Some results on partial differential operators with a single characteristic direction

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Workshop on
functional analysis and operator theory

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1. Surjectivity of differential operators on $\mathscr{E}$ and $\mathscr{D}'$

2. An approximation theorem of Runge type

3. The linear topological invariant (Ω) for kernels

Surjectivity of differential operators on $\mathscr{E}$ and $\mathscr{D}'$

For $P \in \mathbb{C}[X_1, \ldots, X_d]$ set $P(D) := P(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_d})$, let $X \subseteq \mathbb{R}^d$ be open.

Malgrange, 1956: $P(D) : \mathscr{E}(X) \to \mathscr{E}(X)$ surjective $\Leftrightarrow$ X is P-convex for supports, i.e. $\forall\, K \subseteq X$ compact $\exists\, \tilde{K} \subseteq X$ compact

$$\forall\, u \in \mathscr{E}'(X) : \big(\operatorname{supp} \check{P}(D)u \subseteq K \Rightarrow \operatorname{supp} u \subseteq \tilde{K}\big),$$

where $\check{P}(\xi) := P(-\xi)$.

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Convex sets are P-convex for supports for every $P \neq 0$.

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Convex sets are P-convex for supports for every $P \neq 0$.

$P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$, $\deg(P) = m$, *elliptic* $:\Leftrightarrow$
$\forall\, \xi \in \mathbb{R}^d \backslash \{0\} : 0 \neq P_m(\xi) := \sum_{|\alpha| = m} a_\alpha \xi^\alpha$ (P_m *principal part* of P)

For elliptic P every open $X \subseteq \mathbb{R}^d$ is P-convex for supports.

Hörmander, 1962: $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective $\Leftrightarrow$ X is P-convex for supports and P-convex for singular supports, the latter meaning that $\forall\, K \subseteq X$ compact $\exists\, \tilde{K} \subseteq X$ compact

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Convex sets are P-convex for singular supports for every $P \neq 0$.

P *hypoelliptic* $:\Leftrightarrow \forall\, X \subseteq \mathbb{R}^d$ open, $u \in \mathscr{D}'(X) : \big(P(D)u = 0 \Rightarrow u \in \mathscr{E}(X)\big)$

For hypoelliptic P every open $X \subseteq \mathbb{R}^d$ is P-convex for singular supports.

P elliptic $\Rightarrow$ P hypoelliptic

Necessary conditions for P-convexity properties:
$f : X \to \mathbb{R}$ *satisfies the minimum principle in a (fixed) closed subset* F *of* $\mathbb{R}^d$ if for every compact set $K \subseteq F \cap X$ we have

$$\inf_{x \in K} f(x) = \inf_{\partial_F K} f(x),$$

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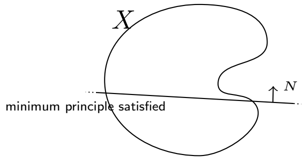
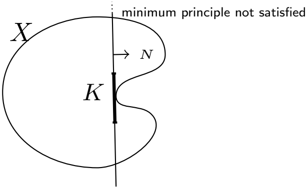
We set $d_X : X \to \mathbb{R}, x \mapsto \text{dist}(x, X^c)$, the *boundary distance of* X.

X is P-convex for supports $\Rightarrow d_X$ satisfies the minimum principle in every characteristic hyperplane H for P, i.e. in

$$H_{N,\beta} = \{x \in \mathbb{R}^d;\, \langle N, x\rangle = \beta\}\, (\beta \in \mathbb{R}, N \in \mathbb{R}^d, |N| = 1, P_m(N) = 0)$$

where $P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ for $P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha, \text{deg}(P) = m$.





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$$0 = \sigma_P(V) := \inf_{t\geq 1} \liminf_{x\in\mathbb{R}^d, |x|\to\infty} \frac{\sup_{\xi\in V, |\xi|\leq t} |P(x+\xi)|}{\sup_{\xi\in\mathbb{R}^d, |\xi|\leq t} |P(x+\xi)|}$$

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$\sigma_P(\xi) := \sigma_P(\mathrm{span}\{\xi\}), \xi \in \mathbb{R}^d \Rightarrow \sigma_P$ homogeneous of degree 0;
P hypoelliptic $\Leftrightarrow \forall\, N \in \mathbb{R}^d, |N| = 1 :\ \sigma_P(N) \neq 0$

Theorem

Given P with principal part P_m, $X \subseteq \mathbb{R}^d$ open. Let $W \subsetneq \mathbb{R}^d$ be a subspace such that d_X satisfies the minimum principle in $x + W^\perp$ for every $x \in \mathbb{R}^d$.

i) If $\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} \subseteq W$ then X is P-convex for supports.

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Sketch of proof: X P-convex for [singular] supports $\Leftrightarrow$

$$\forall\, u \in \mathscr{E}'(X) : \mathrm{dist}([\mathrm{sing}]\mathrm{supp}\, u, \mathbb{R}^d \backslash X) \geq \mathrm{dist}([\mathrm{sing}]\mathrm{supp}\, \check{P}(D)u, \mathbb{R}^d \backslash X)$$

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d_X satisfies minimum principle in $x + W^\perp \Rightarrow \exists \alpha : [0, T] \to X$ cont. piecewise affine:

$\alpha 1)$ $\alpha(0) = x$,

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By $\alpha 2) + \alpha 3)\, \exists\, \varepsilon > 0 :$ $[\text{sing}]\text{supp}\, u \cap B(\alpha(T), \varepsilon) = \emptyset$, $K \cap \big(\alpha([0,T]) + B(0, \varepsilon)\big) = \emptyset$.

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By $\alpha 4)$: $\forall\, \beta \in \mathbb{R}, N \in \mathbb{R}^d, |N| = 1, P_m(N) = 0\, [\sigma_P(N) = 0]$:

$$\Big(H_{N,\beta} \cap \big(\alpha([0,T]) + B(0,\varepsilon)\big) \neq \emptyset \Rightarrow H_{N,\beta} \cap B(\alpha(T),\varepsilon) \neq \emptyset\Big).$$

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$$\Big(H_{N,\beta} \cap \big(\alpha([0,T]) + B(0,\varepsilon)\big) \neq \emptyset \Rightarrow H_{N,\beta} \cap B(\alpha(T), \varepsilon) \neq \emptyset\Big).$$

Consequence of Holmgren's uniqueness theorem [Hörmander's continuation of differentiability theorem]: $[\text{sing}]\text{supp}\, u \cap B(\alpha(0), \varepsilon) = \emptyset$, by $\alpha 1)\, x \notin [\text{sing}]\text{supp}\, u$ ☺

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Corollary

i) Assume $\exists\, N \in \mathbb{R}^d, |N| = 1 : \{\xi \in \mathbb{R}^d; P_m(\xi) = 0\} = \text{span}\{N\}$. Then X is P-convex for supports iff

$$\forall\, \beta \in \mathbb{R} : \; d_X \text{ satisfies minimum principle in } H_{N,\beta},$$

ii) Assume $\exists\, N \in \mathbb{R}^d, |N| = 1 : \{\xi \in \mathbb{R}^d; \sigma_P(\xi) = 0\} = \text{span}\{N\}$. Then $X \subseteq \mathbb{R}^d$ is P-convex for singular supports iff

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An approximation theorem of Runge type

We equip $\mathscr{E}(X)$ with its natural locally convex topology generated by the seminorms

$$\forall\, K \subseteq X \text{ compact}, l \in \mathbb{N}_0 : \; \|f\|_{l,K} := \sup_{|\alpha| \leq l} \sup_{x \in K} |\partial^\alpha f(x)|.$$

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Malgrange 1956: P hypoelliptic $\Rightarrow \mathscr{E}_P(X) = \mathscr{D}'_P(X)$ as locally convex spaces and therefore: topology of $\mathscr{E}_P(X)$ is generated by the seminorms $\{\|\cdot\|_{0,K};\ K \subseteq X \text{ compact}\}$, i.e. the compact-open topology.

Lax-Malgrange Theorem

For $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ open and P elliptic the following are equivalent.

i) The restriction map $r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1}$ has dense range.

ii) X_2 does not contain a compact connected component of $\mathbb{R}^d \backslash X_1$.

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$d = 2, P(D) = \frac{1}{2}(\partial_1 + i\partial_2)$ gives Runge's Approximation Theorem.

Lax-Malgrange Theorem

For $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ open and P elliptic the following are equivalent.

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Consider the class of differential operators $P(D)$ for which

$$\exists\, N \in \mathbb{R}^d, |N| = 1 : \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} = \text{span}\{N\}$$

which contains e.g.

- $P(D) = i\frac{\partial}{\partial t} + \Delta_x$ with $(x,t) = (x_1, \ldots, x_{d-1}, t) \in \mathbb{R}^d$ (time-dependent free Schrödinger operator),
- $P(D) = \frac{\partial}{\partial t} - Q(D_x)$ with $(x,t) = (x_1, \ldots, x_{d-1}, t) \in \mathbb{R}^d$ and elliptic $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ of degree ≥ 2 (non-degenerate parabolic operators).

When has

$$r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1} \text{ resp. } r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$$

dense range?

Theorem

Given P with $\exists\, N \in \mathbb{R}^d, |N| = 1 : \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} = \text{span}\{N\}$ and let $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ be open and P-convex for supports.
Assume

$$\forall\, \beta \in \mathbb{R} : \big(C \text{ compact connected component of } \mathbb{R}^d \backslash X_1 \cap H_{N,\beta} \Rightarrow C \nsubseteq X_2\big),$$

where $H_{N,\beta} = \{x \in \mathbb{R}^d;\, \langle N, x\rangle = \beta\}$. Then, both restriction maps

$$r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1} \text{ and } r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$$

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Corollary

Let

- $P(D) = i\frac{\partial}{\partial t} + \Delta_x$ with $(x,t) = (x_1, \ldots, x_{d-1}, t) \in \mathbb{R}^d$ or
- $P(D) = \frac{\partial}{\partial t} - Q(D_x)$ with $(x,t) = (x_1, \ldots, x_{d-1}, t) \in \mathbb{R}^d$ and elliptic $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ of degree ≥ 2.

Moreover, let $Y_1 \subseteq Y_2 \subseteq \mathbb{R}^{d-1}$, $I \subseteq \mathbb{R}$ be open such that Y_2 does not contain a compact connected component of $\mathbb{R}^{d-1} \backslash Y_1$ then

$$r_{\mathscr{E}} : \mathscr{E}_P(Y_2 \times I) \to \mathscr{E}_P(Y_1 \times I) \text{ and } r_{\mathscr{D}'} : \mathscr{D}'_P(Y_2 \times I) \to \mathscr{D}'_P(Y_1 \times I)$$

have dense range.

The linear topological invariant (Ω) for kernels

Let $P(D) : \mathscr{E}(X) \to \mathscr{E}(X)$ be surjective.
Given a locally convex space F, is $P(D) : \mathscr{E}(X, F) \to \mathscr{E}(X, F)$ surjective?

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"Yes" if F is a Fréchet space (Grothendieck).

In general "No" for $F = E'_b$ being the strong dual of a Fréchet space E (Vogt, 1983).

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Let E be a Fréchet space with a fundamental sequence of seminorms $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$, for $y \in E', n \in \mathbb{N}$, let $\|y\|_n^* := \sup_{x \in E, \|x\|_n \leq 1} |\langle y, x \rangle|$.
E has (Ω) :⇔

$$\forall\, k \in \mathbb{N}\, \exists\, l \geq k\, \forall\, n \geq l\, \exists \lambda \in (0,1), C > 0 :\ \|\cdot\|_l^* \leq C \|\cdot\|_k^{*\,\lambda}\, \|\cdot\|_n^{*\,1-\lambda}.$$

$\mathscr{E}_P(X)$ has (Ω) if

- P is elliptic, X arbitrary (Vogt, 1983)
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For $\alpha, \mathbf{m} \in \mathbb{N}_0^d$ define $|\alpha : \mathbf{m}| := \sum_{j=1}^d \alpha_j/m_j$; P is called *semi-elliptic* if it is possible to write

$$P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha \xi^\alpha$$

such that $\forall\, \xi \in \mathbb{R}^d \backslash \{0\}:\ P^0(\xi) := \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha \neq 0$.

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P elliptic $\Rightarrow$ P semi-elliptic $\Rightarrow$ P hypoelliptic

Examples: $P(\xi) = \sum_{j=1}^{d-1} \xi_j^2 - i\xi_d$, more general $P(\xi) = Q(\xi_1, \ldots, \xi_{d-1}) - i\xi_d$ with $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ elliptic, $\deg(Q) \geq 2$.

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P semi-elliptic $\Leftrightarrow$ $\check{P}$ semi-elliptic

Theorem

Let P be semi-elliptic with principal part P_m and let $X \subseteq \mathbb{R}^d$ be open.
If d_X satisfies the minimum principle in $x + \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}^\perp$ for every $x \in \mathbb{R}^d$ then X is P-convex for supports and $\mathscr{E}_P(X)$ has (Ω).

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Corollary

Let P be semi-elliptic such that $\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} = \mathrm{span}\{N\}$ with $|N| = 1$. Tfae

i) X is P-convex for supports.

ii) X is P-convex for supports and $\mathscr{E}_P(X)$ has (Ω).

iii) $\forall\, \beta \in \mathbb{R} :\ d_X$ satisfies the minimum principle in $H_{N,\beta} = \{x \in \mathbb{R}^d;\, \langle N, x\rangle = \beta\}$.

Representation of $\mathscr{E}_P(X)'$ for hypoelliptic P, X P-convex for supports due to Grothendieck-Köthe enables for a nice interpretation of the above corollary:
E be a fixed temperate fundamental solution for $\check{P}(D)$, $K \subseteq \mathbb{R}^d$ compact

- $u \in \mathscr{E}_{\check{P}}(\mathbb{R}^d \backslash K)$ *regular at infinity w.r.t.* E iff for one (then every) $\psi \in \mathscr{E}(\mathbb{R}^d)$ with $\operatorname{supp} \psi \cap K = \emptyset$ and $\operatorname{supp}(1-\psi)$ compact:
$$E * \check{P}(D)(\psi u) = \psi u.$$

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- $(K_n)_{n \in \mathbb{N}}$ be a compact exhaustion of X. There is $m \in \mathbb{N}_0$ s.th.
$$\forall\, n \in \mathbb{N}, u \in R(\mathbb{R}^d \backslash K_n) : \ \|u\|_{n+1}^* := \sup_{x \in \mathbb{R}^d \backslash K_{n+1}} |u(x)|(1+|x|^2)^{-m} < \infty$$

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$R_n := \{u \in R(\mathbb{R}^d \backslash K_n);\, \|u\|_n^* < \infty\}\big(\supseteq R(\mathbb{R}^d \backslash K_{n-1})\big)$ equipped with $\|\cdot\|_n^*$ is Banach, $\varinjlim_{n\to\infty}(R_n, \|\cdot\|_n^*) \cong \mathscr{E}_P(X)'$ topologically, $(\|\cdot\|_n^*)_{n\in\mathbb{N}}$ dual norms of a fundamental sequence of seminorms on $\mathscr{E}_P(X)$.

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(Ω) for $\mathscr{E}_P(X)$ thus means $\forall\, k \in \mathbb{N}\, \exists\, l \geq k\, \forall\, n \geq l\, \exists \lambda \in (0,1), C > 0\, \forall u \in \cup_r R_r$:
$$\sup_{x \in \mathbb{R}^d \backslash K_l} |u(x)|(1+|x|^2)^{-m} \leq$$
$$C\Big(\sup_{x \in \mathbb{R}^d \backslash K_k} |u(x)|^{\lambda}(1+|x|^2)^{-\lambda m}\Big)\Big(\sup_{x \in \mathbb{R}^d \backslash K_n} |u(x)|^{1-\lambda}(1+|x|^2)^{-(1-\lambda)m}\Big).$$

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