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Algebras of Random Operators Associated to Delone Dynamical Systems ⋆

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Abstract. We carry out a careful study of operator algebras associated with Delone dynamical systems. A von Neumann algebra is defined using noncommutative integration theory. Features of these algebras and the operators they contain are discussed. We restrict our attention to a certain C^*-subalgebra to discuss a Shubin trace formula.

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Introduction

This paper is part of a study of Hamiltonians for aperiodic solids. Among them, special emphasis is laid on models for quasicrystals. To describe aperiodic order, we use Delone (Delaunay) sets. Here we construct and study certain operator algebras which can be naturally associated with Delone sets and reflect the aperiodic order present in a Delone dynamical system. In particular, we use Connes noncommutative integration theory to build a von Neumann algebra. This is achieved in Section 2 after some preparatory definitions and results gathered in Section 1. Let us stress the following facts: it is not too hard to write down explicitly the von Neumann algebra $\mathcal{N}(\Omega, T, \mu)$ of observables, starting from a Delone dynamical system (Ω, T) with an invariant measure μ. As in the case of random operators, the observables are families of operators, indexed by a set Ω of Delone sets. This set represents a type of (aperiodic) order and the ergodic properties of (Ω, T) can often be expressed by combinatorial properties of its elements ω. The latter are thought of as realizations of the type of disorder described by (Ω, T). The algebra $\mathcal{N}(\Omega, T, \mu)$ incorporates this disorder and plays the role of a noncommutative space underlying the algebra of observables. To see that this algebra is in fact a von Neumann algebra is by no means clear. At that point the analysis of Connes [9] enters the picture.

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In order to verify the necessary regularity properties we rely on work done in [29], where we studied topological properties of a groupoid that naturally comes with (Ω, T). Using this, we can construct a measurable (even topological) groupoid. Any invariant measure μ on the dynamical system gives rise to a transversal measure Λ and the points of the Delone sets are used to define a random Hilbert space $\mathcal{H}$. This latter step specifically uses the fact that we are dealing with a dynamical system consisting of point sets and leads to a noncommutative random variable that has no analogue in the general framework of dynamical systems. We are then able to identify $\mathcal{N}(\Omega, T, \mu)$ as $\mathrm{End}_\Lambda(\mathcal{H})$. While in our approach we use noncommutative integration theory to verify that a certain algebra is a von Neumann algebra, we should also like to point out that at the same time we provide interesting examples for the theory. Of course, tilings have been considered in this connection quite from the start as seen on the cover of [10]. However, we emphasize the point of view of concrete operators and thus are led to a somewhat different setup.

The study of traces on this algebra is started in Section 3. Traces are intimately linked to transversal functions on the groupoid. These can also be used to study certain spectral properties of the operator families constituting the von Neumann algebra. For instance, spectral properties are almost surely constant for the members of any such family. This type of results is typical for random operators. In fact, we regard the families studied here in this random context. An additional feature that is met here is the dependence of the Hilbert space on the random parameter $\omega \in \Omega$.

In Section 4 we introduce a C^*-algebra that had already been encountered in a different form in [6, 17]. Our presentation here is geared towards using the elements of the C^*-algebra as tight binding Hamiltonians in a quantum mechanical description of disordered solids (see [6] for related material as well). We relate certain spectral properties of the members of such operator families to ergodic features of the underlying dynamical system. Moreover, we show that the eigenvalue counting functions of these operators are convergent. The limit, known as the integrated density of states, is an object of fundamental importance from the solid state physics point of view. Apart from proving its existence, we also relate it to the canonical trace on the von Neumann algebra $\mathcal{N}(\Omega, T, \mu)$ in case that the Delone dynamical system (Ω, T) is uniquely ergodic. Results of this genre are known as Shubin's trace formula due to the celebrated results from [36].

We conclude this section with two further remarks.

Firstly, let us mention that starting with the work of Kellendonk [17], C^*-algebras associated to tilings have been subject to intense research within the framework of K-theory (see, e.g., [18, 19, 32]). This can be seen as part of a program originally initiated by Bellissard and his coworkers in the study of so-called gap-labelling for almost periodic operators [3–5]. While the C^*-algebras we encounter are essentially the same, our motivation, aims and results are quite different.

Secondly, let us remark that some of the results below have been announced in [28, 29]. A stronger ergodic theorem will be found in [30] and a spectral theoretic application is given in [20].

1. Delone Dynamical Systems and Coloured Delone Dynamical Systems

In this section we recall standard concepts from the theory of Delone sets and introduce a suitable topology on the closed sets in Euclidian space. A slight extension concerns the discussion of *coloured* (*decorated*) *Delone sets*.

A subset ω of $\mathbb{R}^d$ is called a *Delone set* if there exist $0 < r, R < \infty$ such that $2r \leqslant \|x - y\|$ whenever $x, y \in \omega$ with $x \neq y$, and $B_R(x) \cap \omega \neq \emptyset$ for all $x \in \mathbb{R}^d$. Here, the Euclidean norm on $\mathbb{R}^d$ is denoted by $\|\cdot\|$ and $B_s(x)$ denotes the (closed) ball in $\mathbb{R}^d$ around x with radius s. The set ω is then also called an (r, R)-set. We will sometimes be interested in the restrictions of Delone sets to bounded sets. In order to treat these restrictions, we introduce the following definition.

DEFINITION 1.1. (a) A pair (Λ, Q) consisting of a bounded subset Q of $\mathbb{R}^d$ and $\Lambda \subset Q$ finite is called a *pattern*. The set Q is called the *support of the pattern*.

(b) A pattern (Λ, Q) is called a *ball pattern* if $Q = B_s(x)$ with $x \in \Lambda$ for suitable $x \in \mathbb{R}^d$ and $s \in (0, \infty)$.

The pattern (Λ_1, Q_1) is contained in the pattern (Λ_2, Q_2) written as $(\Lambda_1, Q_1) \subset (\Lambda_2, Q_2)$ if $Q_1 \subset Q_2$ and $\Lambda_1 = Q_1 \cap \Lambda_2$. Diameter, volume, etc., of a pattern are defined to be the diameter, volume, etc., of its support. For patterns $X_1 = (\Lambda_1, Q_1)$ and $X_2 = (\Lambda_2, Q_2)$, we define $\sharp_{X_1} X_2$, *the number of occurrences of* X_1 *in* X_2, to be the number of elements in $\{t \in \mathbb{R}^d : \Lambda_1 + t \subset \Lambda_2, Q_1 + t \subset Q_2\}$.

For further investigation we will have to identify patterns that are equal up to translation. Thus, on the set of patterns we introduce an equivalence relation by setting $(\Lambda_1, Q_1) \sim (\Lambda_2, Q_2)$ if and only if there exists a $t \in \mathbb{R}^d$ with $\Lambda_1 = \Lambda_2 + t$ and $Q_1 = Q_2 + t$. In this latter case we write $(\Lambda_1, Q_1) = (\Lambda_2, Q_2) + t$. The class of a pattern (Λ, Q) is denoted by $[(\Lambda, Q)]$. The notions of diameter, volume, occurrence, etc., can easily be carried over from patterns to pattern classes.

Every Delone set ω gives rise to a set of pattern classes, $\mathcal{P}(\omega)$ viz $\mathcal{P}(\omega) = \{[Q \wedge \omega] : Q \subset \mathbb{R}^d$ bounded and measurable$\}$, and to a set of ball pattern classes $\mathcal{P}_B(\omega)) = \{[B_s(x) \wedge \omega] : x \in \omega, s > 0\}$. Here we set $Q \wedge \omega = (\omega \cap Q, Q)$.

For $s \in (0, \infty)$, we denote by $\mathcal{P}_B^s(\omega)$ the set of ball patterns with radius s; note the relation with s-patches as considered in [21]. A Delone set is said to be of *finite local complexity* if for every radius s the set $\mathcal{P}_B^s(\omega)$ is finite. We refer the reader to [21] for a detailed discussion of Delone sets of finite type.

Let us now extend this framework a little, allowing for coloured Delone sets. The alphabet $\mathbb{A}$ is the set of possible *colours* or *decorations*. An $\mathbb{A}$-*coloured Delone set* is a subset $\omega \subset \mathbb{R}^d \times \mathbb{A}$ such that the projection $pr_1(\omega) \subset \mathbb{R}^d$ onto the first coordinate is a Delone set. The set of all $\mathbb{A}$-coloured Delone sets is denoted by $\mathcal{D}_{\mathbb{A}}$.

Of course, we speak of an (r, R)-*set* if $pr_1(\omega)$ is an (r, R)-set. The notions of *pattern, diameter, volume* of pattern, etc., easily extend to coloured Delone sets, e.g.

DEFINITION 1.2. A pair (Λ, Q) consisting of a bounded subset Q of $\mathbb{R}^d$ and $\Lambda \subset Q \times \mathbb{A}$ finite is called an $\mathbb{A}$-*decorated pattern*. The set Q is called the *support of the pattern*.

A coloured Delone set ω is thus viewed as a Delone set $pr_1(\omega)$ whose points $x \in pr_1(\omega)$ are labelled by colours $a \in \mathbb{A}$. Accordingly, the translate $T_t\omega$ of a coloured Delone set $\omega \subset \mathbb{R}^d \times \mathbb{A}$ is given by

$$T_t\omega = \{(x + t, a) : (x, a) \in \omega\}.$$

From [29] we infer the notion of the *natural topology*, defined on the set $\mathcal{F}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$. Since in our subsequent study in [30] the alphabet is supposed to be a finite set, the following construction will provide a suitable topology for coloured Delone sets. Define, for $a \in \mathbb{A}$,

$$p_a\colon \mathcal{D}_{\mathbb{A}} \to \mathcal{F}(\mathbb{R}^d),\ p_a(\omega) = \{x \in \mathbb{R}^d : (x, a) \in \omega\}.$$

The initial topolgy on $\mathcal{D}_{\mathbb{A}}$ with respect to the family $(p_a)_{a\in\mathbb{A}}$ is called the *natural topology* on the set of $\mathbb{A}$-decorated Delone sets. It is obvious that metrizability and compactness properties carry over from the natural topology without decorations to the decorated case.

Finally, the notions of Delone dynamical system and Delone dynamical system of finite local complexity carry over to the coloured case in the obvious manner.

DEFINITION 1.3. Let $\mathbb{A}$ be a finite set. (a) Let Ω be a set of Delone sets. The pair (Ω, T) is called a *Delone dynamical system* (DDS) if Ω is invariant under the shift T and closed in the natural topology.

(a′) Let Ω be a set of $\mathbb{A}$-coloured Delone sets. The pair (Ω, T) is called an $\mathbb{A}$-*coloured Delone dynamical system* ($\mathbb{A}$-DDS) if Ω is invariant under the shift T and closed in the natural topology.

(b) A DDS (Ω, T) is said to be of *finite local complexity* if $\bigcup_{\omega\in\Omega} P_B^s(\omega)$ is finite for every $s > 0$.

(b′) An $\mathbb{A}$-DDS (Ω, T) is said to be of *finite local complexity* if $\bigcup_{\omega\in\Omega} P_B^s(\omega)$ is finite for every $s > 0$.

(c) Let $0 < r, R < \infty$ be given. A DDS (Ω, T) is said to be an (r, R)-*system* if every $\omega \in \Omega$ is an (r, R)-set.

(c′) Let $0 < r, R < \infty$ be given. An $\mathbb{A}$-DDS (Ω, T) is said to be an (r, R)-*system* if every $\omega \in \Omega$ is an (r, R)-set.

(d) The set $\mathcal{P}(\Omega)$ of *pattern classes associated to a DDS* Ω is defined by $\mathcal{P}(\Omega) = \bigcup_{\omega\in\Omega} \mathcal{P}(\omega)$.

In view of the compactness properties known for Delone sets, [29], we get that Ω is compact whenever (Ω, T) is a DDS or an $\mathbb{A}$-DDS.

2. Groupoids and Noncommutative Random Variables

In this section we use concepts from Connes noncommutative integration theory [9] to associate a natural von Neumann algebra with a given DDS (Ω, T). To do so, we introduce

- a suitable *groupoid* $\mathcal{G}(\Omega, T)$,
- a *transversal measure* $\Lambda = \Lambda_\mu$ for a given invariant measure μ on (Ω, T),
- and a Λ-*random Hilbert space* $\mathcal{H} = (\mathcal{H}_\omega)_{\omega\in\Omega}$,

leading to the von Neumann algebra

$$\mathcal{N}(\Omega, T, \mu) := \mathrm{End}_\Lambda(\mathcal{H})$$

of *random operators*, all in the terminology of [9]. Of course, all these objects will now be properly defined and some crucial properties have to be checked. Part of the topological prerequisites have already been worked out in [29]. Note that comparing the latter with the present paper, we put more emphasis on the relation with noncommutative integration theory.

The definition of the groupoid structure is straightforward see also [6], Sect. 2.5. A set $\mathcal{G}$ together with a partially defined associative multiplication $\cdot$: $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, and an inversion $^{-1}$: $\mathcal{G} \to \mathcal{G}$ is called a groupoid if the following holds:

- $(g^{-1})^{-1} = g$ for all $g \in \mathcal{G}$,
- If $g_1 \cdot g_2$ and $g_2 \cdot g_3$ exist, then $g_1 \cdot g_2 \cdot g_3$ exists as well,
- $g^{-1} \cdot g$ exists always and $g^{-1} \cdot g \cdot h = h$, whenever $g \cdot h$ exists,
- $h \cdot h^{-1}$ exists always and $g \cdot h \cdot h^{-1} = g$, whenever $g \cdot h$ exists.

A groupoid is called a topological groupoid if it carries a topology making inversion and multiplication continuous. Here, of course, $\mathcal{G} \times \mathcal{G}$ carries the product topology and $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$ is equipped with the induced topology.

A given groupoid $\mathcal{G}$ gives rise to some standard objects: The subset $\mathcal{G}^0 = \{g \cdot g^{-1} \mid g \in \mathcal{G}\}$ is called the set of *units*. For $g \in \mathcal{G}$, we define its *range* $r(g)$ by $r(g) = g \cdot g^{-1}$ and its *source* by $s(g) = g^{-1} \cdot g$. Moreover, we set $\mathcal{G}^\omega = r^{-1}(\{\omega\})$ for any unit $\omega \in \mathcal{G}^0$. One easily checks that $g \cdot h$ exists if and only if $r(h) = s(g)$.

By a standard construction we can assign a groupoid $\mathcal{G}(\Omega, T)$ to a Delone dynamical system. As a set $\mathcal{G}(\Omega, T)$ is just $\Omega \times \mathbb{R}^d$. The multiplication is given by $(\omega, x)(\omega - x, y) = (\omega, x + y)$ and the inversion is given by $(\omega, x)^{-1} = (\omega - x, -x)$. The groupoid operations can be visualized by considering an element (ω, x) as an arrow $\omega - x \xrightarrow{x} \omega$. Multiplication then corresponds to concatenation of arrows; inversion corresponds to reversing arrows and the set of units $\mathcal{G}(\Omega, T)^0$ can be identified with Ω.

Apparently this groupoid $\mathcal{G}(\Omega, T)$ is a topological groupoid when Ω is equipped with the topology of the previous section and $\mathbb{R}^d$ carries the usual topology.

The groupoid $\mathcal{G}(\Omega, T)$ acts naturally on a certain topological space $\mathcal{X}$. This space and the action of $\mathcal{G}$ on it are of crucial importance in the sequel. The space $\mathcal{X}$ is given by

$$\mathcal{X} = \{(\omega, x) \in \mathcal{G} : x \in \omega\} \subset \mathcal{G}(\Omega, T).$$

In particular, it inherits a topology form $\mathcal{G}(\Omega, T)$. This $\mathcal{X}$ can be used to define a *random variable* or *measurable functor* in the sense of [9]. Following the latter reference, p. 50f, this means that we are given a functor F from $\mathcal{G}$ to the category of measurable spaces with the following properties:

- For every $\omega \in \mathcal{G}^0$ we are given a measure space $F(\omega) = (\mathcal{Y}^\omega, \beta^\omega)$.
- For every $g \in \mathcal{G}$ we have an isomorphism $F(g)$ of measure spaces, $F(g)\colon \mathcal{Y}^{s(g)} \to \mathcal{Y}^{r(g)}$ such that $F(g_1 g_2) = F(g_1)F(g_2)$, whenever $g_1 g_2$ is defined, i.e., whenever $s(g_1) = r(g_2)$.
- A measurable structure on the disjoint union
$$\mathcal{Y} = \bigcup_{\omega\in\Omega} \mathcal{Y}^\omega$$
such that the projection $\pi\colon \mathcal{Y} \to \Omega$ is measurable as well as the natural bijection of $\pi^{-1}(\omega)$ to $\mathcal{Y}^\omega$.
- The mapping $\omega \mapsto \beta^\omega$ is measurable.

We will use the notation $F\colon \mathcal{G} \rightsquigarrow \mathcal{Y}$ to abbreviate the above.

Let us now turn to the groupoid $\mathcal{G}(\Omega, T)$ and the bundle $\mathcal{X}$ defined above. Since $\mathcal{X}$ is closed ([29], Prop. 2.1), it carries a reasonable Borel structure. The projection $\pi\colon \mathcal{X} \to \Omega$ is continuous, in particular measurable. Now, we can discuss the action of $\mathcal{G}$ on $\mathcal{X}$. Every $g = (\omega, x)$ gives rise to a map $J(g)\colon \mathcal{X}^{s(g)} \to \mathcal{X}^{r(g)}$, $J(g)(\omega - x, p) = (\omega, p + x)$. A simple calculation shows that $J(g_1 g_2) = J(g_1)J(g_2)$ and $J(g^{-1}) = J(g)^{-1}$, whenever $s(g_1) = r(g_2)$. Thus, $\mathcal{X}$ is an $\mathcal{G}$-space in the sense of [27]. It can be used as the target space of a measurable functor $F\colon \mathcal{G} \rightsquigarrow \mathcal{X}$. What we still need is a *positive random variable* in the sense of the following definition, taken from [29]. First some notation:

Given a locally compact space Z, we denote the set of continuous functions on Z with compact support by $C_c(Z)$. The support of a function in $C_c(Z)$ is denoted by $\operatorname{supp}(f)$. The topology gives rise to the Borel-σ-algebra. The measurable non-negative functions with respect to this σ-algebra will be denoted by $\mathcal{F}^+(Z)$. The measures on Z will be denoted by $\mathcal{M}(Z)$.

DEFINITION 2.1. Let (Ω, T) be an (r, R)-system.

(a) A choice of measures $\beta\colon \Omega \to \mathcal{M}(\mathcal{X})$ is called *a positive random variable with values in* $\mathcal{X}$ if the map $\omega \mapsto \beta^\omega(f)$ is measurable for every $f \in \mathcal{F}^+(\mathcal{X})$, β^ω

is supported on $\mathcal{X}^\omega$, i.e., $\beta^\omega(\mathcal{X} - \mathcal{X}^\omega) = 0$, $\omega \in \Omega$, and β satisfies the following invariance condition

$$\int_{\mathcal{X}^{s(g)}} f(J(g)p)\, \mathrm{d}\beta^{s(g)}(p) = \int_{\mathcal{X}^{r(g)}} f(q)\, \mathrm{d}\beta^{r(g)}(q)$$

for all $g \in \mathcal{G}$ and $f \in \mathcal{F}^+(\mathcal{X}^{r(g)})$.

(b) A map $\Omega \times C_c(\mathcal{X}) \to \mathbb{C}$ is called a *complex random variable* if there exist an $n \in \mathbb{N}$, positive random variables β_i, $i = 1, \ldots, n$ and $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$ with $\beta^\omega(f) = \sum_{i=1}^n \beta_i^\omega(f)$.

We are now heading towards introducing and studying a special random variable. This variable is quite important as it gives rise to the ℓ^2-spaces on which the Hamiltonians act. Later we will see that these Hamiltonians also induce random variables.

PROPOSITION 2.2. *Let (Ω, T) be an (r, R)-system. Then the map α: $\Omega \to \mathcal{M}(\mathcal{X})$, $\alpha^\omega(f) = \sum_{p\in\omega} f(p)$ is a random variable with values in $\mathcal{X}$. Thus the functor F_α given by $F_\alpha(\omega) = (\mathcal{X}^\omega, \alpha^\omega)$ and $F_\alpha(g) = J(g)$ is measurable.*

Proof. See [29], Corollary 2.6. □

Clearly, the condition that (Ω, T) is an (r, R)-system is used to verify the measurability conditions needed for a random variable. We should like to stress the fact that the above functor given by $\mathcal{X}$ and $\alpha^\bullet$ differs from the canonical choice, possible for any dynamical system. In the special case at hand this canonical choice reads as follows:

PROPOSITION 2.3. *Let (Ω, T) be a DDS. Then the map ν: $\Omega \to \mathcal{M}(\mathcal{G})$, $\nu^\omega(f) = \int_{\mathbb{R}^d} f(\omega, t)\, \mathrm{d}t$ is a* transversal function*, i.e., a random variable with values in $\mathcal{G}$.*

Actually, one should possibly define transversal functions before introducing random variables. Our choice to do otherwise is to underline the specific functor used in our discussion of Delone sets. As already mentioned above, the analogue of the transversal function ν from Proposition 2.3 can be defined for any dynamical system. In fact this structure has been considered by Bellissard and coworkers in a C^*-context. The notion *almost random operators* has been coined for that; see [3] and the literature quoted there.

After having encountered functors from $\mathcal{G}$ to the category of measurable spaces under the header random variable or measurable functor, we will now meet *random Hilbert spaces*. By that one designates, according to [9], a *representation* of $\mathcal{G}$ in the category of Hilbert spaces, given by the following data:

- A measurable family $\mathcal{H} = (\mathcal{H}_\omega)_{\omega\in\mathcal{G}^0}$ of Hilbert spaces.
- For every $g \in \mathcal{G}$ a unitary U_g: $\mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ such that

$$U(g_1g_2) = U(g_1)U(g_2)$$

whenever $s(g_1) = r(g_2)$. Moreover, we assume that for every pair (ξ, η) of measurable sections of $\mathcal{H}$ the function

$$\mathcal{G} \to \mathbb{C},\, g \mapsto (\xi|\eta)(g) := (\xi_{r(g)}|U(g)\eta_{s(g)})$$

is measurable.

Given a measurable functor $F\colon \mathcal{G} \rightsquigarrow \mathcal{Y}$ there is a natural representation $L^2 \circ F$, where $\mathcal{H}_\omega = L^2(\mathcal{Y}^\omega, \beta^\omega)$ and $U(g)$ is induced by the isomorphism $F(g)$ of measure spaces.

Let us assume that (Ω, T) is an (r, R)-system. We are especially interested in the representation of $\mathcal{G}(\Omega, T)$ on $\mathcal{H} = (\ell^2(\mathcal{X}^\omega, \alpha^\omega))_{\omega\in\Omega}$ induced by the measurable functor $F_\alpha\colon \mathcal{G}(\Omega, T) \rightsquigarrow \mathcal{X}$ defined above. The necessary measurable structure is provided by [29], Proposition 2.8. It is the measurable structure generated by $C_c(\mathcal{X})$.

The last item we have to define is a *transversal measure*. We denote the set of nonnegative transversal functions on a groupoid $\mathcal{G}$ by $\mathcal{E}^+(\mathcal{G})$ and consider the unimodular case ($\delta \equiv 1$) only. Following [9], p. 41f, a *transversal measure* Λ is a linear mapping $\Lambda\colon \mathcal{E}^+(\mathcal{G}) \to [0, \infty]$ satisfying

- Λ is *normal*, i.e., $\Lambda(\sup \nu_n) = \sup \Lambda(\nu_n)$ for every increasing sequence (ν_n) in $\mathcal{E}^+(\mathcal{G})$.
- Λ is *invariant*, i.e., for every $\nu \in \mathcal{E}^+(\mathcal{G})$ and every kernel λ with $\lambda^\omega(1) = 1$ we get $\Lambda(\nu * \lambda) = \Lambda(\nu)$.

Given a fixed transversal function ν on $\mathcal{G}$ and an invariant measure μ on $\mathcal{G}^0$ there is a unique transversal measure $\Lambda = \Lambda_\nu$ such that $\Lambda(\nu * \lambda) = \mu(\lambda^\bullet(1))$, see [9], Theorem 3, p. 43. In the next section we will discuss that in a little more detail in the case of DDS groupoids.

We can now put these constructions together.

DEFINITION 2.4. Let (Ω, T) be an (r, R)-system and let μ be an invariant measure on Ω. Denote by $\mathcal{V}_1$ the set of all $f\colon \mathcal{X} \to \mathbb{C}$ which are measurable and satisfy $f(\omega, \cdot) \in \ell^2(\mathcal{X}^\omega, \alpha^\omega)$ for every $\omega \in \Omega$.

A family $(A_\omega)_{\omega\in\Omega}$ of bounded operators $A_\omega\colon \ell^2(\omega, \alpha^\omega) \to \ell^2(\omega, \alpha^\omega)$ is called measurable if $\omega \mapsto \langle f(\omega), (A_\omega g)(\omega)\rangle_\omega$ is *measurable* for all $f, g \in \mathcal{V}_1$. It is called *bounded* if the norms of the A_ω are uniformly bounded. It is called *covariant* if it satisfies the covariance condition

$$H_{\omega+t} = U_t H_\omega U_t^*, \quad \omega \in \Omega, t \in \mathbb{R}^d,$$

where $U_t\colon \ell^2(\omega) \to \ell^2(\omega + t)$ is the unitary operator induced by translation. Now, we can define

$$\mathcal{N}(\Omega, T, \mu) := \{A = (A_\omega)_{\omega\in\Omega} \mid A \text{ covariant, measurable and bounded}\}/\sim,$$

where $\sim$ means that we identify families which agree μ almost everywhere.

As is clear from the definition, the elements of $\mathcal{N}(\Omega, T, \mu)$ are classes of families of operators. However, we will not distinguish too pedantically between classes and their representatives in the sequel.

Remark 2.5. It is possible to define $\mathcal{N}(\Omega, T, \mu)$ by requiring seemingly weaker conditions. Namely, one can consider families (A_ω) that are essentially bounded and satisfy the covariance condition almost everywhere. However, by standard procedures (see [9, 25]), it is possible to show that each of these families agrees almost everywhere with a family satisfying the stronger conditions discussed above.

Obviously, $\mathcal{N}(\Omega, T, \mu)$ depends on the measure class of μ only. Hence, for uniquely ergodic (Ω, T), $\mathcal{N}(\Omega, T, \mu) =: \mathcal{N}(\Omega, T)$ gives a canonical algebra. This case has been considered in [28, 29].

Apparently, $\mathcal{N}(\Omega, T, \mu)$ is an involutive algebra under the obvious operations. Moreover, it can be related to the algebra $\mathrm{End}_\Lambda(\mathcal{H})$ defined in [9] as follows.

THEOREM 2.6. *Let (Ω, T) be an (r, R)-system and let μ be an invariant measure on Ω. Then $\mathcal{N}(\Omega, T, \mu)$ is a weak-$*$-algebra. More precisely,*

$$\mathcal{N}(\Omega, T, \mu) = \mathrm{End}_\Lambda(\mathcal{H}),$$

where $\Lambda = \Lambda_\nu$ and $\mathcal{H} = (\ell^2(\mathcal{X}^\omega, \alpha^\omega))_{\omega\in\Omega}$ are defined as above.

Proof. The asserted equation follows by plugging in the respective definitions. The only thing that remains to be checked is that $\mathcal{H}$ is a square integrable representation in the sense of [9], Definition, p. 80. In order to see this it suffices to show that the functor F_α giving rise to $\mathcal{H}$ is proper. See [9], Proposition 12, p. 81.

This in turn follows by considering the transversal function ν defined in Proposition 2.3 above. In fact, any $u \in C_c(\mathbb{R}^d)^+$ gives rise to the function $f \in \mathcal{F}^+(\mathcal{X})$ by $f(\omega, p) := u(p)$. It follows that

$$(\nu * f)(\omega, p) = \int_{\mathbb{R}^d} u(p+t)\,\mathrm{d}t = \int_{\mathbb{R}^d} u(t)\,\mathrm{d}t,$$

so that $\nu * f \equiv 1$ if the latter integral equals 1 as required by [9], Definition 3, p. 55. $\square$

We can use the measurable structure to identify $L^2(\mathcal{X}, m)$, where $m = \int_\Omega \alpha^\omega \mu(\omega)$ with $\int_\Omega^\oplus \ell^2(\mathcal{X}^\omega, \alpha^\omega)\,\mathrm{d}\mu(\omega)$. This gives the faithful representation

$$\pi\colon \mathcal{N}(\Omega, T, \mu) \to B(L^2(\mathcal{X}, m)), \pi(A) f((\omega, x)) = (A_\omega f_\omega)((\omega, x))$$

and the following immediate consequence.

COROLLARY 2.7. *$\pi(\mathcal{N}(\Omega, T, \mu)) \subset B(L^2(\mathcal{X}, m))$ is a von Neumann algebra.*

Next we want to identify conditions under which $\pi(\mathcal{N}(\Omega, T, \mu))$ is a factor. Recall that a Delone set ω is said to be *nonperiodic* if $\omega + t = \omega$ implies that $t = 0$.

THEOREM 2.8. *Let (Ω, T) be an (r, R)-system and let μ be an ergodic invariant measure on Ω. If ω is nonperiodic for μ-a.e. $\omega \in \Omega$ then $\mathcal{N}(\Omega, T, \mu)$ is a factor.*

Proof. We want to use [9], Corollaire 7, p. 90. In our case $\mathcal{G} = \mathcal{G}(\Omega, T)$, $\mathcal{G}^0 = \Omega$ and

$$\mathcal{G}_\omega^\omega = \{(\omega, t) : \omega + t = \omega\}.$$

Obviously, the latter is trivial, i.e., equals $\{(\omega, 0)\}$ iff ω is nonperiodic. By our assumption this is valid μ-a.s. so that we can apply [9], Corollaire 7, p. 90. Therefore the centre of $\mathcal{N}(\Omega, T, \mu)$ consists of families

$$f = (f(\omega) 1_{\mathcal{H}_\omega})_{\omega \in \Omega},$$

where $f\colon \Omega \to \mathbb{C}$ is bounded, measurable and invariant. Since μ is assumed to be ergodic this implies that $f(\omega)$ is a.s. constant so that the centre of $\mathcal{N}(\Omega, T, \mu)$ is trivial. $\square$

Remark 2.9. Since μ is ergodic, the assumption of nonperiodicity in the theorem can be replaced by assuming that there is a set of positive measure consisting of nonperiodic ω.

Note that the latter result gives an extension of part of what has been announced in [28], Theorem 2.1 and [29], Theorem 3.8. The remaining assertions of [29] will be proved in the following section, again in greater generality.

3. Transversal Functions, Traces and Deterministic Spectral Properties

In the preceding section we have defined the von Neumann algebra $\mathcal{N}(\Omega, T, \mu)$ starting from an (r, R)-system (Ω, T) and an invariant measure μ on (Ω, T). In the present section we will study traces on this algebra. Interestingly, this rather abstract and algebraic enterprise will lead to interesting spectral consequences. We will see that the operators involved share some fundamental properties with 'usual random operators'.

Let us first draw the connection of our families to 'usual random operators', referring to [7, 31, 39] for a systematic account. Generally speaking one is concerned with families $(A_\omega)_{\omega \in \Omega}$ of operators indexed by some probability space and acting on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$ typically. The probability space Ω encodes some statistical properties, a certain kind of disorder that is inspired by physics in many situations. One can view the set Ω as the set of all possible realization of a fixed disordered model and each single ω as a possible realization of the disorder described by Ω. Of course, the information is mostly encoded in a measure on Ω that describes the probability with which a certain realization is picked.

We are faced with a similar situation, one difference being that in any family $A = (A_\omega)_{\omega \in \Omega} \in \mathcal{N}(\Omega, T, \mu)$, the operators A_ω act on the possibly different spaces $\ell^2(\omega)$. Apart from that we have the same ingredients as in the usual random business, where, of course, Delone dynamical systems still bear quite some order. That is, we are in the realm of weakly disordered systems. For a first idea what this might have to do with aperiodically ordered solids, quasicrystals, assume that the points $p \in \omega$ are the atomic positions of a quasicrystal. In a tight binding approach (see [6], Section 4 for why this is reasonable), the Hamiltonian H_ω describing the respective solid would naturally be defined on $\ell^2(\omega)$, its matrix elements $H_\omega(p, q)$, $p, q \in \omega$ describing the diagonal and hopping terms for an electron that undergoes the influence of the atomic constellation given by ω. The definite choice of these matrix elements has to be done on physical grounds. In the following subsection we will propose a C^*-subalgebra that contains what we consider the most reasonable candidates; see also [6, 17]. It is clear, however, that $\mathcal{N}(\Omega, T, \mu)$ is a reasonable framework, since translations should not matter. Put in other words, every reasonable Hamiltonian family $(H_\omega)_{\omega\in\Omega}$ should be covariant.

The remarkable property that follows from this 'algebraic' fact is that certain spectral properties of the H_ω are *deterministic*, i.e., do not depend on the choice of the realization ω μ-a.s.

Let us next introduce the necessary algebraic concepts, taking a second look at transversal functions and random variables with values in $\mathfrak{X}$. In fact, random variables can be integrated with respect to transversal measures by [9], i.e., for a given nonnegative random variable β with values in $\mathfrak{X}$ and a transversal measure Λ, the expression $\int F_\beta \,\mathrm{d}\Lambda$ is well defined. More precisely, the following holds:

LEMMA 3.1. *Let (Ω, T) be an (r, R)-system and μ be T-invariant.*

(a) *Let β be a nonnegative random variable with values in $\mathfrak{X}$. Then $\int_\Omega \beta^\omega(f(\omega, \cdot))\,\mathrm{d}\mu(\omega)$ does not depend on $f \in \mathcal{F}^+(\mathfrak{X})$ provided f satisfies $\int f((\omega + t, x + t)\,\mathrm{d}t = 1$ for every $(\omega, x) \in \mathfrak{X}$ and*

$$\int_\Omega \beta^\omega(f(\omega, \cdot))\,\mathrm{d}\mu(\omega) = \int F_\beta \,\mathrm{d}\Lambda,$$

where $F_\beta\colon \mathcal{G} \rightsquigarrow \mathfrak{X}$ is the measurable functor induced by $F_\beta(\omega) = (\mathfrak{X}^\omega, \beta^\omega)$ and $\Lambda = \Lambda_\nu$ the transversal measure defined in the previous section.

(b) *An analogous statement remains true for a complex random variable $\beta = \sum_k \lambda_k \beta_k$, when we define*

$$\int F_\beta \,\mathrm{d}\Lambda = \sum_k \lambda_k \int F_{\beta_k} \,\mathrm{d}\Lambda$$

and restrict to $f \in \mathcal{F}^+(\mathfrak{X})$ with $\operatorname{supp} f$ *compact.*

Proof. Part (a) is a direct consequence of the definitions and results in [9]. Part (b), then easily follows from (a) by linearity. □

A special instance of the foregoing lemma is given in the following proposition.

PROPOSITION 3.2. *Let (Ω, T) be an (r, R)-system and let μ be T-invariant. If λ is a transversal function on $G(\Omega, T)$ then*

$$\varphi \mapsto \int_\Omega \langle \lambda^\omega, \varphi \rangle \, \mathrm{d}\mu(\omega)$$

defines an invariant functional on $C_c(\mathbb{R}^d)$, i.e., a multiple of the Lebesgue measure. In particular, if μ is an ergodic measure, then either $\lambda^\omega(1) = 0$ a.s. or $\lambda^\omega(1) = \infty$ a.s.

Proof. Invariance of the functional follows by direct checking. By uniqueness of the Haar measure, this functional must then be a multiple of Lebesgue measure. If μ is ergodic, the map $\omega \mapsto \lambda^\omega(1)$ is almost surely constant (as it is obviously invariant). This easily implies the last statement. □

Each random operator gives rise to a random variable as seen in the following proposition whose simple proof we omit.

PROPOSITION 3.3. *Let (Ω, T) be an (r, R)-system and μ be T-invariant. Let $(A_\omega) \in \mathcal{N}(\Omega, T, \mu)$ be given. Then the map $\beta_A \colon \Omega \to \mathcal{M}(\mathcal{X})$, $\beta_A^\omega(f) = \mathrm{tr}(A_\omega M_f)$ is a complex random variable with values in $\mathcal{X}$.*

Now, choose a nonnegative measurable u on $\mathbb{R}^d$ with compact support and $\int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = 1$. Combining the previous proposition with Lemma 3.1, $f(\omega, p) := u(p)$, we infer that the map

$$\tau \colon \mathcal{N}(\Omega, T, \mu) \longrightarrow \mathbb{C}, \tau(A) = \int_\Omega \mathrm{tr}(A_\omega M_u)\,\mathrm{d}\mu(\omega)$$

does not depend on the choice of f viz u as long as the integral is one. Important features of τ are given in the following lemma.

LEMMA 3.4. *Let (Ω, T) be an (r, R)-system and μ be T-invariant. Then the map $\tau \colon \mathcal{N}(\Omega, T, \mu) \to \mathbb{C}$ is continuous, faithful, nonegative on $\mathcal{N}(\Omega, T, \mu)^+$ and satisfies $\tau(A) = \tau(U^*AU)$ for every unitary $U \in \mathcal{N}(\Omega, T, \mu)$ and arbitrary $A \in \mathcal{N}(\Omega, T, \mu)$, i.e., τ is a trace.*

We include the elementary proof, stressing the fact that we needn't rely on the noncommutative framework; see also [27] for the respective statement in a different setting.

Proof. Choosing a continuous u with compact support we see that

$$|\tau(A) - \tau(B)| \leqslant \int \|A_\omega - B_\omega\| \,\mathrm{tr}\, M_u \,\mathrm{d}\mu(\omega) \leqslant \|A - B\| C,$$

where $C > 0$ only depends on u and Ω. On the other hand, choosing u with arbitrary large support we easily infer that τ is faithful. It remains to show the last statement.

According to [12], I.6.1, Cor. 1 it suffices to show $\tau(K^*K) = \tau(KK^*)$ for every $K = (K_\omega)_{\omega\in\Omega} \in \mathcal{N}(\Omega, T, \mu)$. We write $k_\omega(p, q) := (K_\omega\delta_q|\delta_p)$ for the associated kernel and calculate

$$\begin{aligned}
\tau(K^*K) &= \int_\Omega \operatorname{tr}(K_\omega^* K_\omega M_u)\,\mathrm{d}\mu(\omega) \\
&= \int_\Omega \operatorname{tr}(M_{u^{\frac{1}{2}}} K_\omega^* K_\omega M_{u^{\frac{1}{2}}})\,\mathrm{d}\mu(\omega) \\
&= \int_\Omega \sum_{m\in\omega} \|K_\omega M_{u^{\frac{1}{2}}}\delta_m\|^2 \mu(\omega) \\
&= \int_\Omega \sum_{l,m\in\omega} |k_\omega(l, m)|^2 u(m) \int_{\mathbb{R}^d} u(l-t)\,\mathrm{d}t\,\mathrm{d}\mu(\omega),
\end{aligned}$$

where we used that $\int_{\mathbb{R}^d} u(l-t)\,\mathrm{d}t = 1$ for all $l \in \omega$. By covariance and Fubinis theorem we get

$$\cdots = \int_{\mathbb{R}^d} \int_\Omega \sum_{l,m\in\omega} |k_{\omega-t}(l-t, m-t)|^2 u(m) u(l-t)\,\mathrm{d}\mu(\omega)\,\mathrm{d}t.$$

As μ is T-invariant, we can replace $\omega - t$ by ω and obtain

$$\begin{aligned}
&= \int_{\mathbb{R}^d} \int_\Omega \sum_{l,m\in\omega+t} |k_\omega(l-t, m-t)|^2 u(m) u(l-t)\,\mathrm{d}t\,\mathrm{d}\mu(\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \sum_{l,m\in\omega} |k_\omega(l, m)|^2 u(m+t) u(l)\,\mathrm{d}t\,\mathrm{d}\mu(\omega) \\
&= \int_\Omega \operatorname{tr}(K_\omega K_\omega^* M_u)\,\mathrm{d}\mu(\omega)
\end{aligned}$$

by reversing the first steps. □

Having defined τ, we can now associate a canonial measure ρ_A to every self-adjoint $A \in \mathcal{N}(\Omega, T, \mu)$.

DEFINITION 3.5. For $A \in \mathcal{N}(\Omega, T, \mu)$ self-adjoint, and $B \subset \mathbb{R}$ Borel measurable, we set $\rho_A(B) \equiv \tau(\chi_B(A))$, where χ_B is the characteristic function of B.

For the next two results we refer to [27] where the context is somewhat different.

LEMMA 3.6. *Let (Ω, T) be an (r, R)-system and μ be T-invariant. Let $A \in \mathcal{N}(\Omega, T, \mu)$ self-adjoint be given. Then ρ_A is a spectral measure for A. In particular, the support of ρ_A agrees with the spectrum Σ of A and the equality $\rho_A(F) = \tau(F(A))$ holds for every bounded measurable F on $\mathbb{R}$.*

LEMMA 3.7. *Let (Ω, T) be an (r, R)-system and μ be T-invariant. Let μ be ergodic and $A = (A_\omega) \in \mathcal{N}(\Omega, T, \mu)$ be self-adjoint. Then there exists Σ, Σ_{ac}, Σ_{sc}, Σ_{pp}, $\Sigma_{ess} \subset \mathbb{R}$ and a subset $\widetilde{\Omega}$ of Ω of full measure such that $\Sigma = \sigma(A_\omega)$ and*

$\sigma_\bullet(A_\omega) = \Sigma_\bullet$ for $\bullet = ac, sc, pp, ess$ and $\sigma_{\mathrm{disc}}(A_\omega) = \emptyset$ for every $\omega \in \widetilde{\Omega}$. In this case, the spectrum of A is given by Σ.

We now head towards evaluating the trace τ.

DEFINITION 3.8. The number $\int F_\alpha \, \mathrm{d}\Lambda =: D_{\Omega,\mu}$ is called the *mean density* of Ω with respect to μ.

THEOREM 3.9. *Let (Ω, T) be an (r, R)-system and μ be ergodic. If ω is nonperiodic for μ-a.e. $\omega \in \Omega$ then $\mathcal{N}(\Omega, T, \mu)$ is a factor of type II_D, where $D = D_{\Omega,\mu}$, i.e., a finite factor of type* II *and the canonical trace τ satisfies $\tau(1) = D$.*

Proof. We already know that $\mathcal{N}(\Omega, T, \mu)$ is a factor. Using Proposition 3.2 and [9], Cor. 9, p. 51 we see that $\mathcal{N}(\Omega, T, \mu)$ is not of type I. Since it admits a finite faithful trace, $\mathcal{N}(\Omega, T, \mu)$ has to be a finite factor of type II.

Note that Lemma 3.1, the definition of τ and α give the asserted value for $\tau(1)$. □

Remark 3.10. It is a simple consequence of Proposition 4.6 below that

$$D_\omega = \lim_{R\to\infty} \frac{\#(\omega \cap B_R(0))}{|B_R(0)|}$$

exists and equals $D_{\Omega,\mu}$ for almost every $\omega \in \Omega$. Therefore, the preceding result is a more general version of the results announced as [28], Theorem 2.1 and [29], Theorem 3.8, respectively. Of course, existence of the limit is not new. It can already be found, e.g., in [6].

4. The C^*-Algebra Associated to Finite Range Operators and the Integrated Density of States

In this section we study a C^*-subalgebra of $\mathcal{N}(\Omega, T, \mu)$ that contains those operators that might be used as Hamiltonians for quasicrystals. The approach is direct and does not rely upon the framework introduced in the preceding sections.

We define

$$\mathcal{X} \times_\Omega \mathcal{X} := \{(p, \omega, q) \in \mathbb{R}^d \times \Omega \times \mathbb{R}^d : p, q \in \omega\},$$

which is a closed subspace of $\mathbb{R}^d \times \Omega \times \mathbb{R}^d$ for any DDS Ω.

DEFINITION 4.1. A *kernel of finite range* is a function $k \in C(\mathcal{X} \times_\Omega \mathcal{X})$ that satisfies the following properties:

(i) k is bounded.
(ii) k has finite range, i.e., there exists $R_k > 0$ such that $k(p, \omega, q) = 0$, whenever $|p - q| \geqslant R_k$.

(iii) k is invariant, i.e.,

$$k(p+t, \omega+t, q+t) = k(p, \omega, q),$$

for $(p, \omega, q) \in \mathcal{X} \times_\Omega \mathcal{X}$ and $t \in \mathbb{R}^d$.

The set of these kernels is denoted by $\mathcal{K}^{\mathrm{fin}}(\Omega, T)$.

We record a few quite elementary observations. For any kernel $k \in \mathcal{K}^{\mathrm{fin}}(\Omega, T)$ denote by $\pi_\omega k := K_\omega$ the operator $K_\omega \in \mathcal{B}(\ell^2(\omega))$, induced by

$$(K_\omega \delta_q | \delta_p) := k(p, \omega, q) \quad \text{for } p, q \in \omega.$$

Clearly, the family $K := \pi k$, $K = (K_\omega)_{\omega\in\Omega}$, is bounded in the product (equipped with the supremum norm) $\Pi_{\omega\in\Omega}\mathcal{B}(\ell^2(\omega))$. Now, pointwise sum, the convolution (matrix) product

$$(a \cdot b)(p, \omega, q) := \sum_{x\in\omega} a(p, \omega, x) b(x, \omega, q)$$

and the involution $k^*(p, \omega, q) := \overline{k}(q, \omega, p)$ make $\mathcal{K}^{\mathrm{fin}}(\Omega, T)$ into a $*$-algebra. Then, the mapping $\pi\colon \mathcal{K}^{\mathrm{fin}}(\Omega, T) \to \Pi_{\omega\in\Omega}\mathcal{B}(\ell^2(\omega))$ is a faithful $*$-representation. We denote $\mathcal{A}^{\mathrm{fin}}(\Omega, T) := \pi(\mathcal{K}^{\mathrm{fin}}(\Omega, T))$ and call it the *operators of finite range*. The completion of $\mathcal{A}^{\mathrm{fin}}(\Omega, T)$ with respect to the norm $\|A\| := \sup_{\omega\in\Omega} \|A_\omega\|$ is denoted by $\mathcal{A}(\Omega, T)$. It is not hard to see that the mapping π_ω: $\mathcal{A}^{\mathrm{fin}}(\Omega, T) \to \mathcal{B}(\ell^2(\omega))$, $K \mapsto K_\omega$ is a representation that extends by continuity to a representation of $\mathcal{A}(\Omega, T)$ that we denote by the same symbol.

PROPOSITION 4.2. *Let $A \in \mathcal{A}(\Omega, T)$ be given. Then the following holds:*

(a) $\pi_{\omega+t}(A) = U_t \pi_\omega(A) U_t^*$ *for arbitrary $\omega \in \Omega$ and $t \in \mathbb{R}^d$.*
(b) *For $F \in C_c(\mathcal{X})$, the map $\omega \mapsto \langle \pi_\omega(A) F_\omega, F_\omega\rangle_\omega$ is continuous.*

Proof. Both statements are immediate for $A \in \mathcal{A}^{\mathrm{fin}}(\Omega, T)$ and then can be extended to $\mathcal{A}(\Omega, T)$ by density and the definition of the norm. □

We get the following result that relates ergodicity properties of (Ω, T), spectral properties of the operator families from $\mathcal{A}(\Omega, T)$ and properties of the representations π_ω.

THEOREM 4.3. *The following conditions on a DDS (Ω, T) are equivalent:*

(i) *(Ω, T) is minimal.*
(ii) *For any self-adjoint $A \in \mathcal{A}(\Omega, T)$ the spectrum $\sigma(A_\omega)$ is independent of $\omega \in \Omega$.*
(iii) *π_ω is faithful for every $\omega \in \Omega$.*

Proof. (i) $\Rightarrow$ (ii) Choose $\phi \in C(\mathbb{R})$. We then get $\pi_\omega(\phi(A)) = \phi(\pi_\omega(A))$ since π_ω is a continuous algebra homomorphism. Set $\Omega_0 = \{\omega \in \Omega : \pi_\omega(\phi(A)) = 0\}$. By Proposition 4.2(a), Ω_0 is invariant under translations. Moreover, by Proposition 4.2(b) it is closed. Thus, $\Omega_0 = \emptyset$ or $\Omega_0 = \Omega$ by minimality. As ϕ is arbitrary, this gives the desired equality of spectra by spectral calculus.

(ii) $\Rightarrow$ (iii) By (ii) we get that $\|\pi_\omega(A)\|^2 = \|\pi_\omega(A^*A)\|$ does not depend on $\omega \in \Omega$. Thus $\pi_\omega(A) = 0$ for some A implies that $\pi_\omega(A) = 0$ for all $\omega \in \Omega$ whence $A = 0$.

(iii) $\Rightarrow$ (i) Assume that Ω is not minimal. Then we find ω_0 and ω_1 such that $\omega_1 \notin \overline{(\omega_0 + \mathbb{R}^d)}$.

Consequently, there is $r > 0$, $p \in \omega$, $\delta > 0$ such that

$$d_H((\omega_0 - p) \cap B_r(0), (\omega_1 - q) \cap B_r(0)) > 2\delta$$

for all $q \in \omega_1$. Let $\rho \in C(\mathbb{R})$ such that $\rho(t) = 0$ if $t \geqslant 1/2$ and $\rho(0) = 1$. Moreover, let $\psi \in C_c(\mathbb{R}^d)$ such that $\operatorname{supp}\psi \subset B_\delta(0)$ and $\phi \in C_c(\mathbb{R}^d)$ and $\phi = 1$ on $B_{2r}(0)$.

Finally, let

$$\begin{aligned} a(x, \omega, y) \;:=\; &\rho\Biggl(\biggl\| \Bigl(\sum_{p\in\omega} T_p\psi\Bigr) T_x\phi - \Bigl(\sum_{q\in\omega_0} T_q\psi\Bigr) T_y\phi \biggr\|_\infty + \\ &+ \biggl\| \Bigl(\sum_{p\in\omega_0} T_p\psi\Bigr) T_x\phi - \Bigl(\sum_{q\in\omega} T_q\psi\Bigr) T_y\phi \biggr\|_\infty \Biggr). \end{aligned}$$

It is clear that a is a symmetric kernel of finite range and by construction the corresponding operator family satisfies $A_{\omega_1} = 0$ but $A_{\omega_0} \neq 0$, which implies (iii). □

Let us now comment on the relation between the algebra $\mathcal{A}(\Omega, T)$ defined above and the C^*-algebra introduced in [6, 17] for a different purpose and in a different setting. Using the notation from [6] we let

$$\mathcal{Y} = \{\omega \in \Omega : 0 \in \omega\}$$

and

$$G_{\mathcal{Y}} = \{(\omega, t) \in \mathcal{Y} \times \mathbb{R}^d : t \in \omega\} \subset \mathcal{X}.$$

In [6] the authors introduce the algebra $C^*(G_{\mathcal{Y}})$, the completion of $C_c(G_{\mathcal{Y}})$ with respect to the convolution

$$fg(\omega, q) = \sum_{t\in\omega} f(\omega, t) g(\omega - t, q - t)$$

and the norm induced by the representations

$$\Pi_\omega\colon C_c(G_{\mathcal{Y}}) \to \mathcal{B}(\ell^2(\omega)),\ \Pi_\omega(f)\xi(q) = \sum_{t\in\omega} f(\omega - t, t - q)\xi(q), \quad q \in \omega.$$

The following result can be checked readily, using the definitions.

PROPOSITION 4.4. *For a kernel* $k \in \mathcal{K}^{\text{fin}}(\Omega, T)$ *denote* $f_k(\omega, t) := k(0, \omega, t)$. *Then*

$$J\colon \mathcal{K}^{\text{fin}}(\Omega, T) \to C_c(G_{\mathcal{Y}}), k \mapsto f_k$$

is a bijective algebra isomorphism and $\pi_\omega = \Pi_\omega \circ J$ *for all* ω. *Consequently,* $\mathcal{A}(\Omega, T)$ *and* $C^*(G_{\mathcal{Y}})$ *are isomorphic.*

Note that the setting in [6] and here are somewhat different. In the tiling framework, the analogue of these algebras have been considered in [17].

We now come to relate the abstract trace τ defined in the last section with the mean trace per unit volume. The latter object is quite often considered by physicists and bears the name *integrated density of states*. Its proper definition rests on ergodicity. We start with the following preparatory result for which we need the notion of a van Hove sequence of sets.

For $s > 0$ and $Q \subset \mathbb{R}^d$, we denote by $\partial_s Q$ the set of points in $\mathbb{R}^d$ whose distance to the boundary of Q is less than s. A sequence (Q_n) of bounded subsets of $\mathbb{R}^d$ is called a *van Hove sequence* if $|Q_n|^{-1}|\partial_s Q_n| \to 0, n \to 0$ for every $s > 0$.

PROPOSITION 4.5. *Assume that* (Ω, T) *is a uniquely ergodic* (r, R)*-system with invariant probability measure* μ *and* $A \in \mathcal{A}(\Omega, T)$. *Then, for any van Hove sequence* (Q_n) *it follows that*

$$\lim_{n\in\mathbb{N}} \frac{1}{|Q_n|} \operatorname{tr}(A_\omega|_{Q_n}) = \tau(A)$$

for every $\omega \in \Omega$.

Clearly, $A_\omega|_Q$ denotes the restriction of A_ω to the subspace $\ell^2(\omega \cap Q)$ of $\ell^2(\omega)$. Note that this subspace is finite-dimensional, whenever $Q \subset \mathbb{R}^d$ is bounded.

We will use here the shorthand $A_\omega(p, q)$ for the kernel associated with A_ω.

Proof. Fix a nonnegative $u \in C_c(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = 1$ and support contained in $B_r(0)$ and let $f(\omega, p) := u(p)$. Then

$$\begin{aligned}
\tau(A) &= \int_\Omega \operatorname{tr}(A_\omega M_u)\,\mathrm{d}\mu(\omega) \\
&= \int_\Omega \left(\sum_{p\in\omega} A_\omega(p, p)u(p)\right)\mathrm{d}\mu(\omega) \\
&= \int_\Omega F(\omega)\,\mathrm{d}\mu(\omega),
\end{aligned}$$

where

$$F(\omega) := \sum_{p\in\omega} A_\omega(p, p)u(p)$$

is continuous by virtue of [29], Proposition 2.5(a). Therefore, the ergodic theorem for uniquely ergodic systems implies that for every $\omega \in \Omega$:

$$\frac{1}{|Q_n|}\int_{Q_n} F(\omega + t)\,\mathrm{d}t \to \int_{\Omega} F(\omega)\,\mathrm{d}\mu(\omega).$$

On the other hand,

$$\begin{aligned}\frac{1}{|Q_n|}\int_{Q_n} F(\omega + t)\,\mathrm{d}t &= \frac{1}{|Q_n|}\int_{Q_n}\left(\sum_{p\in\omega+t} A_{\omega+t}(p,p)u(p)\right)\mathrm{d}t\\ &= \frac{1}{|Q_n|}\underbrace{\int_{Q_n}\left(\sum_{q\in\omega} A_{\omega}(q,q)u(q+t)\right)\mathrm{d}t}_{I_n}\end{aligned}$$

by covariance of A_ω. Since $\operatorname{supp} u \subset B_r(0)$ and the integral over u equals 1, every $q \in \omega$ such that $q + B_r(0) \subset Q_n$ contributes $A_\omega(q,q)\cdot 1$ in the sum under the integral I_n. For those $q \in \omega$ such that $q + B_r(0) \cap Q_n = \emptyset$, the corresponding summand gives 0. Hence

$$\begin{aligned}\left|\frac{1}{|Q_n|}\left(\sum_{q\in\omega\cap Q_n} A_\omega(q,q) - I_n\right)\right| &\leqslant \frac{1}{|Q_n|}\cdot \#\{q \in \partial_{2r} Q_n\}\cdot \|A_\omega\|\\ &\leqslant C\cdot \frac{|\partial_{2r} Q_n|}{|Q_n|} \to 0\end{aligned}$$

since (Q_n) is a van Hove sequence. □

A variant of this proposition also holds in the measurable situation.

PROPOSITION 4.6. *Let μ be an ergodic measure on (Ω, T). Let $A \in \mathcal{N}(\Omega, T, \mu)$ and an increasing van Hove sequence (Q_n) of compact sets in $\mathbb{R}^d$ with $\mathbb{R}^d = \bigcup Q_n$, $0 \in Q_1$ and $|Q_n - Q_n| \leqslant C|Q_n|$ for some $C > 0$ and all $n \in \mathbb{N}$ be given. Then,*

$$\lim_{n\in\mathbb{N}} \frac{1}{|Q_n|}\operatorname{tr}(A_\omega|_{Q_n}) = \tau(A)$$

for μ-almost every $\omega \in \Omega$.

Proof. The proof follows along similar lines as the proof of the preceding proposition after replacing the ergodic theorem for uniquely ergodic systems by the Birkhoff ergodic theorem. Note that for $A \in \mathcal{N}(\Omega, T, \mu)$, the function F defined there is bounded and measurable. □

In the proof we used ideas of Hof [14]. The following result finally establishes an identity that one might call an abstract Shubin's trace formula. It says that the abstractly defined trace τ is determined by the integrated density of states. The latter is the limit of the following eigenvalue counting measures. Let, for self-adjoint $A \in \mathcal{A}(\Omega, T)$ and $Q \subset \mathbb{R}^d$:

$$\langle \rho[A_\omega, Q], \varphi \rangle := \frac{1}{|Q|} \operatorname{tr}(\varphi(A_\omega|_Q)), \quad \varphi \in C(\mathbb{R}).$$

Its distribution function is denoted by $n[A_\omega, Q]$, i.e., $n[A_\omega, Q](E)$ gives the number of eigenvalues below E per volume (counting multiplicities).

THEOREM 4.7. *Let (Ω, T) be a uniquely ergodic (r, R)-system and μ its ergodic probability measure. Then, for self-adjoint $A \in \mathcal{A}(\Omega, T)$ and any van Hove sequence (Q_n),*

$$\langle \rho[A_\omega, Q_n], \varphi \rangle \to \tau(\varphi(A)) \quad \textit{as } n \to \infty$$

for every $\varphi \in C(\mathbb{R})$ and every $\omega \in \Omega$. Consequently, the measures $\rho_\omega^{Q_n}$ converge weakly to the measure ρ_A defined above by $\langle \rho_A, \varphi \rangle := \tau(\varphi(A))$, for every $\omega \in \Omega$.

Proof. Let $\varphi \in C(\mathbb{R})$ and (Q_n) be a van Hove sequence. From Proposition 4.5, applied to $\varphi(A) = (\varphi(A_\omega))_{\omega \in \Omega}$, we already know that

$$\lim_{n \in \mathbb{N}} \frac{1}{|Q_n|} \operatorname{tr}(\varphi(A_\omega)|_{Q_n}) = \tau(\varphi(A))$$

for arbitrary $\omega \in \Omega$. Therefore, it remains to show that

$$\lim_{n \in \mathbb{N}} \frac{1}{|Q_n|} \big(\operatorname{tr}(\varphi(A_\omega)|_{Q_n}) - \operatorname{tr}(\varphi(A_\omega|_{Q_n})) \big) = 0. \tag{$*$}$$

This latter property is stable under uniform limits of functions φ, since both $\varphi(A_\omega|_{Q_n})$ and $\varphi(A_\omega)|_{Q_n}$ are operators of rank dominated by $c \cdot |Q_n|$.

It thus suffices to consider a polynomial φ.

Now, for a fixed polynomial φ with degree N, there exists a constant $C = C(\varphi)$ such that

$$\|\varphi(A) - \varphi(B)\| \leqslant C \|A - B\| (\|A\| + \|B\|)^N$$

for any A, B on an arbitrary Hilbert space. In particular,

$$\frac{1}{|Q_n|} \big| \operatorname{tr}(\varphi(A_\omega)|_{Q_n}) - \operatorname{tr}(\varphi(B_\omega)|_{Q_n}) \big| \leqslant C \|A_\omega - B_\omega\| (\|A_\omega\| + \|B_\omega\|)^N$$

and

$$\frac{1}{|Q_n|} \big| \operatorname{tr}(\varphi(A_\omega|_{Q_n})) - \operatorname{tr}(\varphi(B_\omega|_{Q_n})) \big| \leqslant C \|A_\omega - B_\omega\| (\|A_\omega\| + \|B_\omega\|)^N$$

for all A_ω and B_ω.

Thus, it suffices to show $(*)$ for a polynomial φ and $A \in \mathcal{A}^{\text{fin}}(\Omega, T)$, as this algebra is dense in $\mathcal{A}(\Omega, T)$. Let such A and φ be given.

Let R_a the range of the kernel $a \in C(\mathcal{X} \times_\Omega \mathcal{X})$ corresponding to A. Since the kernel of A^k is the k-fold convolution product $b := a \cdots a$ one can easily verify that the range of A^k is bounded by $N \cdot R_a$. Thus, for all $p, q \in \omega \cap Q_n$ such that the distance of p, q to the complement of Q_n is larger than $N \cdot R_a$, the kernels of $A_\omega^k|_{Q_n}$ and $(A|_{Q_n})^k$ agree for $k \leqslant N$. We get:

$$((\varphi(A_\omega)|_{Q_n})\delta_q|\delta_p) = b(p, \omega, q) = (\varphi(A_\omega|_{Q_n})\delta_q|\delta_p).$$

Since this is true outside $\{q \in \omega \cap Q_n : \mathrm{dist}(q, Q_n^c) > N \cdot R_a\} \subset \partial_{N \cdot R_a} Q_n$ the matrix elements of $(\varphi(A_\omega)|_{Q_n})$ and $\varphi(A_\omega|_{Q_n})$ differ at at most $c \cdot |\partial_{N \cdot R_a} Q_n|$ sites, so that

$$|\mathrm{tr}(\varphi(A_\omega)|_{Q_n}) - \mathrm{tr}(\varphi(A_\omega|_{Q_n}))| \leqslant C \cdot |\partial_{N \cdot R_a} Q_n|.$$

Since (Q_n) is a van Hove sequence, this gives the desired convergence. □

The above statement has many precursors: [2–4, 31, 36] in the context of almost periodic, random or almost random operators on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$. It generalizes results by Kellendonk [17] on tilings associated with primitive substitutions. Its proof relies on ideas from [2–4, 17] and [14]. Nevertheless, it is new in the present context.

For completeness reasons, we also state the following result.

THEOREM 4.8. *Let (Ω, T) be an (r, R)-system with an ergodic probabiltiy measure μ. Let $A \in \mathcal{A}(\Omega, T)$ be self-adjoint (Q_n) be an increasing van Hove sequence (Q_n) of compact sets in $\mathbb{R}^d$ with $\bigcup Q_n = \mathbb{R}^d$, $0 \in Q_1$ and $|Q_n - Q_n| \leqslant C|Q_n|$ for some $C > 0$ and all $n \in \mathbb{N}$. Then,*

$$\langle \rho[A_\omega, Q_n], \varphi\rangle \to \tau(\varphi(A)) \quad \text{as } n \to \infty$$

for μ-almost every $\omega \in \Omega$. Consequently, the measures $\rho_\omega^{Q_n}$ converge weakly to the measure ρ_A defined above by $\langle \rho_A, \varphi\rangle := \tau(\varphi(A))$, for μ-almost every $\omega \in \Omega$.

The *Proof* follows along similar lines as the proof of the previous theorem with two modifications: Instead of Proposition 4.5, we use Proposition 4.6; and instead of dealing with arbitrary polynomials we choose a countable set of polynomials which is dense in $C_c([-\|A\| - 2, \|A\| + 2])$.

The primary object from the physicists point of view is the finite volume limit:

$$N[A](E) := \lim_{n \to \infty} n[A_\omega, Q_n](E)$$

known as the integrated density of states. It has a striking relevance as the number of energy levels below E per unit volume, once its existence and independence of ω are settled.

The last two theorems provide the mathematically rigorous version. Namely, the distribution function $N_A(E) := \rho_A(-\infty, E]$ of ρ_A is the right choice. It gives a limit of finite volume counting measures since

$$\rho[A_\omega, Q_n] \to \rho_A \quad \text{weakly as } n \to \infty.$$

Therefore, the desired independence of ω is also clear. Moreover, by standard arguments we get that the distribution functions of the finite volume counting functions converge to N_A at points of continuity of the latter.

In [30] we present a much stronger result for uniquely ergodic minimal DDS that extends results for one-dimensional models by the first named author, [26]. Namely we prove that the distribution functions converge uniformly, uniform in ω. The above result can then be used to identify the limit as given by the tace τ. Let us stress the fact that unlike in usual random models, the function N_A does exhibit discontinuities in general, as explained in [20].

Let us end by emphasizing that the assumptions we posed are met by all the models that are usually considered in connection with quasicrystals. In particular, included are those Delone sets that are constructed by the cut-and-project method as well as models that come from primitive substitution tilings.

References

1. Anderson, J. E. and Putnam, I. F.: Topological invariants for substitution tilings and their associated C^*-algebras, *Ergodic Theory Dynam. Systems* **18**(3) (1998), 509–537.
2. Avron, J. and Simon, B.: Almost periodic Schrödinger operators, II: The integrated density of states, *Duke Math. J.* **50** (1982), 369–391.
3. Bellissard, J., Lima, R. and Testard, D.: Almost periodic Schrödinger operators, In: *Mathematics + Physics*, Vol. 1, World Scientific, Singapore, 1995, pp. 1–64.
4. Bellissard, J.: K-theory of C^*-algebras in solid state physics, In: *Statistical Mechanics and Field Theory: Mathematical Aspects (Groningen, 1985)*, Lecture Notes in Phys. 257, Springer, Berlin, 1986, pp. 99–156.
5. Bellissard, J.: Gap labelling theorems for Schrödinger operators, In: M. Walsdschmidt, P. Moussa, J. M. Luck and C. Itzykson (eds), *From Number Theory to Physics*, Springer, Berlin, 1992, pp. 539–630.
6. Bellissard, J., Hermann, D. J. L. and Zarrouati, M.: Hulls of aperiodic solids and gap labelling theorem, In: *Directions in Mathematical Quasicrystals*, CRM Monogr. Ser. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 207–258.
7. Carmona, R. and Lacroix, J.: *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston, 1990.
8. Coburn, L. A., Moyer, R. D. and Singer, I. M.: C^*-algebras of almost periodic pseudo-differential operators, *Acta Math.* **130** (1973), 279–307.
9. Connes, A.: Sur la théorie non commutative de l'intégration, In: *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, Lecture Notes in Math. 725, Springer, Berlin, 1979, pp. 19–143.
10. Connes, A.: *Géometrie non commutative*.
11. Delaunay, B. [Delone, B. N.]: Sur la sphére vide, *Izvestia Akad. Nauk SSSR Otdel. Mat. Sov. Nauk* **7** (1934), 793–800.
12. Dixmier, J.: *Von Neumann Algebras,* North-Holland, Amsterdam, 1981.
13. Geerse, C. P. M. and Hof, A.: Lattice gas models on self-similar aperiodic tilings, *Rev. Math. Phys.* **3** (1991), 163–221.
14. Hof, A.: Some remarks on discrete aperiodic Schrödinger operators, *J. Statist. Phys.* **72** (1993), 1353–1374.
15. Hof, A.: A remark on Schrödinger operators on aperiodic tilings, *J. Statist. Phys.* **81** (1996), 851–855.
16. Janot, C.: *Quasicrystals: A Primer*, Oxford Univ. Press, Oxford, 1992.

17. Kellendonk, J.: Noncommutative geometry of tilings and gap labelling, *Rev. Math. Phys.* **7** (1995), 1133–1180.
18. Kellendonk, J.: The local structure of tilings and their integer group of coinvariants, *Comm. Math. Phys.* **187** (1997), 115–157.
19. Kellendonk, J. and Putnam, I. F.: Tilings; C^*-algebras, and K-theory, In: *Directions in Mathematical Quasicrystals*, CRM Monogr. Ser. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 177–206.
20. Klassert, S., Lenz, D. and Stollmann, P.: Discontinuities of the integrated density of states for random operators on Delone sets, *Comm. Math. Phys.*, to appear.
21. Lagarias, J. C.: Geometric models for quasicrystals I. Delone sets of finite type, *Discrete Comp. Geom.* **21** (1999), 161–191.
22. Lagarias, J. C.: Geometric models for quasicrystals II. Local rules under isometries, *Discrete Comp. Geom.* **21** (1999), 345–372.
23. Lagarias, J. C. and Pleasants, P. A. B.: Repetitive Delone sets and quasicrystals, *Ergodic Theory Dynam. Systems*, to appear.
24. J.-Y. Lee, Moody, R. V. and Solomyak, B.: Pure point dynamical and diffraction spectra, *Ann. H. Poincaré* **3** (2001), 1003–1018.
25. Lenz, D.: Random operators and crossed products, *Math. Phys. Anal. Geom.* **2** (1999), 197–220.
26. Lenz, D.: Uniform ergodic theorems on subshifts over a finite alphabet, *Ergodic Theory Dynam. Systems* **22** (2002), 245–255.
27. Lenz, D., Peyerimhof, N. and Veselic, I.: Von Neumann algebras, groupoids and the integrated density of states, `eprint: arXiv math-ph/0203026`.
28. Lenz, D. and Stollmann, P.: Delone dynamical systems, groupoid von Neuman algebras and Hamiltonians for quasicrystals, *C.R. Acad. Sci. Paris, Ser. I* **334** (2002), 1–6.
29. Lenz, D. and Stollmann, P.: Delone dynamical systems and associated random operators, *Proc. OAMP*, to appear, `eprint: arXiv math-ph/0202142`.
30. Lenz, D. and Stollmann, P.: An ergodic theorem for Delone dynamical systems and existence of the density of states, in preparation.
31. Pastur, L. and Figotin, A.: *Spectra of Random and Almost Periodic Operators*, Springer, Berlin, 1992.
32. Putnam, I. F.: The ordered K-theory of C^*-algebras associated with substitution tilings, *Comm. Math. Phys.* **214** (2000), 593–605.
33. Schlottmann, M.: Generalized model sets and dynamical systems, In: M. Baake and R. V. Moody (eds), *Directions in Mathematical Quasicrystals*, CRM Monogr. Ser., Amer. Math. Soc., Providence, RI, 2000, pp. 143–159.
34. Senechal, M.: *Quasicrystals and Geometry*, Cambridge Univ. Press, Cambridge, 1995.
35. Shechtman, D., Blech, I., Gratias, D. and Cahn, J. W.: Metallic phase with long-range orientational order and no translation symmetry, *Phys. Rev. Lett.* **53** (1984), 1951–1953.
36. Shubin, M.: The spectral theory and the index of elliptic operators with almost periodic coefficients, *Russian Math. Surveys* **34** (1979).
37. Solomyak, B.: Dynamics of self-similar tilings, *Ergodic Theory Dynam. Systems* **17** (1997), 695–738.
38. Solomyak, B.: Spectrum of a dynamical system arising from Delone sets, In: J. Patera (ed.), *Quasicrystals and Discrete Geometry*, Fields Institute Monogr. 10, Amer. Math. Soc., Providence, RI, 1998, pp. 265–275.
39. Stollmann, P.: *Caught by Disorder: Bound States in Random Media*, Progr. in Math. Phys. 20, Birkhäuser, Boston, 2001.