High dimensional approximation with trigonometric polynomials

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joint work with L. Kämmerer and T. Volkmer

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Content

Multivariate trigonometric polynomials $p(\mathbf{x}) = \sum_{\mathbf{k}\in I} \hat{p}_{\mathbf{k}}\, \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$

Fast evaluation at rank-1 lattice nodes $(p(\mathbf{x}_j))_{j=0}^{M-1}$

Fast, exact, stable reconstruction of $\hat{p}_{\mathbf{k}}$, $\mathbf{k} \in I$ ($I \subset \mathbb{Z}^d$ known)

Reconstruction of functions $f \in \mathcal{A}^\omega(\mathbb{T}^d)$

by sampling at rank-1 lattice nodes

Dimension incremental reconstruction ($I \subset \mathbb{Z}^d$ unknown)

Summary

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- approximate f by multivariate trigonometric polynomial p supported on $\mathrm{I} \subset \mathbb{Z}^d,\ |\mathrm{I}| < \infty$,

$$p(\mathbf{x}) := \sum_{\mathbf{k}\in\mathrm{I}} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \hat{p}_{\mathbf{k}} \in \mathbb{C}$$

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- e.g., approximate f using its Fourier partial sum $S_{\mathrm{I}} f$,

$$p(\mathbf{x}) = S_{\mathrm{I}} f(\mathbf{x}) := \sum_{\mathbf{k} \in \mathrm{I}} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}, \qquad \mathrm{I} \subset \mathbb{Z}^d, \ |\mathrm{I}| < \infty,$$

  where the Fourier coefficients of f are given by

$$\hat{f}_{\mathbf{k}} = \int_{\mathbb{T}^d} f(\mathbf{x}) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}} \mathrm{d}\mathbf{x}, \qquad \mathbf{k} \in \mathbb{Z}^d$$

Content - Function Spaces and Index Sets I

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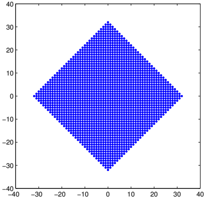

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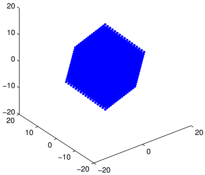

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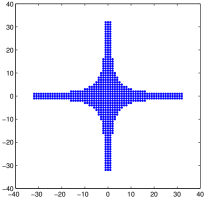

hyperbolic cross

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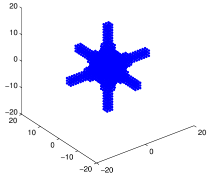

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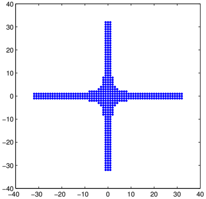

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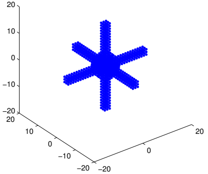

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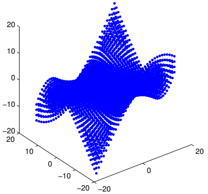

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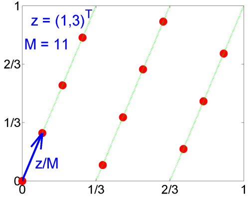
- large dimension d

Trig. polynomials - Fast evaluation at rank-1 lattices

- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\Lambda(\mathbf{z}, M) = \{\mathbf{x}_j = \frac{j\mathbf{z}}{M} \text{ mod } \mathbf{1};\ j = 0, \dots, M-1\}$$



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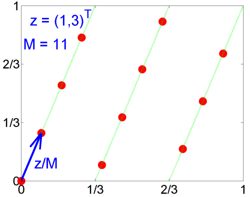

- reformulation

$$p(\mathbf{x}_j) = \sum_{\mathbf{k}\in \mathrm{I}} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k}\cdot\mathbf{x}_j} = \sum_{l=0}^{M-1} \left(\sum_{\substack{\mathbf{k}\in \mathrm{I} \\ \mathbf{k}\cdot\mathbf{z} \equiv l \pmod M}} \hat{p}_{\mathbf{k}} \right) \mathrm{e}^{2\pi \mathrm{i} \frac{j \mathbf{k}\cdot\mathbf{z}}{M}} = \sum_{l=0}^{M-1} \hat{g}_l \mathrm{e}^{2\pi \mathrm{i} \frac{jl}{M}}$$

- complexity $\mathcal{O}(M \log M + d|\mathrm{I}|)$ applying one-dimensional FFT

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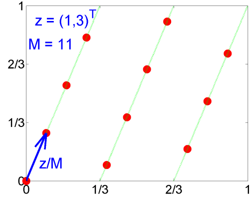

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Aim: stable and unique reconstruction of $\hat{p}_{\mathbf{k}}$

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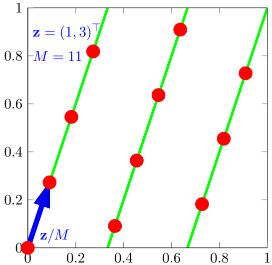

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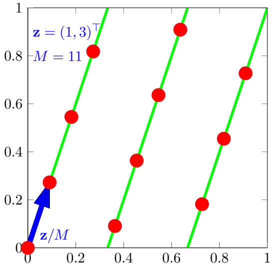

$$\hat{p}_{\mathbf{k}} = \int_{\mathbb{T}^d} p(\mathbf{x}) \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} \approx \mathrm{Q}(p(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot(\cdot)}) = \underbrace{\frac{1}{M}\sum_{j=0}^{M-1} p(\mathbf{x}_j)\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}}_{:=\hat{\tilde{p}}_{\mathbf{k}}}$$

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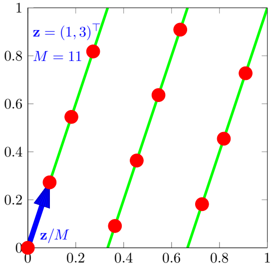

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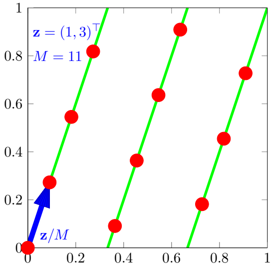

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$\Rightarrow\ \hat{\tilde{p}}_{\mathbf{k}} = \hat{p}_{\mathbf{k}} \Leftrightarrow \mathbf{k}_1 \cdot \mathbf{z} \not\equiv \mathbf{k}_2 \cdot \mathbf{z} \pmod{M}$ for all $\mathbf{k}_1 \neq \mathbf{k}_2 \in \mathrm{I}$

Trig. polynomials - Fast, exact, stable reconstruction

Theorem (Kämmerer 2012)

Let $\mathrm{I} \subset \{\mathbf{k}+\mathbf{h} : \mathbf{h} \in [0, M-1]^d \cap \mathbb{Z}^d\}$ *for a fixed* $\mathbf{k} \in \mathbb{Z}^d$ *and* $|\mathrm{I}| \geq 4$. *Then there exists a rank-1 lattice* $\Lambda(\mathbf{z}, M)$ *that allows the reconstruction of all trigonometric polynomials with frequencies supported on* I, *where*

- *the lattice size* M *can be determined by* $M \leq |\mathrm{I}|^2$,
- *the generating vector* $\mathbf{z}$ *can be found using a component–by–component*[Sloan, Reztsov] *search,*
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- *the generating vector* $\mathbf{z}$ *can be found using a component–by–component*[Sloan, Reztsov] *search,*
- *the reconstruction is unique & perfectly stable.*

- adapt results from numerical integration (Cools, Kuo, Nuyens 2010)

Trig. polynomials - Fast, exact, stable reconstruction

Theorem (Kämmerer 2012)

Let $\mathrm{I} \subset \{\mathbf{k} + \mathbf{h} : \mathbf{h} \in [0, M-1]^d \cap \mathbb{Z}^d\}$ *for a fixed* $\mathbf{k} \in \mathbb{Z}^d$ *and* $|\mathrm{I}| \geq 4$. *Then there exists a rank-1 lattice* $\Lambda(\mathbf{z}, M)$ *that allows the reconstruction of all trigonometric polynomials with frequencies supported on* I, *where*

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- name $\Lambda(\mathbf{z}, M)$ reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I})$ for I

$$\hat{p}_{\mathbf{k}} = \hat{\tilde{p}}_{\mathbf{k}} = M^{-1} \sum_{j=0}^{M-1} p\left(\frac{j\mathbf{z}}{M}\right) \mathrm{e}^{-2\pi \mathrm{i} j \frac{\mathbf{k}\cdot\mathbf{z}}{M}}, \quad \text{for all } \mathbf{k} \in \mathrm{I}$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- rank-1 lattice: $\mathbf{x}_j = \frac{j\mathbf{z}}{M} \bmod \mathbf{1};\ j = 0, \ldots, M-1$
- approximative reconstruction of the Fourier coefficients of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ applying a lattice rule

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Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- L_∞ error of the exact Fourier partial sum

$$\|f - S_{\mathrm{I}} f | L_\infty(\mathbb{T}^d)\| = \| \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \mathrm{I}} \hat{f}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}} | L_\infty(\mathbb{T}^d)\| \leq \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus \mathrm{I}} |\hat{f}_{\mathbf{k}}|$$

$$\leq \frac{1}{\min_{\mathbf{k} \in \mathbb{Z}^d \setminus \mathrm{I}} \omega(\mathbf{k})} \sum_{\mathbf{k} \in \mathbb{Z}^d} \omega(\mathbf{k}) |\hat{f}_{\mathbf{k}}| = \frac{1}{\min_{\mathbf{k} \in \mathbb{Z}^d \setminus \mathrm{I}} \omega(\mathbf{k})} \|f | \mathcal{A}^\omega(\mathbb{T}^d)\|$$

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- choose $\mathrm{I} = \mathrm{I}_N^d := \{\mathbf{k} \in \mathbb{Z}^d : \ \omega(\mathbf{k}) \leq N\}$

$$\|f - S_{\mathrm{I}_N^d} f | L_\infty(\mathbb{T}^d)\| \leq \frac{1}{N} \|f | \mathcal{A}^\omega(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

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$$\|f - S_{\mathrm{I}_N^d} f | L_\infty(\mathbb{T}^d)\| \leq \frac{1}{N} \|f | \mathcal{A}^\omega(\mathbb{T}^d)\|$$

- cardinality of I_N^d mainly depends on ω
  - our interest: $|\mathrm{I}_N^d| \ll N^d$, e.g. $|\mathrm{I}_N^d| \sim N^r$, $1 \leq r \leq 2$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - rank-1 lattice nodes

- approximation $\tilde{S}_{\mathrm{I}} f$ of f

$$\tilde{S}_{\mathrm{I}} f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathrm{I}} \hat{\tilde{\hat{f}}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}\in\mathrm{I}} \left(\sum_{\substack{\mathbf{h}\in\mathbb{Z}^d \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{M}}} \hat{f}_{\mathbf{k}+\mathbf{h}} \right) \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

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- error estimation for $\mathrm{I} \subset \mathbb{Z}^d$ and corresponding reconstructing rank-1 lattice $\Lambda(\mathbf{z}, M, \mathrm{I})$

$$\|S_{\mathrm{I}} f - \tilde{S}_{\mathrm{I}} f | L_\infty(\mathbb{T}^d)\| = \| \sum_{\mathbf{k}\in\mathrm{I}} \sum_{\substack{\mathbf{h}\in\mathbb{Z}^d\setminus\{\mathbf{0}\} \\ \mathbf{h}\cdot\mathbf{z}\equiv 0 \pmod{M}}} \hat{f}_{\mathbf{k}+\mathbf{h}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}} | L_\infty(\mathbb{T}^d)\|$$

$$\leq \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\mathrm{I}} |\hat{f}_{\mathbf{k}}| \leq \frac{1}{\min_{\mathbf{k}\in\mathbb{Z}^d\setminus\mathrm{I}} \omega(\mathbf{k})} \|f | \mathcal{A}^\omega(\mathbb{T}^d)\|$$

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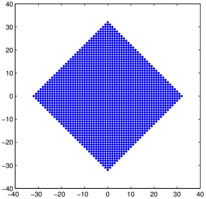
Lemma

Let $\mathrm{I}_N^d := \{\mathbf{k} \in \mathbb{Z}^d : \ \omega(\mathbf{k}) \le N\}$ *and a corresponding reconstructing rank-1 lattice* $\Lambda(\mathbf{z}, M, \mathrm{I})$ *be given. We estimate the approximation error by*

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L_\infty(\mathbb{T}^d)\| \le \frac{2}{N} \|f | \mathcal{A}^\omega(\mathbb{T}^d)\|.$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - rank-1 lattice nodes



Example: ℓ_1-ball
weights: $\omega(\mathbf{k}) := \omega^{\alpha}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^{\alpha}$

Sobolev space:

$$\mathcal{A}^{\alpha,0}(\mathbb{T}^d) := \left\{ f\colon \|f|\mathcal{A}^{\alpha,0}(\mathbb{T}^d)\| := \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha}(\mathbf{k})|\hat{f}_{\mathbf{k}}| < \infty \right\}$$

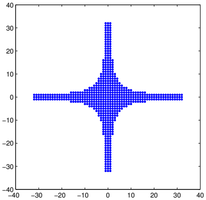
index set: $|\mathrm{I}_N^d| \in \mathcal{O}\left(N^d\right)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d\colon \; \max(1, \|\mathbf{k}\|_1) \; \leq N \right\}$$

error estimate:

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L_{\infty}(\mathbb{T}^d)\| \leq 2N^{-\alpha} \|f|\mathcal{A}^{\alpha,0}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - rank-1 lattice nodes



Example: hyperbolic cross

weights: $\omega(\mathbf{k}) := \omega^{\beta}(\mathbf{k}) := \prod_{s=1}^{d} \max(1, |k_s|)^{\beta}$

Sobolev space:

$$\mathcal{A}^{0,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{A}^{0,\beta}(\mathbb{T}^d)\| := \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\beta}(\mathbf{k})|\hat{f}_{\mathbf{k}}| < \infty \right\}$$

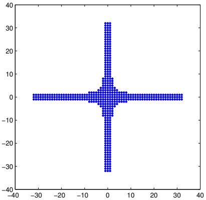
index set: $|\mathrm{I}_N^d| \in \mathcal{O}\left(N \log^{d-1} N\right)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \leq N \right\}$$

error estimate:

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L_{\infty}(\mathbb{T}^d)\| \leq 2N^{-\beta} \|f|\mathcal{A}^{0,\beta}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - rank-1 lattice nodes



Example: energy norm based hyperbolic cross

weights: $\omega^{\alpha,\beta}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^{\alpha} \prod_{s=1}^{d} \max(1, |k_s|)^{\beta}$

Sobolev space:

$$\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\| := \sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})|\hat{f}_{\mathbf{k}}| < \infty \right\}$$

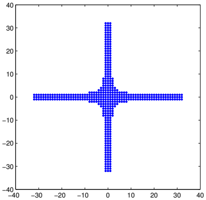
index set: for $0 < -\alpha < \beta$, $|\mathrm{I}_N^d| \in \mathcal{O}(N)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha}{\alpha+\beta}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{\beta}{\alpha+\beta}} \leq N \right\}$$

error estimate:

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L_\infty(\mathbb{T}^d)\| \leq 2N^{-\alpha-\beta} \|f|\mathcal{A}^{\alpha,\beta}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^{\omega}(\mathbb{T}^d)$ - rank-1 lattice nodes



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Hilbert space:

$$\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \colon \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| := \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\mathbf{k})^2 |\hat{f}_{\mathbf{k}}|^2} < \infty \right\}$$

index set: for $0 < r - \alpha < \beta - t$, $|\mathrm{I}_N^d| \in \mathcal{O}(N)$

$$\mathrm{I}_N^d := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \max(1, \|\mathbf{k}\|_1)^{\frac{\alpha-r}{\alpha-r+\beta-t}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{\beta-t}{\alpha-r+\beta-t}} \leq N \right\}$$

error estimate: $\beta > t \geq 0$, $\lambda > 1/2$

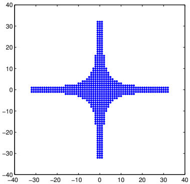
$$\|f - \tilde{S}_{\mathrm{I}_N^d} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq c\, N^{r-\alpha+t-\beta} \|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test

- function $f \in \mathcal{H}^{0,\frac{7}{2}-\epsilon}(\mathbb{T}^d)$, $\epsilon > 0$,

$$f(\mathbf{x}) := \prod_{s=1}^{d} \left(4 + \operatorname{sgn}\left(x_s - \frac{1}{2}\right) \sin(2\pi x_s)^3 + \operatorname{sgn}\left(x_s - \frac{1}{2}\right) \sin(2\pi x_s)^4\right),$$

- hyperbolic cross index set $\mathrm{I}_N^d := \left\{\mathbf{k} \in \mathbb{Z}^d \colon \prod_{s=1}^{d} \max(1, |k_s|) \le N\right\}$



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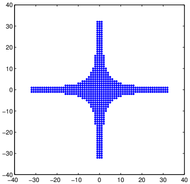
- error estimate: $\tilde{\epsilon} > 0$, $\lambda > 1/2$

$$\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| \leq c\, N^{-3+\tilde{\epsilon}} \|f|\mathcal{H}^{0,3-\tilde{\epsilon}+\lambda}(\mathbb{T}^d)\|$$



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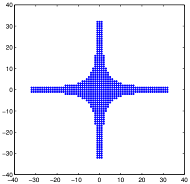
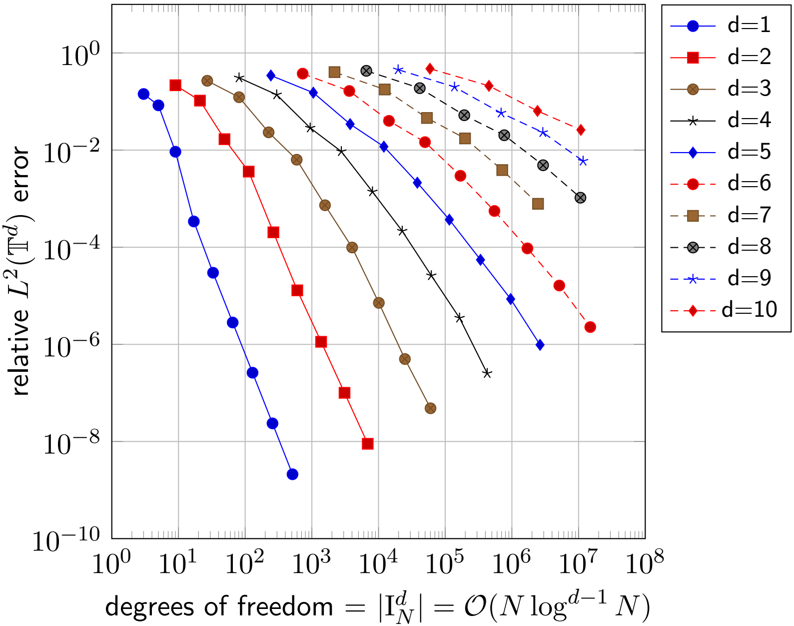


- we compute the relative $L^2(\mathbb{T}^d) = \mathcal{H}^{0,0}(\mathbb{T}^d)$
  - i.e., $\|f - \tilde{S}_{\mathrm{I}_N^d} f | L^2(\mathbb{T}^d)\| / \|f|L^2(\mathbb{T}^d)\|$
  - corresponds to the above error estimate with $r = t = 0$ up to a "constant" since

$$\frac{\|f - \tilde{S}_{\mathrm{I}_N^d} f|\mathcal{H}^{0,0,\gamma}(\mathbb{T}^d)\|}{\|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|} = \underbrace{\frac{\|f|L^2(\mathbb{T}^d)\|}{\|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\|}}_{\le 1} \frac{\|f - \tilde{S}_{\mathrm{I}_N^d} f|L^2(\mathbb{T}^d)\|}{\|f|L^2(\mathbb{T}^d)\|}$$

Reconstruction of $f \in \mathcal{A}^\omega(\mathbb{T}^d)$ - Numerical test



Dimension incremental reconstruction

until now: approximation of function $f(\mathbf{x}) \approx \sum_{\mathbf{k}\in \mathrm{I}} \hat{\tilde{f}}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{x}}$

- given frequency index set I
- compute $\hat{\tilde{f}}_{\mathbf{k}}$ from samples along reconstructing rank-1 lattice

- search for location I of largest Fourier coefficients of f (and compute $\hat{\tilde{f}}_{\mathbf{k}}$, $\mathbf{k} \in I$)
- search domain: (possibly) large index set $\Gamma \subset \mathbb{Z}^d$, e.g., full grid $\hat{G}_N^d := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \|\boldsymbol{k}\|_\infty \leq N\}$, ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)

Dimension incremental reconstruction

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next:

- search for location I of largest Fourier coefficients of f (and compute $\hat{\tilde{f}}_{\mathbf{k}}$, $\mathbf{k} \in I$)
- search domain: (possibly) large index set $\Gamma \subset \mathbb{Z}^d$, e.g., full grid $\hat{G}_N^d := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \|\boldsymbol{k}\|_\infty \leq N\}$, ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)
- several existing methods from compressed sensing, e.g.
  - Gilbert, Guha, Indyk, Muthukrishnan, Strauss 2002
  - Iwen, Gilbert, Strauss 2007

Dimension incremental reconstruction

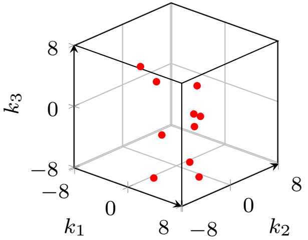
until now: approximation of function $f(\mathbf{x}) \approx \sum_{\mathbf{k}\in\mathrm{I}} \hat{\hat{f}}_{\mathbf{k}} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$

- given frequency index set I
- compute $\hat{\hat{f}}_{\mathbf{k}}$ from samples along reconstructing rank-1 lattice

next:

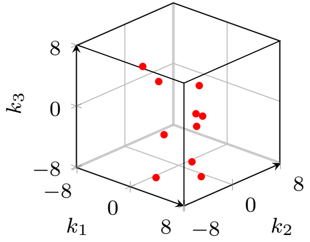
- search for location I of largest Fourier coefficients of f (and compute $\hat{\hat{f}}_{\mathbf{k}}$, $\mathbf{k} \in I$)
- search domain: (possibly) large index set $\Gamma \subset \mathbb{Z}^d$, e.g., full grid $\hat{G}_N^d := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \|\boldsymbol{k}\|_\infty \leq N\}$, ($|\hat{G}_{64}^{10}| \approx 1.28 \cdot 10^{21}$)
- several existing methods from compressed sensing, e.g.
  - Gilbert, Guha, Indyk, Muthukrishnan, Strauss 2002
  - Iwen, Gilbert, Strauss 2007
- our method
  - compute (projected) Fourier coefficients from sampling values and determine frequency locations dimension incremental
  - use reconstructing rank-1 lattices ($\Rightarrow$ 1-dim iFFT)

Dimension incremental reconstruction - Method

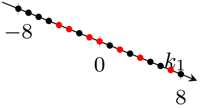


$\operatorname{supp}\hat{p} \subset \Gamma = \hat{G}_8^3$

Dimension incremental reconstruction - Method

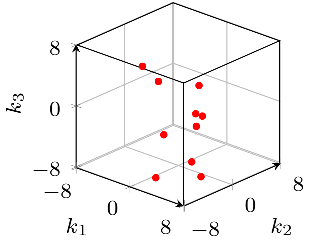


$\operatorname{supp}\hat{p} \subset \Gamma = \hat{G}_8^3$

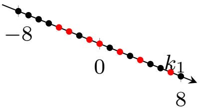


frequency candidates

Dimension incremental reconstruction - Method

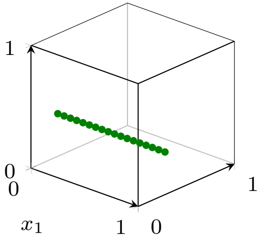


$\operatorname{supp} \hat{p} \subset \Gamma = \hat{G}_8^3$



frequency candidates

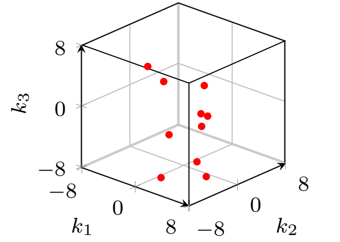
construct $\longrightarrow$ sampling set



sampling nodes

$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

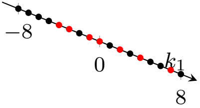
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \Gamma = \hat{G}_8^3$

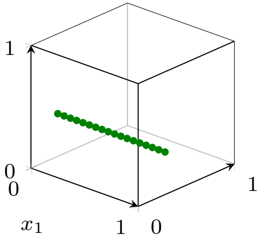
$$\tilde{\hat{p}}_{k_1} := \frac{1}{17} \sum_{\ell=0}^{16} p\left(\begin{pmatrix} \ell/17 \\ x_2' \\ x_3' \end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$

$k_1 = -8, \ldots, 8$



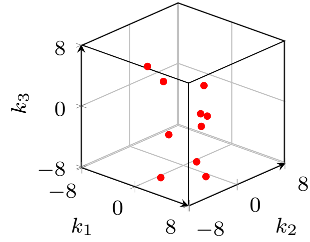
frequency candidates

$\xleftarrow[\text{iFFT}]{\text{1-dim}}$



sampling nodes
$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

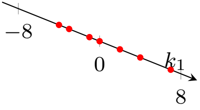
Dimension incremental reconstruction - Method



$\operatorname{supp}\hat{p} \subset \Gamma = \hat{G}_8^3$

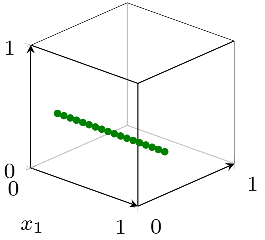
$$\tilde{\hat{p}}_{k_1} := \frac{1}{17}\sum_{\ell=0}^{16} p\left(\begin{pmatrix}\ell/17\\ x_2'\\ x_3'\end{pmatrix}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k_1}{17}}$$
$$= \sum_{\substack{(h_2,h_3)\in\{-8,\ldots,8\}^2\\ (k_1,h_2,h_3)^\top\in\operatorname{supp}\hat{p}}} \hat{p}_{\begin{pmatrix}k_1\\h_2\\h_3\end{pmatrix}} \mathrm{e}^{2\pi\mathrm{i}(h_2x_2'+h_3x_3')},$$

$k_1 = -8, \ldots, 8$



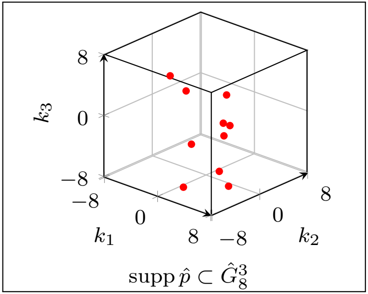
detected frequencies $I^{(1)}$

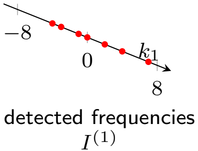
1-dim
$\longleftarrow$
iFFT



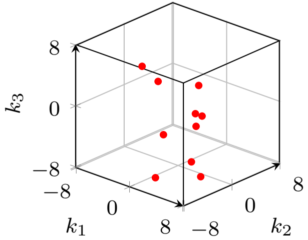
sampling nodes
$(\frac{\ell}{17}, x_2', x_3')^\top$, $\ell = 0, \ldots, 16$

Dimension incremental reconstruction - Method

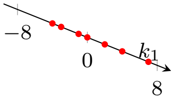




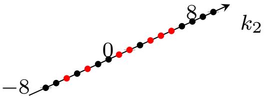
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

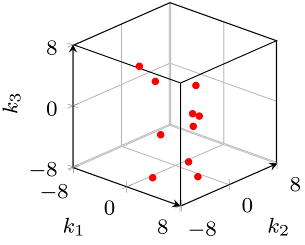


detected frequencies
$I^{(1)}$

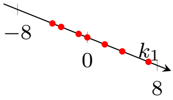


frequency candidates

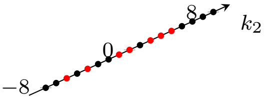
Dimension incremental reconstruction - Method



$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$

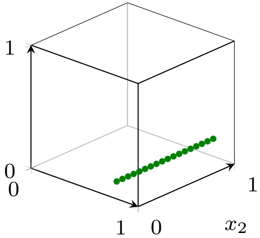


detected frequencies
$I^{(1)}$



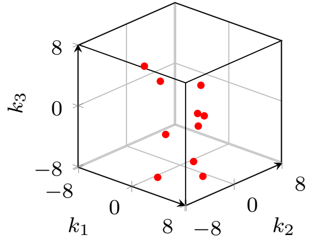
frequency candidates

construct
$\longrightarrow$
sampling set

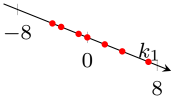


sampling nodes

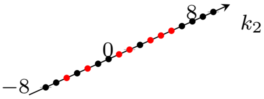
Dimension incremental reconstruction - Method



$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

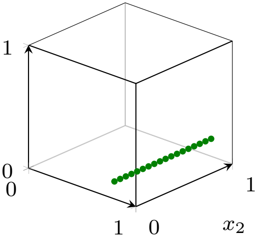


detected frequencies
$I^{(1)}$



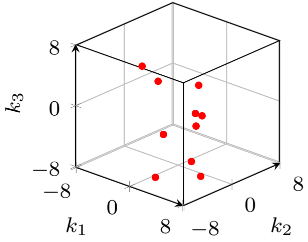
frequency candidates

1-dim
←
iFFT

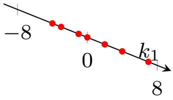


sampling nodes

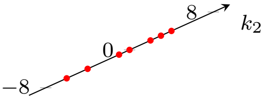
Dimension incremental reconstruction - Method



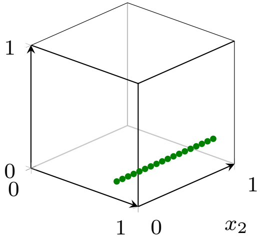
$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$



detected frequencies $I^{(1)}$

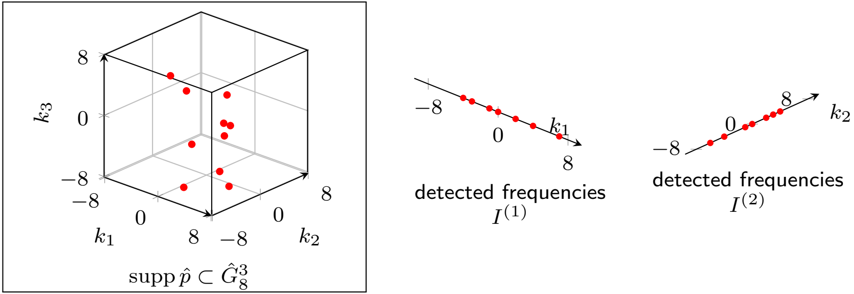


detected frequencies $I^{(2)}$

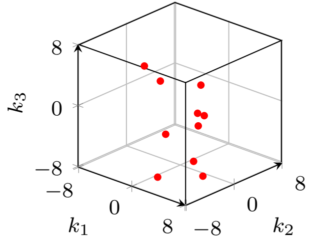


sampling nodes

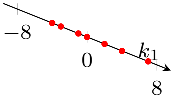
Dimension incremental reconstruction - Method



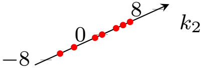
Dimension incremental reconstruction - Method



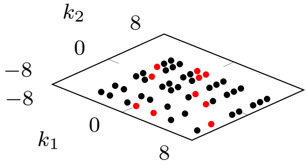
$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$



detected frequencies
$I^{(1)}$

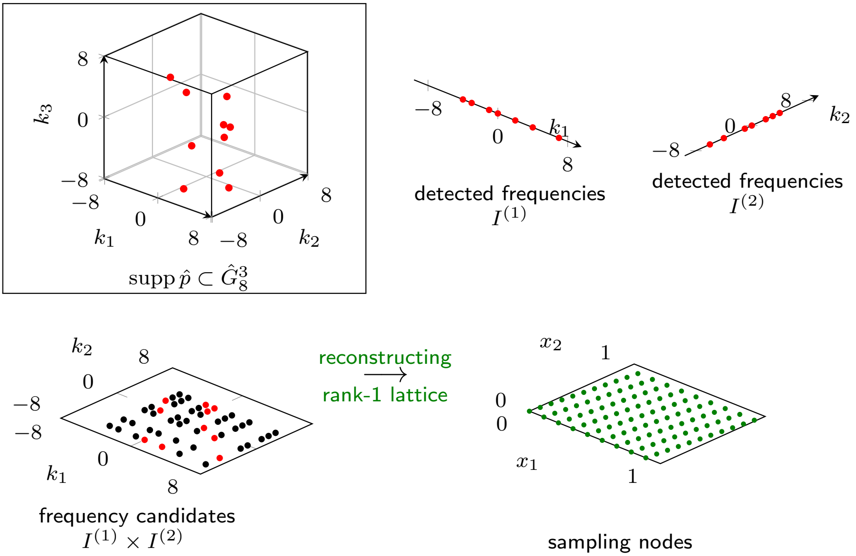


detected frequencies
$I^{(2)}$

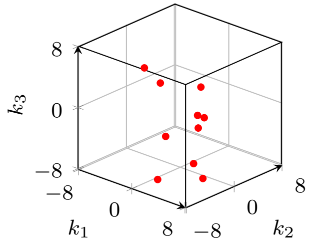


frequency candidates
$I^{(1)} \times I^{(2)}$

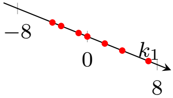
Dimension incremental reconstruction - Method



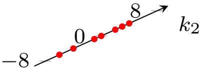
Dimension incremental reconstruction - Method



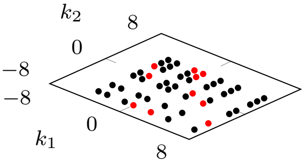
$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$



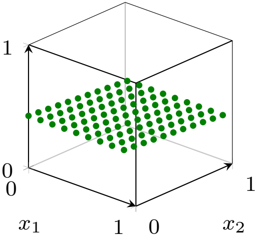
detected frequencies
$I^{(1)}$



detected frequencies
$I^{(2)}$

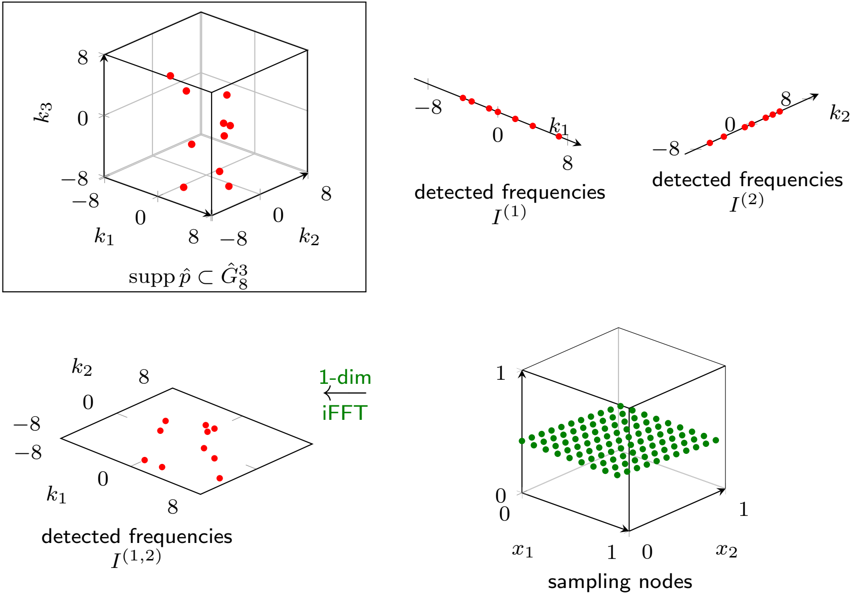


frequency candidates
$I^{(1)} \times I^{(2)}$

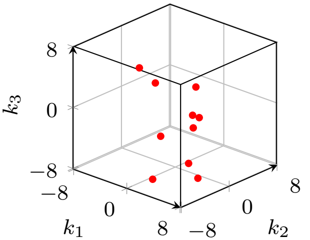


sampling nodes

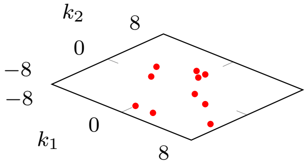
Dimension incremental reconstruction - Method



Dimension incremental reconstruction - Method

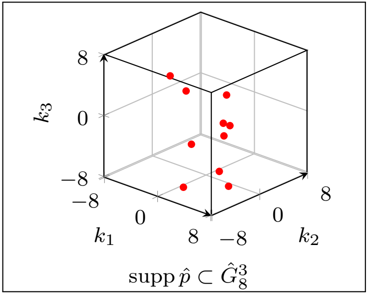


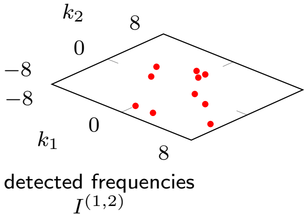
$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$

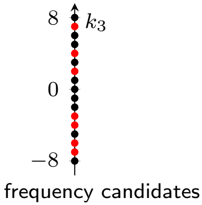


detected frequencies $I^{(1,2)}$

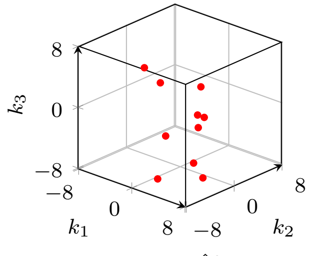
Dimension incremental reconstruction - Method



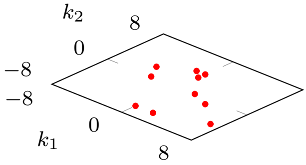




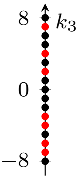
Dimension incremental reconstruction - Method



$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$

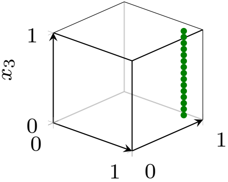


detected frequencies $I^{(1,2)}$



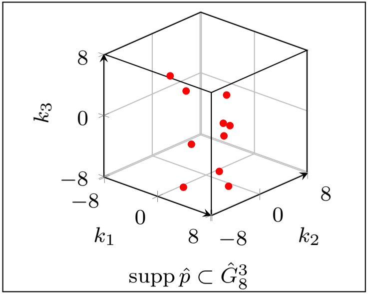
frequency candidates

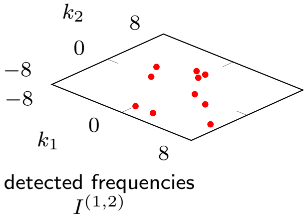
construct $\longrightarrow$ sampling set

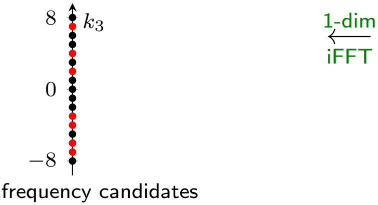


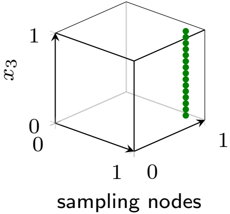
sampling nodes

Dimension incremental reconstruction - Method

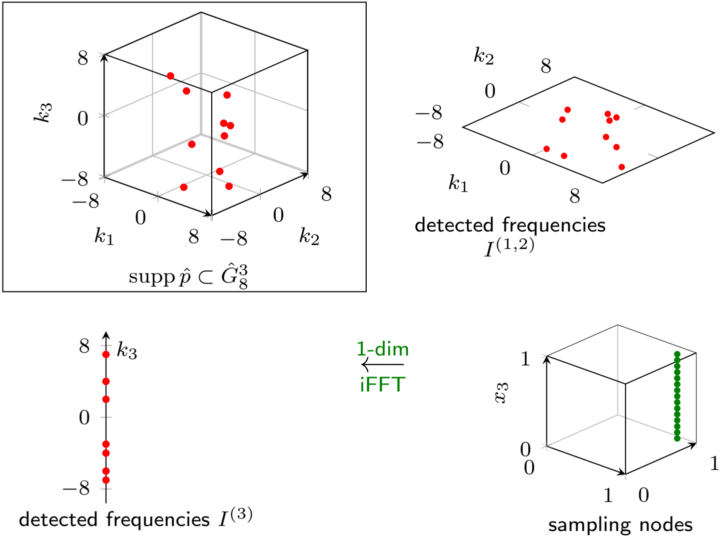




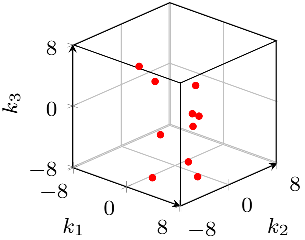




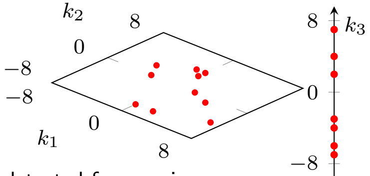
Dimension incremental reconstruction - Method



Dimension incremental reconstruction - Method



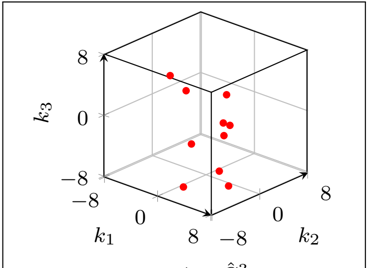
$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$



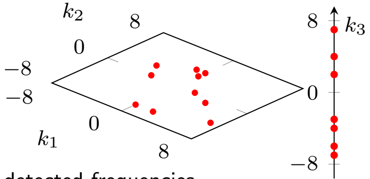
detected frequencies $I^{(1,2)}$

$I^{(3)}$

Dimension incremental reconstruction - Method

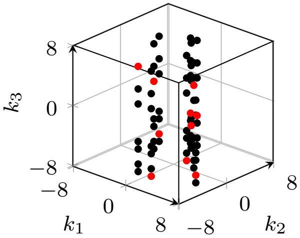


$\operatorname{supp}\hat{p} \subset \hat{G}_8^3$



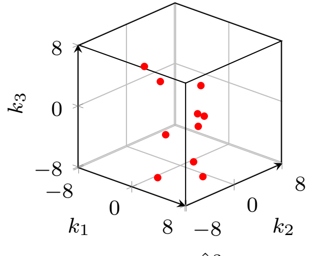
detected frequencies
$I^{(1,2)}$

$I^{(3)}$

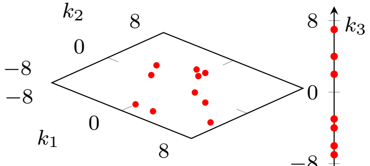


frequency candidates
$I^{(1,2)} \times I^{(3)}$

Dimension incremental reconstruction - Method

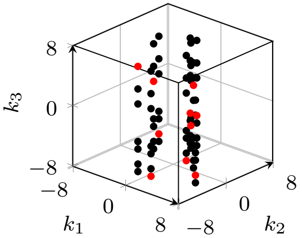


$\operatorname{supp} \hat{p} \subset \hat{G}_8^3$



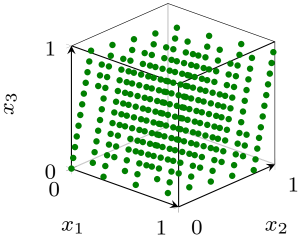
detected frequencies
$I^{(1,2)}$

$I^{(3)}$



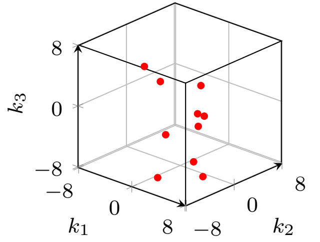
frequency candidates
$I^{(1,2)} \times I^{(3)}$

reconstructing
$\longrightarrow$
rank-1 lattice



sampling nodes

Dimension incremental reconstruction - Method



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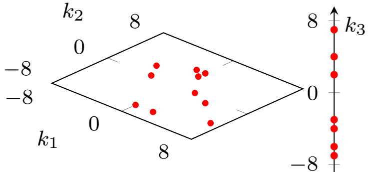

detected frequencies $I^{(1,2)}$

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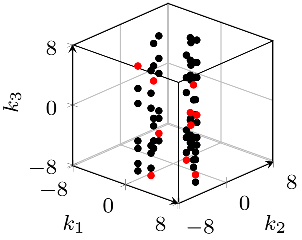

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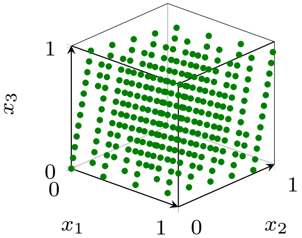
1-dim $\longleftarrow$ iFFT



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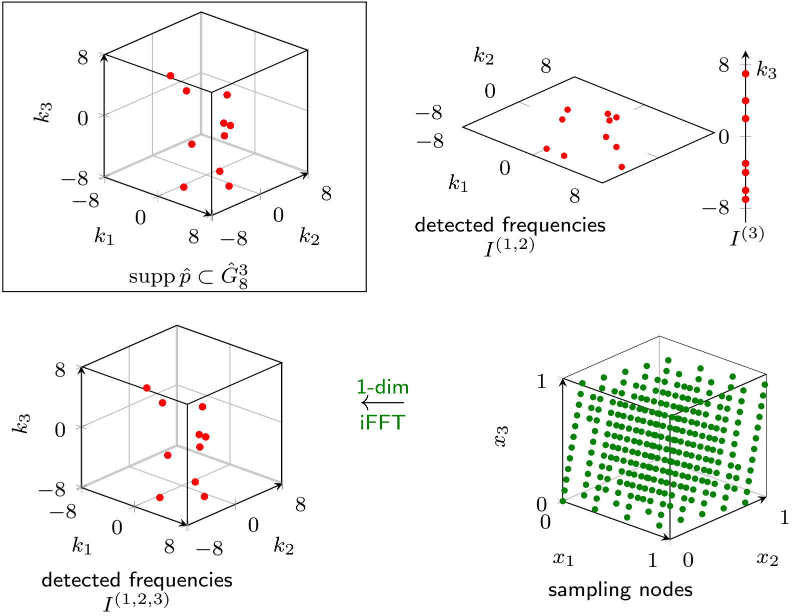

Dimension incremental reconstruction - Example

- B-spline $N_m(x) := \sum_{k\in\mathbb{Z}} C_m \operatorname{sinc}\left(\frac{\pi}{m}k\right)^m \cos(\pi k)\,\mathrm{e}^{2\pi\mathrm{i}kx}$, $\|N_m|L^2(\mathbb{T})\| = 1$, $|\hat{N}_m(k)| \sim |k|^{-m}$
- $f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} N_2(x_t) + \prod_{t\in\{2,5,6,10\}} N_4(x_t) + \prod_{t\in\{4,7,9\}} N_6(x_t)$

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- results for dimension incremental algorithm with $\Gamma = \hat{G}_{64}^{10}$ (tests repeated 10 times):

threshold	#samples	$\lvert I\rvert$	rel. L_2-error
1.0e-02	254 530	491	1.4e-01
1.0e-03	2 789 050	1 121	1.1e-02
1.0e-04	17 836 042	3 013	1.7e-03
1.0e-05	82 222 438	7 163	4.7e-04

Summary

approximate reconstruction of high-dimensional periodic functions $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ by sampling along rank-1 lattice nodes

- perfectly stable, computation only uses single 1-dim iFFT + scalar products for arbitrary index set I_N^d
- samples: $\mathcal{O}\left(|\mathrm{I}_N^d|^2\right)$, arithm. complexity: $\mathcal{O}\left(|\mathrm{I}_N^d|^2 \log |\mathrm{I}_N^d|\right)$
- oversampling factor up to $|\mathrm{I}_N^d|$
- observed oversampling factor lower for realistic problem sizes
- theoretical estimates for approximation error
- numerical tests encourage theoretical results

Kämmerer, L., Potts, D., Volkmer, T.
Approximation of multivariate functions by trigonometric polynomials based on rank-1 lattice sampling.
DFG-Schwerpunktprogramm 1324, Preprint 145, 2013.
(http://www.tu-chemnitz.de/∼potts)

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Potts, D., Volkmer, T.
Sparse high-dimensional FFT based on rank-1 lattice sampling.
DFG-Schwerpunktprogramm 1324, Preprint 171, 2014.
(http://www.tu-chemnitz.de/∼potts)