Chaotic C_0-semigroups induced by semiflows in Lebesgue and Sobolev spaces

Thomas Kalmes

Trier University

Based on joint work with E. Mangino (Universitá del Salento) and J. Aroza (Universitat Politécnica de Valencia)

Workshop on Functional Analysis Valencia 2013
On the occasion of the 60th birthday of Andreas Defant

Valencia, 3-6 June, 2013

X separable Banach space, T C_0-semigroup on X

T *hypercyclic* $:\Leftrightarrow \exists x \in X$: $\mathrm{orb}(T,x) := \{T(s)x; s \geq 0\}$ dense in X

$\mathrm{HC}(T) := \{x \in X; \overline{\mathrm{orb}(T,x)} = X\}$

X separable Banach space, T C_0-semigroup on X

T *hypercyclic* $:\Leftrightarrow \exists x \in X$: $\mathrm{orb}(T,x) := \{T(s)x; s \geq 0\}$ dense in X

$\mathrm{HC}(T) := \{x \in X; \overline{\mathrm{orb}(T,x)} = X\}$

$\mathrm{Per}(T) := \{x \in X; \exists s > 0 : T(s)x = x\}$

T *chaotic* $:\Leftrightarrow$ T hypercyclic and $\overline{\mathrm{Per}(T)} = X$

X separable Banach space, T C_0-semigroup on X

T *hypercyclic* $:\Leftrightarrow \exists x \in X\colon \mathrm{orb}(T,x) := \{T(s)x; s \geq 0\}$ dense in X

$\mathrm{HC}(T) := \{x \in X; \overline{\mathrm{orb}(T,x)} = X\}$

$\mathrm{Per}(T) := \{x \in X; \exists s > 0 : T(s)x = x\}$

T *chaotic* $:\Leftrightarrow$ T hypercyclic and $\overline{\mathrm{Per}(T)} = X$

T *frequently hypercyclic* $:\Leftrightarrow \exists\, x \in \mathrm{HC}(T)\,\forall\,\emptyset \neq U \subseteq X$ open:

$$\liminf_{t\to\infty} \frac{\lambda(\{s \in [0,t];\ T(s)x \in U\})}{t} > 0.$$

Let S be a C_0-semigroup on the Banach space Y.

T and S are *conjugate* if there is a homeomorphism $\Phi : X \to Y$ such that

$$\forall\, r \geq 0 : \Phi \circ T(r) = S(r) \circ \Phi.$$

Let S be a C_0-semigroup on the Banach space Y.

T and S are *conjugate* if there is a homeomorphism $\Phi : X \to Y$ such that

$$\forall\, r \geq 0 : \Phi \circ T(r) = S(r) \circ \Phi.$$

T and S conjugate $\Rightarrow$ T is (frequently) hypercyclic/chaotic iff S is (frequently) hypercyclic/chaotic.

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \tag{1}$$

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \tag{1}$$

General assumption: $\forall\, x \in \Omega :\ [0, \infty) \subseteq J(x)$
(for all initial values the solution is defined for any $t \geq 0$

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \tag{1}$$

General assumption: $\forall\, x \in \Omega :\ [0, \infty) \subseteq J(x)$
(for all initial values the solution is defined for any $t \geq 0 \Leftrightarrow: (\Omega, F)$ has A. De.f.an.t

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \tag{1}$$

General assumption: $\forall\, x \in \Omega :\ [0, \infty) \subseteq J(x)$
(for all initial values the solution is defined for any $t \geq 0 \Leftrightarrow$: (Ω, F) has A. De.f.an.$t \Leftrightarrow$: (Ω, F) has A. D.)

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \tag{1}$$

General assumption: $\forall\, x \in \Omega :\ [0, \infty) \subseteq J(x)$
(for all initial values the solution is defined for any $t \geq 0 \Leftrightarrow$: (Ω, F) has A. De.f.an.t $\Leftrightarrow$: (Ω, F) has A. D.)

$\Omega = (a, b)$ bounded, $F \in C^1([a, b])$ with $F(a) \geq 0, F(b) \leq 0$
$\Rightarrow (\Omega, F)$ has A. D.

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \tag{1}$$

General assumption: $\forall\, x \in \Omega :\ [0, \infty) \subseteq J(x)$
(for all initial values the solution is defined for any $t \geq 0 \Leftrightarrow$: (Ω, F) has A. De.f.an.t $\Leftrightarrow$: (Ω, F) has A. D.)

Denoting solution of (1) by $\varphi(\cdot, x)$ we have:

$\forall\, s \geq 0 : \varphi(s, \cdot) : \Omega \to \Omega$ injective

$\forall\, s \geq 0, x \in \varphi(s, \Omega) :\ [-s, \infty) \subseteq J(x)$ and $\varphi(-s, x) = \varphi(s, \cdot)^{-1}(x)$

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \tag{1}$$

General assumption: $\forall\, x \in \Omega :\ [0, \infty) \subseteq J(x)$
(for all initial values the solution is defined for any $t \geq 0 \Leftrightarrow: (\Omega, F)$ has A. De.f.an.$t \Leftrightarrow: (\Omega, F)$ has A. D.)

Denoting solution of (1) by $\varphi(\cdot, x)$ we have:

$\forall\, s \geq 0 : \varphi(s, \cdot) : \Omega \to \Omega$ injective

$\forall\, s \geq 0, x \in \varphi(s, \Omega) :\ [-s, \infty) \subseteq J(x)$ and $\varphi(-s, x) = \varphi(s, \cdot)^{-1}(x)$

$\forall\, x \in \Omega, s, r \in J(x) :\ \varphi(s + r, x) = \varphi(s, \varphi(r, x))$

$\Omega \subseteq \mathbb{R}$ open interval, $F \in C^1(\Omega)$ real valued, $x \in \Omega$

Let $J(x)$ be the domain of the unique maximal solution of

$$\frac{d\varphi}{dt} = F(\varphi),\ \varphi(0) = x \qquad (1)$$

General assumption: $\forall\, x \in \Omega :\ [0, \infty) \subseteq J(x)$
(for all initial values the solution is defined for any $t \geq 0$ $\Leftrightarrow$: (Ω, F) has A. De.f.an.t $\Leftrightarrow$: (Ω, F) has A. D.)

Denoting solution of (1) by $\varphi(\cdot, x)$ we have:

$\forall\, s \geq 0 : \varphi(s, \cdot) : \Omega \to \Omega$ injective

$\forall\, s \geq 0, x \in \varphi(s, \Omega) :\ [-s, \infty) \subseteq J(x)$ and $\varphi(-s, x) = \varphi(s, \cdot)^{-1}(x)$

$\forall\, x \in \Omega, s, r \in J(x) :\ \varphi(s + r, x) = \varphi(s, \varphi(r, x))$

$\forall\, s \geq 0 :\ \varphi(s, \cdot) : \Omega \to \Omega$ and $\varphi(-s, \cdot) : \varphi(s, \Omega) \to \Omega$ are C^1

$h \in C(\Omega)$, set for $s \geq 0$

$$h_s : \Omega \to \mathbb{C}, h_s(x) := \exp(\int_0^s h(\varphi(t,x))dt) \ (\Rightarrow |h_s| = 1 \text{ if } \operatorname{Re} h = 0)$$

$h \in C(\Omega)$, set for $s \geq 0$

$$h_s : \Omega \to \mathbb{C}, h_s(x) := \exp(\int_0^s h(\varphi(t,x))dt) \; (\Rightarrow |h_s| = 1 \text{ if } \operatorname{Re} h = 0)$$

For $f : \Omega \to \mathbb{C}$ set

$$\forall s \geq 0, x \in \Omega : \qquad (T(s)f)(x) := h_s(x) f(\varphi(s,x))$$
$$\Rightarrow \; (T(s+r)f)(x) = (T(s)T(r)f)(x), \; (T(0)f)(x) = f(x)$$

$h \in C(\Omega)$, set for $s \geq 0$

$$h_s : \Omega \to \mathbb{C}, h_s(x) := \exp(\int_0^s h(\varphi(t,x))dt) \; (\Rightarrow |h_s| = 1 \text{ if } \operatorname{Re} h = 0)$$

For $f : \Omega \to \mathbb{C}$ set

$$\forall s \geq 0, x \in \Omega : \; u(s,x) := (T(s)f)(x) := h_s(x) f(\varphi(s,x))$$
$$\Rightarrow \; (T(s+r)f)(x) = (T(s)T(r)f)(x), \; (T(0)f)(x) = f(x)$$

Note: if $f, h \in C^1(\Omega)$ then

$$\frac{\partial}{\partial s} u(s,x) = F(x) \frac{\partial}{\partial x} u(s,x) + h(x) u(s,x), \; u(0,x) = f(x).$$

Let (Ω, F) have A. D. , $h \in C(\Omega)$.
For $\rho : \Omega \to (0, \infty)$ measurable, $1 \leq p < \infty$, $L^p_\rho(\Omega) := L^p(\Omega, \rho\, d\lambda)$
$(L^p(\Omega) := L^p_1(\Omega))$.

Let (Ω, F) have A. D. , $h \in C(\Omega)$.
For $\rho : \Omega \to (0,\infty)$ measurable, $1 \leq p < \infty$, $L^p_\rho(\Omega) := L^p(\Omega, \rho\, d\lambda)$ $(L^p(\Omega) := L^p_1(\Omega))$.

ρ is called p-admissible for F and h if there are $M \geq 1, \omega \in \mathbb{R}$ with

$$\forall\, s \geq 0\, \forall\, x \in \Omega : \, |h_s(x)|^p \rho(x) \leq M e^{\omega s} \rho(\varphi(s,x)) |\partial_2 \varphi(s,x)|$$

$\Rightarrow T_{F,h}(s) f := T(s) f := h_s(\cdot) f(\varphi(s,\cdot))$ defines a C_0-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$, its generator being an extension of

$$C^1_c(\Omega) \to L^p_\rho(\Omega), u \mapsto F u' + h u.$$

Let (Ω, F) have A. D. , $h \in C(\Omega)$.
For $\rho : \Omega \to (0, \infty)$ measurable, $1 \leq p < \infty$, $L^p_\rho(\Omega) := L^p(\Omega, \rho\, d\lambda)$ $(L^p(\Omega) := L^p_1(\Omega))$.

ρ is called p-admissible for F and h if there are $M \geq 1, \omega \in \mathbb{R}$ with

$$\forall\, s \geq 0\, \forall\, x \in \Omega : \; |h_s(x)|^p \rho(x) \leq M e^{\omega s} \rho(\varphi(s, x)) |\partial_2 \varphi(s, x)|$$

$\Rightarrow T_{F,h}(s)f := T(s)f := h_s(\cdot) f(\varphi(s, \cdot))$ defines a C_0-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$, its generator being an extension of

$$C^1_c(\Omega) \to L^p_\rho(\Omega), u \mapsto Fu' + hu.$$

ρ p-admissible for F and $h \Leftrightarrow \rho\, p$-admissible for F and $\mathrm{Re}\, h$!

$(h_s(x) = \exp(\int_0^s h(\varphi(t, x))\, dt))$

Let (Ω, F) have A. D. , $h \in C(\Omega)$.
For $\rho : \Omega \to (0, \infty)$ measurable, $1 \leq p < \infty$, $L^p_\rho(\Omega) := L^p(\Omega, \rho\, d\lambda)$ ($L^p(\Omega) := L^p_1(\Omega)$).

ρ is called p-admissible for F and h if there are $M \geq 1, \omega \in \mathbb{R}$ with

$$\forall\, s \geq 0\, \forall\, x \in \Omega : \; |h_s(x)|^p \rho(x) \leq M e^{\omega s} \rho(\varphi(s,x)) |\partial_2 \varphi(s,x)|$$

$\Rightarrow T_{F,h}(s)f := T(s)f := h_s(\cdot) f(\varphi(s,\cdot))$ defines a C_0-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$, its generator being an extension of

$$C^1_c(\Omega) \to L^p_\rho(\Omega), u \mapsto Fu' + hu.$$

ρ p-admissible for F and $h \Leftrightarrow$ ρ p-admissible for F and $\mathrm{Re}\, h$!

$\rho = 1$ is p-admissible (for any $1 \leq p < \infty$) if $\mathrm{Re}\, h$ is bounded above and F' is bounded below.

Let (Ω, F) have A. D. , $h \in C(\Omega)$, and ρ p-admissible for F and h.

Let (Ω, F) have A. D. , $h \in C(\Omega)$, and ρ p-admissible for F and h.

For $s \geq 0$ set $\rho_{s,p} : \Omega \to [0, \infty)$,

$$\rho_{s,p}(x) := \chi_{\varphi(s,\Omega)}(x)|h_s(\varphi(-s,x))|^p \rho(\varphi(-s,x))|\partial_2\varphi(-s,x)|$$

and $\rho_{-s,p} : \Omega \to [0, \infty)$,

$$\rho_{-s,p}(x) := |h_s(x)|^{-p} \rho(\varphi(s,x))|\partial_2\varphi(s,x)|$$

Let (Ω, F) have A. D. , $h \in C(\Omega)$, and ρ p-admissible for F and h.

For $s \geq 0$ set $\rho_{s,p} : \Omega \to [0, \infty)$,

$$\rho_{s,p}(x) := \chi_{\varphi(s,\Omega)}(x)|h_s(\varphi(-s,x))|^p \rho(\varphi(-s,x))|\partial_2\varphi(-s,x)|$$

and $\rho_{-s,p} : \Omega \to [0, \infty)$,

$$\rho_{-s,p}(x) := |h_s(x)|^{-p} \rho(\varphi(s,x))|\partial_2\varphi(s,x)|$$

$\mathrm{Re}\, h = 0 \Rightarrow |h_s| = 1 \Rightarrow \forall\, 1 \leq p < \infty :\ \rho_{s,p} = \rho_{s,1}, \rho_{-s,p} = \rho_{-s,1}$

$\rho_{s,1}$ is λ-density of the image measure $\left(\rho\, d\lambda\right)^{\varphi(s,\cdot)}$

Theorem (Aroza, Mangino, K.)

Let (Ω, F) have A. D. , $h \in C(\Omega)$ real valued, ρ p-admissible for F and h. Tfae.

i) $T_{F,h}$ is chaotic in $L^p_\rho(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are m different connected components $C_1, \dots, C_m$ of $\Omega \backslash \{F = 0\}$, for λ^m-almost every $(x_1, \dots, x_m) \in \Pi_{j=1}^m C_j$ there is $s > 0$ such that
$$\forall\, 1 \leq j \leq m : \sum_{l \in \mathbb{Z}} \rho_{ls,p}(x_j) < \infty.$$

Relaxing "$h \in C(\Omega)$ real valued":

Lemma (Aroza, Mangino, K.)

Let (Ω, F) have A. D. , $h \in C(\Omega)$, and ρ p-admissible for F and h. Assume that

a) $\forall x \in \{F = 0\} : h(x) \in \mathbb{R}$, b) $\frac{\operatorname{Im} h}{F} \in L^1(\Omega)$.

Then the C_0-semigroups $T_{F,h}$ and $T_{F,\operatorname{Re} h}$ on $L^p_\rho(\Omega)$ are conjugate (via a continuous, bijective multiplication operator).

Relaxing "$h \in C(\Omega)$ real valued":

Lemma (Aroza, Mangino, K.)

Let (Ω, F) have A. D. , $h \in C(\Omega)$, and ρ p-admissible for F and h. Assume that

a) $\forall\, x \in \{F = 0\} : h(x) \in \mathbb{R}$, b) $\frac{\text{Im}\, h}{F} \in L^1(\Omega)$.

Then the C_0-semigroups $T_{F,h}$ and $T_{F,\text{Re}\, h}$ on $L^p_\rho(\Omega)$ are conjugate (via a continuous, bijective multiplication operator).

Corollary (Aroza, Mangino, K.)

... Tfae.

i) $T_{F,h}$ is chaotic in $L^p_\rho(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and for any m different components $C_1, \dots,$ $C_m \subseteq \Omega\backslash\{F = 0\}$, for λ^m-almost all $(x_1, \dots, x_m) \in \Pi_{j=1}^m C_j$

$$\exists\, s > 0\, \forall\, 1 \leq j \leq m : \sum_{l \in \mathbb{Z}} \rho_{ls,p}(x_j) < \infty.$$

Lemma

Let (Ω, F) have A. D. , ρ p-admissible for F and $h \in C(\Omega)$.

a) If F' and $\mathrm{Re}\, h$ are bounded the following are equivalent for $x \in \Omega \backslash \{F = 0\}$.

   i) $\exists s > 0 : \sum_{l \in \mathbb{Z}} \rho_{ls,p}(x) < \infty$.
   ii) $\forall s > 0 : \sum_{l \in \mathbb{Z}} \rho_{ls,p}(x) < \infty$.
   iii) $\int_{\mathbb{R}} \rho_{s,p}(x)\, d\lambda(s) < \infty$.

Lemma

Let (Ω, F) have A. D. , ρ p-admissible for F and $h \in C(\Omega)$.

a) If F' and $\operatorname{Re} h$ are bounded the following are equivalent for $x \in \Omega \backslash \{F = 0\}$.
   i) $\exists s > 0 : \sum_{l \in \mathbb{Z}} \rho_{ls,p}(x) < \infty$.
   ii) $\forall s > 0 : \sum_{l \in \mathbb{Z}} \rho_{ls,p}(x) < \infty$.
   iii) $\int_{\mathbb{R}} \rho_{s,p}(x)\, d\lambda(s) < \infty$.

b) For $x \in \Omega \backslash \{F = 0\}$ denote by $C(x)$ the connected component of $\Omega \backslash \{F = 0\}$ containing x. Then

$$\int_{\mathbb{R}} \rho_{s,p}(x)\, d\lambda(s) = \frac{1}{|F(x)|} \int_{C(x)} \exp(-p \int_x^y \frac{\operatorname{Re} h(w)}{F(w)}\, dw) \rho(y)\, d\lambda(y).$$

Theorem

Let (Ω, F) have A. D. , $h \in C(\Omega)$ with F' and $\mathrm{Re}\, h$ bounded, $\frac{\mathrm{Im}\, h}{F} \in L^1(\Omega), h(x) \in \mathbb{R}$ for every $x \in \{F = 0\}$.
Moreover, let ρ be p-admissible for F and h. Tfae.

i) $T_{F,h}$ is chaotic in $L^p_\rho(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and for every connected component C of $\Omega\backslash\{F = 0\}$ there is $x \in C$ with

$$\int_C \exp(-p \int_x^y \frac{\mathrm{Re}\, h(w)}{F(w)}\, dw)\rho(y)\, d\lambda(y) < \infty.$$

Theorem

Let (Ω, F) have A. D. , $h \in C(\Omega)$ with F' and $\operatorname{Re} h$ bounded, $\frac{\operatorname{Im} h}{F} \in L^1(\Omega), h(x) \in \mathbb{R}$ for every $x \in \{F = 0\}$.
Moreover, let ρ be p-admissible for F and h. Tfae.

i) $T_{F,h}$ is chaotic in $L^p_\rho(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and for every connected component C of $\Omega \backslash \{F = 0\}$ there is $x \in C$ with

$$\int_C \exp(-p \int_x^y \frac{\operatorname{Re} h(w)}{F(w)} \, dw) \rho(y) \, d\lambda(y) < \infty.$$

In case of $p = 1$ or $\operatorname{Re} h = 0$ the above are also equivalent to

iii) $T_{F,h}$ satisfies the Frequent Hypercyclicity Criterion.

From now on $\Omega = (a, b)$ bounded open interval, $1 \leq p < \infty$.

$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ u' \in L^p(\Omega)\}$

$W_*^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega);\ u(a) = 0\}$

$\Rightarrow W^{1,p}(\Omega) = W_*^{1,p}(\Omega) \oplus \text{span}\{1\}$ topological direct sum

From now on $\Omega = (a, b)$ bounded open interval, $1 \leq p < \infty$.

$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ u' \in L^p(\Omega)\}$

$W^{1,p}_*(\Omega) := \{u \in W^{1,p}(\Omega);\ u(a) = 0\}$

$\Rightarrow W^{1,p}(\Omega) = W^{1,p}_*(\Omega) \oplus \text{span}\{1\}$ topological direct sum

$\Phi : L^p(\Omega) \to W^{1,p}_*(\Omega), \Phi(f)(x) := \int_a^x f(y)dy$ is continuous linear, bijective with continuous inverse $\Phi^{-1}(u) = u'$.

Let $((a,b),F)$ have A. D. with $F \in C^1([a,b]), F(a) = 0, \gamma \in \mathbb{R}$
$\Rightarrow \rho = 1$ p-admissible for F and $F' + \gamma$.

Let $((a,b),F)$ have A. D. with $F\in C^1([a,b]),F(a)=0,\gamma\in\mathbb{R}$
$\Rightarrow \rho=1$ p-admissible for F and $F'+\gamma$.

$\Rightarrow T_{F,F'+\gamma}$ well-defined C_0-semigroup in $L^p(\Omega)$

Let $((a,b),F)$ have A. D. with $F \in C^1([a,b]), F(a)=0, \gamma \in \mathbb{R}$
$\Rightarrow \rho = 1$ p-admissible for F and $F' + \gamma$.

$\Rightarrow T_{F,F'+\gamma}$ well-defined C_0-semigroup in $L^p(\Omega)$

$\Rightarrow S_{F,\gamma} := (\Phi \circ T_{F,F'+\gamma}(r) \circ \Phi^{-1})_{r\geq 0}$ defines C_0-semigroup in $W_*^{1,p}(\Omega)$ which is chaotic/(frequently) hypercyclic iff $T_{F,F'+\gamma}$ is

Let $((a,b),F)$ have A. D. with $F \in C^1([a,b]), F(a) = 0, \gamma \in \mathbb{R}$
$\Rightarrow \rho = 1$ p-admissible for F and $F' + \gamma$.

$\Rightarrow T_{F,F'+\gamma}$ well-defined C_0-semigroup in $L^p(\Omega)$

$\Rightarrow S_{F,\gamma} := (\Phi \circ T_{F,F'+\gamma}(r) \circ \Phi^{-1})_{r\geq 0}$ defines C_0-semigroup in $W_*^{1,p}(\Omega)$ which is chaotic/(frequently) hypercyclic iff $T_{F,F'+\gamma}$ is

$\forall\, x \in \Omega, r \geq 0, f \in W_*^{1,p}(\Omega) : (S_{F,\gamma}(r)f)(x) = e^{\gamma r} f(\varphi(r,x))$

Let $((a,b),F)$ have A. D. with $F \in C^1([a,b]), F(a)=0, \gamma \in \mathbb{R}$
$\Rightarrow \rho = 1$ p-admissible for F and $F' + \gamma$.

$\Rightarrow T_{F,F'+\gamma}$ well-defined C_0-semigroup in $L^p(\Omega)$

$\Rightarrow S_{F,\gamma} := (\Phi \circ T_{F,F'+\gamma}(r) \circ \Phi^{-1})_{r\geq 0}$ defines C_0-semigroup in $W^{1,p}_*(\Omega)$ which is chaotic/(frequently) hypercyclic iff $T_{F,F'+\gamma}$ is

$\forall\, x \in \Omega, r \geq 0, f \in W^{1,p}_*(\Omega) : (S_{F,\gamma}(r)f)(x) = e^{\gamma r} f(\varphi(r,x))$

$W^{1,p}(\Omega) = W^{1,p}_*(\Omega) \oplus \text{span}\{1\}$, $e^{\gamma r}\,(1 \circ \varphi(r,\cdot)) = e^{\gamma r}\,1$

$\Rightarrow (S_{F,\gamma}(r)f)(x) = e^{\gamma r} f(\gamma(r,x))$ defines C_0-semigroup $S_{F,\gamma}$ in $W^{1,p}(\Omega)$

Theorem (Aroza, Mangino, K.)

Let $F \in C^1([a,b])$ with $F(a) = 0, (\Omega = (a,b), F)$ have A. D. , and $\gamma \in \mathbb{R}$. The generator of the C_0-semigroup $S_{F,\gamma}$ in $W^{1,p}(\Omega)$ is given by

$$A : \{u \in W^{1,p}(\Omega); Fu'' \in L^p(\Omega)\} \to W^{1,p}(\Omega), Au = Fu' + \gamma u$$

and $W_*^{1,p}(\Omega)$ is $S_{F,\gamma}$-invariant.
$S_{F,\gamma}$ is not hypercyclic in $W^{1,p}(\Omega)$.

Theorem (Aroza, Mangino, K.)

Let $F \in C^1([a,b])$ with $F(a) = 0, (\Omega = (a,b), F)$ have A. D. , and $\gamma \in \mathbb{R}$. The generator of the C_0-semigroup $S_{F,\gamma}$ in $W^{1,p}(\Omega)$ is given by

$$A : \{u \in W^{1,p}(\Omega); Fu'' \in L^p(\Omega)\} \to W^{1,p}(\Omega), Au = Fu' + \gamma u$$

and $W^{1,p}_*(\Omega)$ is $S_{F,\gamma}$-invariant.
$S_{F,\gamma}$ is not hypercyclic in $W^{1,p}(\Omega)$.

$h \in W^{1,\infty}(\Omega) \Rightarrow M_h : W^{1,p}(\Omega) \to W^{1,p}(\Omega), M_h u := hu$
continuous, linear

Theorem (Aroza, Mangino, K.)

Let $F \in C^1([a,b])$ with $F(a) = 0, (\Omega = (a,b), F)$ have A. D. , and $\gamma \in \mathbb{R}$. The generator of the C_0-semigroup $S_{F,\gamma}$ in $W^{1,p}(\Omega)$ is given by

$$A : \{u \in W^{1,p}(\Omega); Fu'' \in L^p(\Omega)\} \to W^{1,p}(\Omega), Au = Fu' + \gamma u$$

and $W^{1,p}_*(\Omega)$ is $S_{F,\gamma}$-invariant.
$S_{F,\gamma}$ is not hypercyclic in $W^{1,p}(\Omega)$.

$h \in W^{1,\infty}(\Omega) \Rightarrow M_h : W^{1,p}(\Omega) \to W^{1,p}(\Omega), M_h u := hu$ continuous, linear $\Rightarrow A + M_h$ generates C_0-semigroup $S_{F,h+\gamma}$ given by $(S_{F,h+\gamma}(r)f)(x) = e^{\gamma r} h_r(x) f(\varphi(r,x))$

Lemma (Aroza, Mangino, K.)

Let $F \in C^1([a,b])$ with $F(a) = 0, (\Omega = (a,b), F)$ have A. D. , and $h \in W^{1,\infty}(\Omega)$. Assume that

i) $\forall\, x \in \{F = 0\} :\ h(x) = h(a) \in \mathbb{R}$,

ii) $\frac{h-h(a)}{F} \in L^\infty(\Omega)$.

Then the C_0-semigroups $S_{F,h}$ and $S_{F,h(a)}$ in $W^{1,p}(\Omega)$, resp. $W_*^{1,p}(\Omega)$, are conjugate (via a continuous, bijective multiplication operator).

Theorem

Let $F \in C^1([a,b])$ with $F(a) = 0, (\Omega = (a,b), F)$ have A. D. , and $h \in W^{1,\infty}(\Omega)$.
Assume $\frac{h-h(a)}{F} \in L^\infty(\Omega)$ and $h(x) = h(a) \in \mathbb{R}$ for $x \in \{F = 0\}$.
Tfae.

i) $S_{F,h}$ is chaotic in $W_*^{1,p}(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and $\forall\, C \subseteq \Omega \backslash \{F = 0\}$ connected component $\exists\, x \in C$:

$$\int_C \exp(-p \int_x^y \frac{F'(w) + h(a)}{F(w)} dw) d\lambda(y) < \infty.$$

Moreover, if $p = 1$ or $h(a) = 0$, the above are also equivalent to

iii) $S_{F,h}$ satisfies the Frequent Hypercyclicity Criterion in $W_*^{1,p}(\Omega)$.

References

- J. Aroza, E. Mangino, T. Kalmes, *Chaotic C_0-semigroups induced by semiflows in Lebesgue and Sobolev spaces*, preprint, submitted to J. Math. Anal. Appl.
- T. Kalmes, *Hypercyclic, mixing, and chaotic C_0-semigroups induced by semiflows*, Ergod. Th. Dynam. Sys. (2007), 27, 1599-1631.
- E. Mangino, A. Peris, *Frequently hypercyclic semigroups*, Stud. Math. (2011), 202, 227-242.