On P-convexity and P-Runge pairs for certain linear partial differential operators

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For arbitrary $P \in \mathbb{C}[X_1, \dots, X_d] \backslash \{0\}, d \geq 2$, consider

$$P(D) := P(-i\nabla) = P(-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_d});$$

e.g. $\Delta = P(D)$ with $P(\xi) = -|\xi|^2 = -\sum_{j=1}^d \xi_j^2$.

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Study $P(D)$ on (subspaces) of $\mathscr{D}'(X), X \subseteq \mathbb{R}^d$ open:

i) Given f, is $P(D)u = f$ in X solvable?

ii) Properties of $\{u \in \mathscr{D}'(X);\ P(D)u = 0\}$ (existence of Schauder bases, approximability etc.)?

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Motivation for i) with P of arbitrary order: Solve a linear system of pde's:

$$\begin{pmatrix} P_{1,1}(D) & \ldots & P_{1,n}(D) \\ \vdots & \ddots & \vdots \\ P_{n,1}(D) & \ldots & P_{n,n}(D) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$\rightsquigarrow$ i) for $P(D)$ with $P(x_1, \ldots, x_d) := \det\big(P_{l,k}(x_1, \ldots, x_d)\big)_{1 \leq l,k \leq n}$

1. General solvability of linear partial differential equations and P-convexity
2. An approximation theorem of Runge type
3. Vogt's property (Ω) for kernels of hypoelliptic operators

General solvability of linear partial differential equations and P-convexity

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Notation: For $P \in \mathbb{C}[X_1, \ldots, X_d]$ set $\check{P}(\xi) := P(-\xi)$.

$\mathscr{D}(X)$ smooth functions with compact support in X

$\mathscr{E}'(X)$ distributions with compact support in X

Theorem (Malgrange, 1956)

Tfae:

i) $P(D) : C^\infty(X) \to C^\infty(X)$ is surjective.

ii) X is P-convex for supports, i.e.

$$\forall\, K \Subset X\, \exists\, \tilde{K} \Subset X\, \forall\, u \in \mathscr{E}'(X) : \left(\operatorname{supp} \check{P}(D)u \subseteq K \Rightarrow \operatorname{supp} u \subseteq \tilde{K}\right).$$

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Theorem of supports: $\forall\, u \in \mathscr{E}'(X):\ \operatorname{conv}\big(\operatorname{supp} u\big) = \operatorname{conv}\big(\operatorname{supp} \check{P}(D)u\big)$
$\Rightarrow$ every convex X is P-convex for supports (Recall: $P \neq 0$!)

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Theorem (Ehrenpreis): $\forall\, u \in \mathscr{E}'(X) :\ \text{conv}\big(\text{sing supp}\, u\big) = \text{conv}\big(\text{sing supp}\, \check{P}(D)u\big)$
$\Rightarrow$ every convex X is P-convex for singular supports (Recall: $P \neq 0$!)
(and P-convex for supports)

Evaluation of P-convexity properties?

For P tfae:

i) Every open $X \subseteq \mathbb{R}^d$ is P-convex for supports.

ii) P is elliptic, i.e. if $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha, \sum_{|\alpha|=m} |a_\alpha| > 0$, then

$$\forall\, \xi \in \mathbb{R}^d \backslash \{0\}; 0 \neq P_m(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha \,(\textit{principal part of } P).$$

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For $d = 2$ there is a geometric characterization of P-convexity for singular supports (K., 2011) which proves a conjecture of Tréves (1966):

$$X (\subseteq \mathbb{R}^2)\, P\text{-convex for supports} \Rightarrow X\, P\text{-convex for singular supports.}$$

Necessary conditions for P-convexity properties (Hörmander):

Notation: $f : X \to \mathbb{R}$ *satisfies the minimum principle in a closed set* $F \subseteq \mathbb{R}^d$ $:\Leftrightarrow$ for every compact set $K \subseteq F \cap X$ we have

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Let $d_X : X \to \mathbb{R}, x \mapsto \mathsf{dist}(x, \mathbb{R}^d \backslash X)$ be the *boundary distance of* X.

X is P-convex for supports $\Rightarrow$ d_X satisfies the minimum principle in every characteristic hyperplane H for P, i.e. in

$$H = x + \left(\mathsf{span}\{N\}\right)^{\perp} \; (x \in \mathbb{R}^d, N \in \mathbb{R}^d \backslash \{0\}, P_m(N) = 0).$$

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$$x + (\mathrm{span}\{N\})^\perp \ (x \in \mathbb{R}^d, N \in \mathbb{R}^d\backslash\{0\}, \sigma_P(N) = 0),$$

where

$$\sigma_P(N) = \inf_{t\geq 1} \liminf_{|\xi|\to\infty} \frac{\sup\{|P(\xi+\alpha N)|;\ \alpha\in\mathbb{R}, |\alpha|\leq t\}}{\sup\{|P(\xi+x)|;\ x\in\mathbb{R}^d, |x|\leq t\}} (\in [0,1]).$$

Theorem

Let $W \subsetneq \mathbb{R}^d$ be a subspace such that d_X satisfies the minimum principle in $x + W^\perp$ for every $x \in \mathbb{R}^d$.

i) If $\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} \subseteq W$ then X is P-convex for supports.

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Corollary

Let $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ be elliptic, $\deg(Q) =: m > r \in \mathbb{N}_0$, $c \in \mathbb{C}$, and set $P(x_1, \ldots, x_d) := Q(x_1, \ldots, x_{d-1}) - c(ix_d)^r$ so that $P(D) = Q(D_{d-1}) - c\frac{\partial^r}{\partial x_d^r}$.

Then, for $X \subseteq \mathbb{R}^d$ open, tfae:

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In particular, the operator

$$P(D) = Q(D_{d-1}) - c\frac{\partial^r}{\partial x_d^r} : C^\infty(Y \times I) \to C^\infty(Y \times I)$$

is surjective for every open $Y \subseteq \mathbb{R}^{d-1}, I \subseteq \mathbb{R}$.

An approximation theorem of Runge type

Runge's Approximation Theorem

For $X_1 \subseteq X_2 \subseteq \mathbb{C}$ open the following are equivalent.

i) For every $g \in \mathscr{H}(X_1)$, for every compact $K \subseteq X_1$, and for every $\varepsilon > 0$ there is $f \in \mathscr{H}(X_2)$ such that

$$\varepsilon > \sup_{z \in K} |f(z) - g(z)| =: \|f - g\|_{0,K},$$

i.e. $r : \mathscr{H}(X_2) \to \mathscr{H}(X_1), f \mapsto f_{|X_1}$ has dense range when $\mathscr{H}(X_1)$ is equipped with the compact-open topology (topology of local uniform convergence); (X_1, X_2) is a *Runge pair*.

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Lax-Malgrange Theorem: Precisely the same equivalence is true for $\mathbb{R}^d$ instead of $\mathbb{C}$ and holomorphic functions $\mathscr{H}(X)$ replaced by

$$C_P^\infty(X) := \{u \in C^\infty(X);\ P(D)u = 0\},$$

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What about non-elliptic P? First, we equip $C_P^\infty(X)$ and $\mathscr{D}'_P(X)$ with topologies:

Equip $C^\infty(X)$ with topology generated by the seminorms

$$\|\cdot\|_{l,K}: C^\infty(X) \to [0,\infty), f \mapsto \max_{\alpha\in\mathbb{N}_0^d, |\alpha|\leq l} \max_{x\in K} |\partial^\alpha f(x)| \, (l \in \mathbb{N}_0, K \Subset X)$$

$\Rightarrow$ $C^\infty(X)$ Fréchet space and $P(D)$ is continuous on $C^\infty(X)$

$\Rightarrow$ $C_P^\infty(X) = \{u \in C^\infty(X);\, P(D)u = 0\}$ is a closed subspace of $C^\infty(X)$, hence a Fréchet space (with the subspace topology)

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Equip $\mathscr{D}'(X)$ - the topological dual space of $\mathscr{D}(X)$ - with the strong dual topology, i.e. topology of uniform convergence on bounded subsets of $\mathscr{D}(X)$

$\Rightarrow$ $\mathscr{D}'(X)$ is a complete locally convex space and $P(D)$ is continuous on $\mathscr{D}'(X)$

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If P is hypoelliptic: $C_P^\infty(X) = \mathscr{D}_P'(X)$ and the subspace topologies from $C^\infty(X)$ and $\mathscr{D}'(X)$ coincide

Equip $C^\infty(X)$ with topology generated by the seminorms

$$\|\cdot\|_{l,K}: C^\infty(X)\to[0,\infty), f\mapsto \max_{\alpha\in\mathbb{N}_0^d,|\alpha|\leq l}\max_{x\in K}|\partial^\alpha f(x)|\,(l\in\mathbb{N}_0, K\Subset X)$$

$\Rightarrow$ $C^\infty(X)$ Fréchet space and $P(D)$ is continuous on $C^\infty(X)$

$\Rightarrow$ $C_P^\infty(X)=\{u\in C^\infty(X);\, P(D)u=0\}$ is a closed subspace of $C^\infty(X)$, hence a Fréchet space (with the subspace topology)

Equip $\mathscr{D}'(X)$ - the topological dual space of $\mathscr{D}(X)$ - with the strong dual topology, i.e. topology of uniform convergence on bounded subsets of $\mathscr{D}(X)$

$\Rightarrow$ $\mathscr{D}'(X)$ is a complete locally convex space and $P(D)$ is continuous on $\mathscr{D}'(X)$

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Theorem

Given $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ open, X_2 P-convex for supports. Tfae.

i) X_1 is P-convex for supports and $r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$ has dense range.

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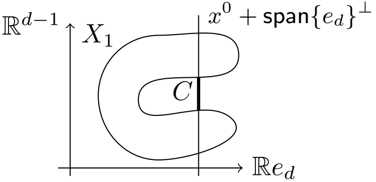

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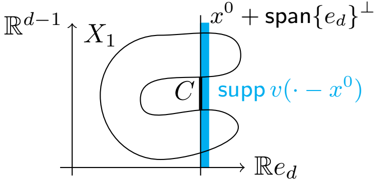

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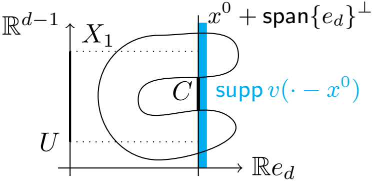

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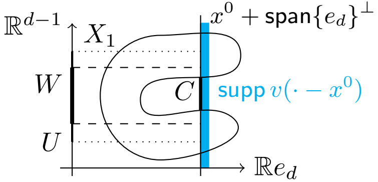

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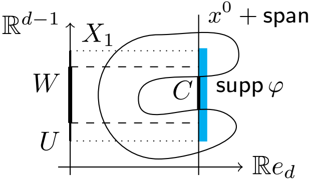

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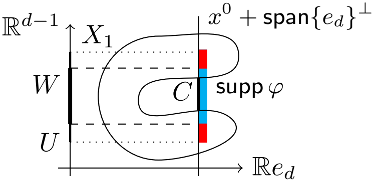

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Theorem

Let P be of degree m s.th. e_d is characteristic for P but e_1 is not; thus, there are $R_k \in \mathbb{C}[X_1, \ldots, X_{d-1}], 0 \leq k \leq m$, such that $R_m = c \in \mathbb{C}\backslash\{0\}$:

$$\forall\, x \in \mathbb{R}^d :\ P(x) = \sum_{k=0}^{m} R_k(x_2, \ldots, x_d) x_1^k.$$

Assume, $\deg_{x_d} R_k < m - k, 0 \leq k \leq m - 1$. Then $(*)$ holds.

Theorem

Let $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ be elliptic, $\deg(Q) =: m > r \in \mathbb{N}_0, c \in \mathbb{C}$ and set $P(x_1, \ldots, x_d) := Q(x_1, \ldots, x_{d-1}) - c(i\, x_d)^r$ so that $P(D) = Q(D_{d-1}) - c\frac{\partial^r}{\partial x_d^r}$.

Then, for open sets $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ which are P-convex for supports, tfae:

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For open $Y_1 \subseteq Y_2 \subseteq \mathbb{R}^{d-1}, I_1 \subseteq I_2 \subseteq \mathbb{R}$ the sets $X_j := Y_j \times I_j, j = 1, 2$, are P-convex for supports and i) simplifies to

i') $\mathbb{R}^{d-1} \backslash Y_1$ has no compact connected component $C \subseteq Y_2$.

Vogt's property (Ω) for kernels of hypoelliptic operators

Let $P(D) : C^\infty(X) \to C^\infty(X)$ be surjective ($\Leftrightarrow X$ P-convex for supports).
Given a locally convex space E, is $P(D) : C^\infty(X, E) \to C^\infty(X, E)$ surjective?

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Motivation: $\Lambda \subseteq \mathbb{R}^q$ open, $(f_\lambda)_{\lambda\in\Lambda} \in C^\infty(X)^\Lambda$ s.th.

$$\forall\, x \in X: \lambda \mapsto f_\lambda(x) \text{ continuous/smooth/real analytic/etc.}$$

Are there solutions $P(D)u_\lambda = f_\lambda, \lambda \in \Lambda$, s.th.

$$\forall\, x \in X: \lambda \mapsto u_\lambda(x) \text{ continuous/smooth/real analytic/etc. ?}$$

Choose $E = C(\Lambda)/C^\infty(\Lambda)/\mathscr{A}(\Lambda)$/etc.

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"Yes" if E is a Fréchet space (Grothendieck, 1955).

In general "No" for $E = F_b'$, the strong dual of a Fréchet space F (Vogt, 1983).

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"Yes" in case of $E = s'$ if and only if $C_P^\infty(X)$ has (Ω) (Vogt, 1983).
(Recall: $s := \{(x_j)_{j\in\mathbb{N}} \in \mathbb{C}^\mathbb{N};\, \forall\, k \in \mathbb{N}: \sum_{j=1}^\infty |x_j|^2 j^{2k} < \infty\}$ so that
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"Yes" if $C_P^\infty(X)$ has (Ω) and $E = F_b'$ is the strong dual of a nuclear Fréchet space F with (DN) (Vogt, 1983).

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"Yes" if E is a Fréchet space (Grothendieck, 1955).

In general "No" for $E = F_b'$, the strong dual of a Fréchet space F (Vogt, 1983).

"Yes" in case of $E = s'$ if and only if $C_P^\infty(X)$ has (Ω) (Vogt, 1983).
(Recall: $s := \{(x_j)_{j\in\mathbb{N}} \in \mathbb{C}^\mathbb{N};\, \forall\, k \in \mathbb{N} : \sum_{j=1}^\infty |x_j|^2 j^{2k} < \infty\}$ so that
$s' = \{(x_j)_{j\in\mathbb{N}} \in \mathbb{C}^\mathbb{N};\, \exists\, k \in \mathbb{N} : \sum_{j=1}^\infty |x_j|^2 j^{-2k} < \infty\}$)

"Yes" if $C_P^\infty(X)$ has (Ω) and $E = F_b'$ is the strong dual of a nuclear Fréchet space F with (DN) (Vogt, 1983).

$C_P^\infty(X)$ has (Ω) if and only if

$\exists\, T : s \to C_P^\infty(X)$ linear, continuous, surjective (Vogt, Wagner 1980)

(Thus: $C_P^\infty(X)$ has $(\Omega) \Rightarrow C_P^\infty(X)$ has a countable (absolute) Schauder basis)

G Fréchet space and $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$ a sequence of seminorms generating its topology

(E.g. P hypoelliptic, $G = C_P^\infty(X)$, $\|u\|_k := \|u\|_{0,K_k} := \sup_{x \in K_k} |u(x)|$, for a compact exhaustion $(K_k)_{k\in\mathbb{N}}$ of X).

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G has $(\Omega) :\Leftrightarrow \forall\, k \in \mathbb{N}\, \exists\, l \geq k\, \forall\, n \geq l\, \exists\, \lambda > 0, C \geq 0\, \forall\, \varepsilon > 0$

$$\forall\, u \in G\, \exists\, v_\varepsilon \in G, \|u - v_\varepsilon\|_k \leq \varepsilon :\ \|v_\varepsilon\|_n \leq C\varepsilon^{-\lambda}\|u\|_l^{1+\lambda}.$$

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$C_P^\infty(X)$ has (Ω) if

- P is elliptic, X arbitrary (Vogt, 1983)
- P is hypoelliptic, X convex (Vogt, 1983)

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However, for all $d \geq 3$ there are hypoelliptic P and $X \subseteq \mathbb{R}^d$ P-convex for supports such that $C_P^\infty(X)$ does not have (Ω) (K. 2012).

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G has $(\Omega) :\Leftrightarrow \forall\, k \in \mathbb{N}\, \exists\, l \geq k\, \forall\, n \geq l\, \exists\, \lambda > 0, C \geq 0\, \forall\, \varepsilon > 0$

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However, for all $d \geq 3$ there are hypoelliptic P and $X \subseteq \mathbb{R}^d$ P-convex for supports such that $C_P^\infty(X)$ does not have (Ω) (K. 2012).

G has $(DN) :\Leftrightarrow \exists\, k \in \mathbb{N}\, \forall\, l \geq k\, \exists\, n \geq l, C \geq 0 :$

$$\forall\, u \in E : \|u\|_l^2 \leq C\|u\|_k\, \|u\|_n.$$

Nuclear Fréchet spaces with (DN): $\mathscr{S}(\mathbb{R}^d)$; $C_{2\pi}^\infty(\mathbb{R}^d)$; $\mathscr{H}(\mathbb{C}^d)$; $\mathscr{D}(K)$; s; $C^\infty(\overline{X}), X$ bounded with $\partial X\, C^1$; $C^\infty(M), M$ compact manifold

Theorem

Let $P(x_1, \dots, x_d) = Q(x_1, \dots, x_{d-1}) - c(i\,x_d)^r$, where $Q \in \mathbb{C}[X_1, \dots, X_{d-1}]$ is elliptic of degree m s.th. $Q_m \in \mathbb{R}[X_1, \dots, X_{d-1}]$, $m > r \in \mathbb{N}$, r odd, $c \in \mathbb{R}$. Then, for $X \subseteq \mathbb{R}^d$ open, tfae:

i) X is P-convex for supports.

ii) X is P-convex for supports and $C_P^\infty(X)$ has (Ω).

iii) $\forall\,\alpha \in \mathbb{R}:\ d_X$ satisfies the minimum principle in $\mathbb{R}^{d-1} \times \{\alpha\}$.

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iii) $\forall\, \alpha \in \mathbb{R} :\ d_X$ satisfies the minimum principle in $\mathbb{R}^{d-1} \times \{\alpha\}$.

In particular, for all $Y \subseteq \mathbb{R}^{d-1}, I \subseteq \mathbb{R}$ open

$$C^\infty(Y \times I, F') \to C^\infty(Y \times I, F'), u \mapsto Q(D_{d-1})u - c\frac{\partial^r}{\partial x_d^r}u$$

is surjective for every nuclear Fréchet space F with (DN).

Thank you for your attention!

T. Kalmes, *Surjectivity of differential operators and linear topological invariants for spaces of zero solutions*, Rev. Mat. Compl. 32 (2019), no. 1, 37-55.

T. Kalmes, *An approximation theorem of Runge type for certain non-elliptic partial differential operators*, arXiv-preprint 1804.08099, 2018.