Hypercyclic, mixing, and chaotic semigroups generated by first order differential operators

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First example of hypercyclic C_0-semigroup due to Rolewicz (1969): $a > 1,\ \mu_a$ the Borel measure on $\mathbb{R}$ with Lebesgue density $\rho(x) = a^{-|x|}$
$\Rightarrow$ the closure of

$$B : C_c^1(\mathbb{R}) \to L^p(\mathbb{R}, \mu_a), f \mapsto f'$$

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$1 \leq p < \infty$.
(T is the left translation semigroup $T(t)f = f(\cdot + t)$)

$\rho : \mathbb{R} \to (0,\infty)$ measurable, μ Borel measure on $\mathbb{R}$ with Lebesgue density ρ.

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generates C_0-semigroup T on $L^p(\mathbb{R},\mu)\,(1 \leq p < \infty)$ iff

$$\exists\, M \geq 1, \omega \in \mathbb{R} \;\; \forall t \geq 0 : \;\; \rho(x) \leq Me^{t\omega}\rho(x+t)$$

for almost all $x \in \mathbb{R}$.

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T hypercyclic $\Leftrightarrow\ \forall x \in \mathbb{R}\, \exists (t_n)_{n\in\mathbb{N}} \uparrow \infty :\ \lim_{n\to\infty} \rho(x \pm t_n) = 0$

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$$T \text{ mixing} \Leftrightarrow\;\; \lim_{t\to\pm\infty} \rho(t) = 0$$

Matsui, Takeo, Yamada (2003):

$$T \text{ chaotic} \Leftrightarrow\;\; \forall x \in \mathbb{R}\; \exists t > 0 :\;\; \sum_{k\in\mathbb{Z}} \rho(x + kt) < \infty$$

Questions

Given $\Omega \subset \mathbb{R}^d$ open, $F : \Omega \to \mathbb{R}^d$ continuous.
Moreover, μ locally finite, regular Borel measure on Ω.
When does the "operator"

$$Af := \sum_{i=1}^{d} F_i \frac{\partial}{\partial x_i} f = \langle F, \nabla f \rangle$$

generate a C_0-semigroup T on $L^p(\Omega, \mu)\,(1 \leq p < \infty)$?
When is T hypercyclic, mixing, or chaotic?

Assume that for each $x \in \Omega$ there is unique solution $t \mapsto \varphi_x(t) =: \varphi(t,x)$ of the initial value problem

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which is defined for all $t \geq 0$.

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$\Rightarrow T(0)f = f$ and because $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$ we have $T(t+s)f = T(t)T(s)f$ and for fixed $x \in \Omega$

$$\frac{d}{dt}T(t)f(x)|_{t=0} = \sum_{j=1}^{d} F_j(x)\frac{\partial}{\partial x_j}f(x).$$

Good candidate for the C_0-semigroup T generated by

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From now on we make the general assumption that this is the case.
($\Rightarrow$ $\varphi(t,\cdot)$ is one-to-one and continuous $\forall\, t \geq 0$)

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For $t \geq 0$ we define the Borel measure μ_t on Ω as

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Theorem

For $1 \leq p < \infty$ the following are equivalent:

i) The family of mappings

$$T(t) : L^p(\Omega,\mu) \to L^p(\Omega,\mu), f \mapsto f(\varphi(t,\cdot)),\ t \geq 0,$$

is well-defined and defines a C_0-semigroup.

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ii) μ_t has a μ-density $g_t \in L^\infty(\mu)$ for each $t \geq 0$ and there are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|g_t\|_\infty \leq Me^{t\omega}$ for all $t \geq 0$.

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$\Rightarrow$ $\varphi(t,\cdot)$ continuously differentiable for all $t\geq 0$;
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Assume μ has Lebesgue density ρ
$\Rightarrow$ μ_t has Lebesgue density
$\rho_t(x)=1\!\!1_{\varphi(t,\Omega_t)}(x)\rho(\varphi(-t,x))|\det D_x\varphi(-t,x)|$

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Corollary

Let $F : \Omega \to \mathbb{R}^d$ be continuously differentiable, $\rho : \Omega \to (0, \infty)$ and $\mu(dx) = \rho(x)\lambda(dx)$. Then, the following are equivalent:

i) μ is admissible for F

ii) There are $M \geq 1, \omega \in \mathbb{R}$ such that for all $t \geq 0$

$$\rho(x) \leq Me^{t\omega}\rho(\varphi(t,x))|\det D_x\varphi(t,x)| \text{ for almost all } x \in \Omega$$

Theorem

Let μ be admissible for F and T the C_0-semigroup on $L^p(\Omega, \mu)$ given by $T(t)f = f(\varphi(t, \cdot))$.

i) The generator A of T is an extension of

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i) The generator A of T is an extension of

$$B : C_c^1(\Omega) \to L^p(\Omega, \mu), f \mapsto \langle F, \nabla f \rangle.$$

ii) If F is continuously differentiable and if for every $x \in \Omega$ the unique solution $t \mapsto \varphi(t, x)$ of

$$y' = F(y), y(0) = x$$

not only is defined for all $t \geq 0$ but for all $t \in \mathbb{R}$ then the generator A of T is given by the closure of

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Already introduced the Borel measures

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Closed subsets of Ω generate the Borel σ-algebra over $\Omega, \varphi(t,\cdot)$ one-to-one $\Rightarrow \varphi(t,B)$ is Borel subset of Ω whenever B is.

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$$\mu_{-t}(B) := \mu(\varphi(t,B))$$

is well-defined Borel measure on Ω

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i) T is hypercyclic.

ii) T is weakly mixing.

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iii) $\forall\, K \subset \Omega$ compact $\exists\, (L_n)_{n\in\mathbb{N}}$ measurable subsets of K, $(t_n)_{n\in\mathbb{N}} \in [0, \infty)^{\mathbb{N}}$:

$$\lim_{n\to\infty} \mu(K \backslash L_n) = 0$$

and

$$\lim_{n\to\infty} \mu_{t_n}(L_n) = \lim_{n\to\infty} \mu_{-t_n}(L_n) = 0.$$

Corollary (Case $d = 1$)

$\Omega \subset \mathbb{R}$, F continuously differentiable. Let for every $x \in \Omega$ the unique solution $t \mapsto \varphi(t, x)$ of

$$y' = F(y), y(0) = x$$

not only is defined for all $t \geq 0$ but for all $t \in \mathbb{R}$.
Let $\rho : \Omega \to (0, \infty)$ be measurable such that $d\mu = \rho\, d\lambda$ is admissible for F.
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i) T given by $T(t)f = f(\varphi(t, \cdot))$ is hypercyclic on $L^p(\Omega, \mu)$

ii) If $\Omega_1, \ldots, \Omega_m$ are different components of $\Omega \backslash \{F = 0\}$ and $x_j \in \Omega_j$, there is $(t_n)_{n \in \mathbb{N}}$ such that for all $j = 1, \ldots, m$

$$\lim_{n \to \infty} \rho(\varphi(t_n, x_j)) \frac{\partial}{\partial x} \varphi(t_n, x_j) = 0$$

$$\lim_{n \to \infty} \rho(\varphi(-t_n, x_j)) \frac{\partial}{\partial x} \varphi(-t_n, x_j) = 0.$$

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$\Omega = \mathbb{R}$, $F \equiv 1 \Rightarrow \varphi(t,x) = x + t$, $\frac{\partial}{\partial x}\varphi \equiv 1$
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$\Rightarrow T$ hypercyclic on $L^p(\mathbb{R}, \rho\, d\lambda)$ iff for each $x \in \mathbb{R}$ there is $(t_n)_{n\in\mathbb{N}}$

$$\rho(\varphi(t_n,x))\frac{\partial}{\partial x}\varphi(t_n,x) = \rho(x+t_n) \to 0$$
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generates a hypercyclic C_0-semigroup on $L^p(\mathbb{R},\mu)$ iff for all $(x_1,x_2) \in (-\infty,0)\times(0,\infty)$ there is $(t_n)_{n\in\mathbb{N}}$ such that

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(take e.g. $\rho(x) := \frac{1}{\pi}(1+x^2)^{-1}$)

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$\Omega \subset \mathbb{R}$, F continuously differentiable. Let for every $x \in \Omega$ the unique solution $t \mapsto \varphi(t,x)$ of

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$$y' = F(y), y(0) = x$$

not only is defined for all $t \geq 0$ but for all $t \in \mathbb{R}$.
Let $\rho : \Omega \to (0, \infty)$ be measurable such that $d\mu = \rho\, d\lambda$ is admissible for F.
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ii) If $\Omega_1, \ldots, \Omega_m$ are different components of $\Omega \backslash \{F = 0\}$ and $x_j \in \Omega_j$, there is $t > 0$ such that for all $j = 1, \ldots, m$

$$\sum_{k \in \mathbb{Z}} \rho(\varphi(kt, x_j)) \left| \frac{\partial}{\partial x} \varphi(kt, x_j) \right| < \infty$$

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replace measures μ_t, μ_{-t} by

$$\mu_{p,t}(B) = \int_{\varphi(t,\cdot)^{-1}(B)} h_t^p \, d\mu$$

and

$$\mu_{p,-t}(B) := \int_{\varphi(t,B)} h_t(\varphi(-t,\cdot))^{-p} \, d\mu \quad \text{resp.}$$

where $h_t(x) := \exp(\int_0^t h(\varphi(s,x))\, ds)$

- theorem characterizing hypercyclicity/transitivity valid for arbitrary families of (weighted) composition operators on $L^p(X,\mu)$ with X locally compact, σ-compact, Hausdorff

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- analogue results for spaces of continuous functions $C_{0,\rho}(\Omega)$