv [B(0,\rho_k)\wedge\omega^{(k)}],$$

where $\omega^{(k)} \in \Omega^{(k)}$ is defined via colouring; see the paragraphs preceding Lemma 3.13 in Section 3 and ρ_k is given by Proposition 3.14. Then $B^{(k)}$ belongs to $\mathcal{P}_B(\Omega^{(k)})$ for every $k \in \mathbb{N}$ and

$$s^k \equiv s(B^{(k)}) = \rho_k, \quad k \in \mathbb{N}.$$

Thus, Proposition 3.14 applies. The proof then proceeds along the same lines as above, with Ω replaced by $\Omega^{(k)}$ and Proposition 3.12 replaced by Proposition 3.14 at the corresponding places. □

Remark 3. Using what could be called the *k-cells*, $C_j^{(k)}$, $k \in \mathbb{N}$, $j = 1, \dots, N(k)$, from the preceding proof we have actually proved that

$$\lim_{k\to\infty} \sum_{j=1}^{N(k)} f(B_j^{(k)})F(C_j^{(k)}) = \lim_{n\to\infty} \frac{F(P_n)}{|P_n|}.$$

Proof of Corollary 2.4. We use the notation of the corollary. Apparently, the reasoning yielding (i) $\Longrightarrow$ (ii) in the foregoing proof remains valid for arbitrary seminorms $\|\cdot\|$. Thus, if F is almost additive with respect to seminorms $\|\cdot\|_\iota$, $\iota \in \mathcal{I}$, then $(|P_n|^{-1}F(P_n))$ is a Cauchy sequence with respect to $\|\cdot\|_\iota$ for every $\iota \in \mathcal{I}$. The corollary now follows from completeness. □

Proof of Corollary 2.6. This can be shown by mimicking the arguments in the above proof. Alternatively, one can define the function $\widetilde{F} : \mathcal{P} \longrightarrow \mathcal{B}$ by setting $\widetilde{F}(P) := F(Q, \omega)$, where (Q, ω) is arbitrary with $P = [\omega \wedge Q]$. This definition may seem very arbitrary. However, by (A0), it is not hard to see that $\widetilde{F}(P)$ is (up to a boundary term) independent of the actual choice of Q and ω. By the same kind of reasoning, one infers that $\widetilde{F}$ is almost additive. Now existence of the limits $|P_n|^{-1}\widetilde{F}(P_n)$ follows for arbitrary van Hove sequences (P_n). Invoking (A0) once more yields the corollary. □

5 Uniform convergence of the integrated density of states

This section is devoted to a proof of Theorems 2 and 3. We need some preparation.

Lemma 5.1. *Let B and C be selfadjoint operators in a finite-dimensional Hilbert space. Then $|n(B)(E) - n(B + C)(E)| \leq \operatorname{rank}(C)$ for every $E \in \mathbb{R}$, where $n(D)$ denotes the eigenvalue counting function of D, i.e., $n(D)(E) \equiv \sharp\{$eigenvalues of D not exceeding $E\}$.*

Proof. This is a consequence of the minmax principle; see Theorem 4.3.6 in [17] for details. □

Proposition 5.2. *Let U be a subspace of the finite-dimensional Hilbert space X with inclusion $j : U \longrightarrow X$ and orthogonal projection $p : X \longrightarrow U$. Then,*

$$|n(A)(E) - n(pAj)(E)| \leq 4 \cdot \operatorname{rank}(1 - j \circ p)$$

for every selfadjoint operator A on X.

Proof. Let $P : X \longrightarrow X$ be the orthogonal projection onto U, i.e., $P = j \circ p$. Set $P^\perp \equiv 1 - P$ and denote the range of $P^\perp$ by $U^\perp$. By

$$A - PAP = P^\perp AP + PAP^\perp + P^\perp AP^\perp$$

and the foregoing lemma, we have $|n(A)(E) - n(PAP)(E)| \leq 3\,\mathrm{rank}(P^\perp)$. As obviously

$$PAP = pAj \oplus 0_{U^\perp},$$

with the zero operator $0_{U^\perp} : U^\perp \longrightarrow U^\perp$, $f \mapsto 0$, we also have

$$|n(PAP)(E) - n(pAj)(E)| \leq \dim(U^\perp).$$

As $\dim U^\perp = \mathrm{rank}(P^\perp)$, we are done. □

Lemma 5.3. *Let (Ω, T) be an (r, R)-system, $\omega \in \Omega$ and Q a bounded subset of $\mathbb{R}^d$. Then*

$$\sharp Q \cap \omega \leq \frac{1}{|B(0,r)|} |Q^r|.$$

Proof. As (Ω, T) is an (r, R)-system, balls with radius r around different points in ω are disjoint. The lemma follows. □

Our main tool is the following consequence of the previous two results.

Proposition 5.4. *Let (Ω, T) be an (r, R)-system. Let $Q, Q_j \subset \mathbb{R}^d$, $j = 1, \ldots, n$ be given with $Q = \bigcup_{j=1}^n Q_j$ and the Q_j pairwise disjoint up to their boundaries. Set $\delta(\omega, s) \equiv |\dim \ell^2(Q_s \cap \omega) - \dim \ell^2(\bigcup_{j=1}^n (Q_{j,s} \cap \omega))|$ for $\omega \in \Omega$ and $s > 0$ arbitrary. Then*

$$\begin{aligned}\delta(\omega, s) &\leq |\dim \ell^2(Q \cap \omega) - \dim \ell^2(\bigcup_{j=1}^n (Q_{j,s} \cap \omega))| \\ &\leq \frac{1}{|B(0,r)|} \sum_{j=1}^n |Q_j^r \setminus Q_{j,s+r}|.\end{aligned}$$

Proof. Obviously,

$$\dim \ell^2(\bigcup_{j=1}^n (Q_{j,s} \cap \omega)) \leq \dim \ell^2(Q_s \cap \omega) \leq \dim \ell^2(Q \cap \omega).$$

Now the first inequality is clear, and the second follows by

$$\begin{aligned}\dim \ell^2(Q \cap \omega) - \dim \ell^2(\bigcup_{j=1}^n Q_{j,s} \cap \omega) &\leq \sum_{j=1}^n \sharp((Q_j \setminus Q_{j,s}) \cap \omega) \\ &\leq \frac{1}{|B(0,r)|} \sum_{j=1}^n |Q_j^r \setminus Q_{j,s+r}|.\end{aligned}$$

Here the last inequality follows by the foregoing lemma. □

We are now able to prove Theorem 2.

Proof of Theorem 2. We have to exhibit $b : \mathcal{P}(\Omega) \longrightarrow (0,\infty)$, and $D > 0$ such that (A1) , (A2) and (A3) of Definition 2.3 are satisfied. Set

$$D \equiv \frac{2}{|B(0,r)|}$$

and define b by

$$b(P) \equiv \frac{8}{|B(0,r)|}|Q^r \setminus Q_{R^A+r}|$$

whenever $P \in \mathcal{P}(\Omega)$ with $P = [(Q,\Lambda)]$. Clearly, b is well-defined. Moreover, (A4) follows by the very definition of b and the van Hove property.

Now (A2) is satisfied, as

$$\|F^A(P)\| = \|n(A_\omega, Q_{R^A})\|_\infty \leq \sharp(Q_{R^A} \cap \omega) \leq \frac{1}{B(0,r)}|Q^r| \leq D|P| + b(P),$$

for $P = [Q \wedge \omega]$. (A3) can be proved by a similar argument. It remains to show (A1). Let $P = \bigoplus_{j=1}^n P_j$. Then, there exists $\omega \in \Omega$ and bounded measurable sets Q, Q_j, $j = 1,\ldots,n$ in $\mathbb{R}^d$ with Q_j pairwise disjoint up to their boundaries and $Q = \bigcup_{j=1}^n Q_j$ such that

$$P = [Q \wedge \omega] \quad \text{and} \quad P_j = [Q_j \wedge \omega], \quad j = 1,\ldots,n.$$

As A is an operator of finite range, it follows from the definition of R^A that

$$A_\omega|_{\bigcup_{j=1}^n Q_{j,R^A}} = \bigoplus_{j=1}^n A_\omega|_{Q_{j,R^A}};$$

in particular,

$$\sum_{j=1}^n n(A_\omega, Q_{j,R^A}) = n(A_\omega, \bigcup_{j=1}^n Q_{j,R^A}). \tag{5.1}$$

Thus, we can calculate

$$
\begin{aligned}
\|F^A(P) - \sum_{j=1}^{n} F(P_j)\| &= \|n(A_\omega, Q_{R^A}) - \sum_{j=1}^{n} n(A_\omega, Q_{j,R^A})\|_\infty \\
&= \|n(A_\omega, Q_{R^A}) - n(A_\omega, \bigcup_{j=1}^{n} Q_{j,R^A})\|_\infty \\
\text{(Prop 5.2)} \quad &\le 4(\dim \ell^2(Q_{R^A}) - \dim \ell^2(\bigcup_{j=1}^{n} Q_{j,R^A}) \\
\text{(Prop 5.4)} \quad &\le \frac{4}{|B(0,r)|} \sum_{j=1}^{n} |Q_j^r \setminus Q_{j,R^A+r}| \\
&\le \sum_{j=1}^{n} b(P_j).
\end{aligned}
$$

This finishes the proof. □

Theorem 3 is an immediate consequence of Theorems 1 and 2, once we have proved the following lemma.

Lemma 5.5. *Let (Ω, T) be a strictly ergodic (r, R)-system. Let A be a finite range operator with range R^A. Then*

$$\|n(A_\omega, Q) - F^A([\omega \wedge Q])\|_\infty \le 4|B(0,r)|^{-1}|Q^r \setminus Q_{R^A+r}|$$

for all $\omega \in \Omega$ and all bounded subsets Q in $\mathbb{R}^d$.

Proof. By definition of F^A, we have

$$\|n(A_\omega, Q) - F^A([\omega \wedge Q])\|_\infty = \|n(A_\omega, Q) - n(A_\omega, Q_{R^A})\|_\infty.$$

Invoking Proposition 5.2, we see that the difference is bounded by $4\sharp(Q \setminus Q_{R^A}) \wedge \omega$. The result now follows by Lemma 5.3. □

Proof of Theorem 3. Let (Q_n) be a van Hove sequence. Then $([Q_n \wedge \omega])$ is a van Hove sequence in $\mathcal{P}(\Omega)$ independent of ω. Thus, $|Q_n|^{-1}F^A([Q_n \wedge \omega])$ converges uniformly in $\omega \in \Omega$ by Theorems 1 and 2. The theorem now follows from the foregoing lemma. □

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