General solvability of linear partial differential equations

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1. Introduction

In many mathematical models linear partial differential operators (DOP for short) show up, e.g.

$$\Delta = \Delta_x = \sum_{j=1}^{d} \partial_j^2 \qquad \text{(Laplace operator)},$$

$$\partial_t - \Delta_x \qquad \text{(Heat operator)},$$

$$\partial_t^2 - \Delta_x \qquad \text{(Wave operator)},$$

$$-i\partial_t - \Delta_x \qquad \text{(time dependent free Schrödinger operator)},$$

$$\tfrac{1}{2}(\partial_1 + i\partial_2) \qquad \text{(Cauchy Riemann operator)}.$$

For general $P \in \mathbb{C}[X_1, \ldots, X_d]$ set

$$P(\partial) := P(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}).$$

(E.g. $\Delta = P_L(\partial)$ for $P_L(\xi) = \sum_{j=1}^d \xi_j^2$

$\frac{\partial}{\partial t} - \Delta_x = P_H(\partial)$ for $P_H(\xi_1, \ldots, \xi_d) = \xi_1 - \sum_{j=2}^d \xi_j^2$

$\frac{\partial^2}{\partial t^2} - \Delta_x = P_W(\partial)$ for $P_W(\xi_1, \ldots, \xi_d) = \xi_1^2 - \sum_{j=2}^d \xi_j^2$.)

$-i\frac{\partial}{\partial t} - \Delta_x = P_S(\partial)$ for $P_S(\xi_1, \ldots, \xi_d) = -i\xi_1 - \sum_{j=2}^d \xi_j^2$

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For $X \subseteq \mathbb{R}^d$ open and connected, f given solve $P(\partial)u = f$ in X.

Possible for every f from a fixed space of functions? "Solution" in which sense; classical, distributional?

Let $P \in \mathbb{C}[X_1, \ldots, X_d] \backslash \{0\}$ and let $X \subseteq \mathbb{R}^d$ be open and connected.

i) When is $P(\partial) : C^\infty(X) \to C^\infty(X)$ surjective?

ii) When is $C^\infty(X) \subseteq P(\partial)(\mathscr{D}'(X))$?

iii) When is $P(\partial) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective?

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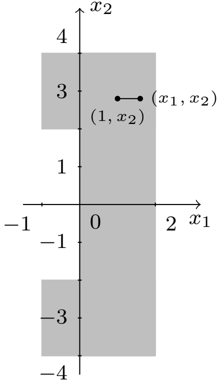
Answers will depend on combined properties of P and X.

Example:

$$X = \Big((0,2) \times (-4,4)\Big) \cup \Big((-1,1) \times (-4,-2)\Big) \cup \Big((-1,1) \times (2,4)\Big)$$

$P_1(\xi_1, \xi_2) = \xi_1 \;\Rightarrow P_1(\partial) = \partial_1$;

given $f \in C^\infty(X) \;\Rightarrow$
$u(x_1, x_2) := \int_1^{x_1} f(t, x_2)\, dt \;\in C^\infty(X)$
satisfies $\partial_1 u = f$



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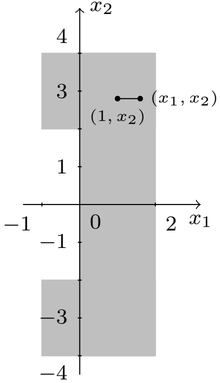

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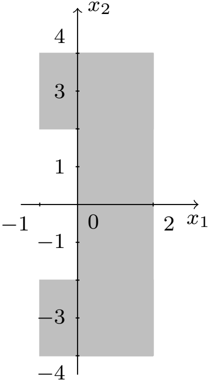

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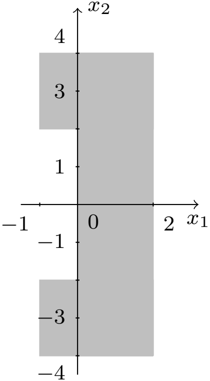
choose $\eta\in C^\infty(\mathbb{R})$ with $\eta(t)=0$ for $t\notin[-1,1]$ and $\int_{-1}^1 \eta(t)\,dt>0$; set

$$f(x_1,x_2)=\begin{cases}\frac{\eta(x_2)}{x_1}, & \text{if } x_1>0\\ 0, & \text{if } x_1\le 0\end{cases}$$

$\Rightarrow f\in C^\infty(X)$



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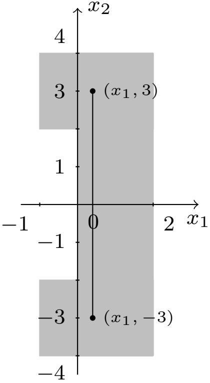
$\Rightarrow f \in C^\infty(X)$; suppose $\exists\, u \in C^1(X) : \partial_2 u = f$;

for $x_1 \in (0,2)$ we then have

$$u(x_1,3) - u(x_1,-3) = \int_{-3}^{3} \partial_2 u(x_1,t)dt = \frac{1}{x_1}\int_{-1}^{1}\eta(t)\,dt \to_{x_1\to 0} \infty$$



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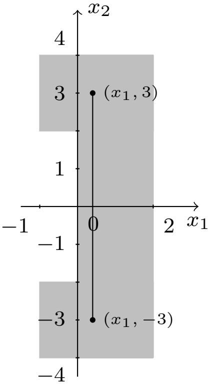
$\Rightarrow P_2(\partial) : C^1(X) \to C^\infty(X)$ not surjective



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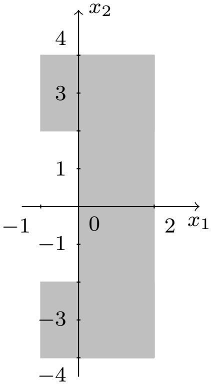

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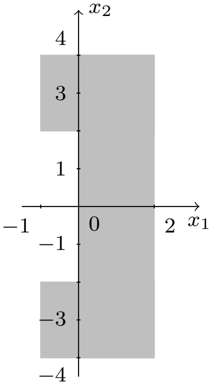
Is it possible to "see" this without calculation? What about $P_2(\partial)$ if we allow for more general solutions of $P_2(\partial)u = f, f \in C^\infty(X)$, than $u \in C^1(X)$?



2. Some Functional Analysis

E be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

a) $p : E \to [0, \infty)$ is called a seminorm if

   i) $\forall\, x \in E, \alpha \in \mathbb{K} :\ p(\alpha x) = |\alpha|\, p(x)$,

   ii) $\forall\, x, y \in E :\ p(x + y) \leq p(x) + p(y)$.

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d) A lcs $(E, \mathscr{P})$ is called separated if

$$\forall\, x \in E\ \exists\, p \in \mathscr{P} : p(x) > 0.$$

Examples:

a) Every normed space is a lcs.
b) $(E, \mathscr{P})$ lcs, $F \subseteq E$ subspace $\Rightarrow$ $(F, \mathscr{P}_{|F}) := (F, \mathscr{P})$, where $\mathscr{P}_{|F} := \{p_{|F};\, p \in \mathscr{P}\}$ is lcs

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c) $X \subseteq \mathbb{R}^d$ open, $K \Subset X (:\Leftrightarrow K \subseteq X$ compact$)$, $l \in \mathbb{N}_0$

$$\|\cdot\|_{l,K} : C^\infty(X) \to [0,\infty), f \mapsto \sup_{\alpha \in \mathbb{N}_0^d, |\alpha| \le l} \sup_{x \in K} |\partial^\alpha f(x)|$$

defines a seminorm on $C^\infty(X)$ and

$$\mathscr{P}_{\infty,c} := \{\|\cdot\|_{l,K};\, l \in \mathbb{N}_0, K \Subset X\}$$

is a directed family of seminorms on $C^\infty(X)$. For the (separated) lcs $(C^\infty(X), \mathscr{P}_{\infty,c})$ we write $\mathscr{E}(X)$.

Examples continued:

d) $(E_1, \mathscr{P}_1), (E_2, \mathscr{P}_2)$ lcs, $p \in \mathscr{P}_1, q \in \mathscr{P}_2$

$$p \oplus q : E_1 \times E_2 \to [0, \infty), (x_1, x_2) \mapsto p(x_1) + q(x_2)$$

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a directed family of seminorms on $E_1 \times E_2$
$\Rightarrow (E_1 \times E_2, \mathscr{P}_1 \oplus \mathscr{P}_2)$ lcs, separated if both $(E_1, \mathscr{P}_1)$ and $(E_2, \mathscr{P}_2)$ are separated

$(E, \mathscr{P})$ lcs, $U \subseteq E$ is called open (in $(E, \mathscr{P})$) $:\Leftrightarrow$

$$\forall\, x \in U\, \exists\, p \in \mathscr{P}, \varepsilon > 0 :\ B_p(x, \varepsilon) \subseteq U,$$

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$$E \times E \to E, (x, y) \mapsto x + y \text{ and } \mathbb{K} \times E \to E, (\lambda, x) \mapsto \lambda x$$

are both continuous

$(E, \mathscr{P})$ be a lcs

2.1 Proposition

For a seminorm $q : E \to [0, \infty)$ the following are equivalent:

i) q is continuous.

ii) q is continuous in 0.

iii) $\exists\, p \in \mathscr{P}, C > 0 \,\forall\, x \in E :\; q(x) \leq Cp(x)$.

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$cs(E, \mathscr{P}) := cs(E) := \{q : E \to [0, \infty);\ q$ continuous seminorm$\}$

$\mathscr{P}_0 \subseteq cs(E, \mathscr{P})$ is called fundamental system of seminorms iff

$$\forall\, q \in \mathscr{P}\, \exists\, p \in \mathscr{P}_0, C > 0\, \forall\, x \in E :\ q(x) \leq Cp(x)$$

Then, by the previous proposition, for a seminorm r on E we have

$$r \in cs(E, \mathscr{P}) \Leftrightarrow \exists\, p \in \mathscr{P}_0, C > 0 :\ r \leq Cp$$

Example:
$X \subseteq \mathbb{R}^d$ open with compact exhaustion $(K_n)_{n\in\mathbb{N}_0}$. Recall

$$\forall\, l \in \mathbb{N}_0,\ K \Subset X, f \in C^\infty(X) : \|f\|_{l,K} = \sup_{|\alpha|\leq l} \sup_{x\in K} |\partial^\alpha f(x)|,$$

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Then $\{\|\cdot\|_{n,K_n}; n \in \mathbb{N}_0\}$ is a countable fundamental system of seminorms for $\mathscr{E}(X)$.

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$(E, \mathscr{P})$ be a lcs. Tfae:

i) There is a semi-metric ρ on E generating the same topology on E as $\mathscr{P}$.

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If ii) holds with $\mathscr{P}_0 = \{p_n; n \in \mathbb{N}\}$ (w.l.o.g. $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$) then

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$(E, \mathscr{P})$ is called semi-metrizable iff the equivalent conditions of proposition 2.2 are satisfied, metrizable iff semi-metrizable and separated.

$(E, \mathscr{P})$ lcs, $(x_n)_{n\in\mathbb{N}}$ sequence in $E, x_0 \in E$

- $(x_n)_{n\in\mathbb{N}}$ converges to x_0 in $(E, \mathscr{P}) \Leftrightarrow \forall\, p \in \mathscr{P}$: $\lim_{n\to\infty} p(x_n - x_0) = 0$ (exercise)

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$(E, \mathscr{P})$ is called Fréchet space $:\Leftrightarrow$ $(E, \mathscr{P})$ is metrizable and sequentially complete, i.e. every Cauchy sequence converges

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Let $(E, \mathscr{P})$ and $(F, \mathscr{Q})$ be lcs, $\mathscr{Q}_0$ a fundamental system of seminorms for $(F, \mathscr{Q})$, and let $A : E \to F$ be linear. Tfae:

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Thus, a linear $u : E \to \mathbb{K}$ belongs to E' iff

$$\exists\, p \in cs(E), C > 0\, \forall\, x \in E:\ |u(x)| \leq Cp(x).$$

Examples:

$X \subseteq \mathbb{R}^d$ open, we denote the dual of $\mathscr{E}(X)$ by $\mathscr{E}'(X)$

a) For each $x \in X$ we have $\delta_x \in \mathscr{E}'(X)$, where

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More generally, for $P \in \mathbb{C}[X_1, \ldots, X_d]$ the mapping

$$P(\partial) : \mathscr{E}(X) \to \mathscr{E}(X), f \mapsto P(\partial) f$$

is well-defined, linear, and continuous (exercise).

For a lcs E and $p \in cs(E)$ we set

$$p^* : E' \to [0, \infty], p^*(u) := \sup\{|u(x)|;\, p(x) \leq 1\}$$

and observe that for $u \in E'$ we have $p^*(u) < \infty$ iff

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By proposition 3 it follows

$$E' = \bigcup_{p \in cs(E)} \{p^* < \infty\}.$$

A lcs E is called a Schwartz space $:\Leftrightarrow$
$\forall\, p \in cs(E)\, \exists\, q \in cs(E)$ with $p \leq Cq$:

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For $A \in L(E, F)$ the mapping $A^t : F' \to E', u \mapsto u \circ A$ is well-defined and linear. A^t is called transpose of A. It is easily seen that $A \mapsto A^t$ is linear.

The following theorem will be used in the process to finding a characterisation of surjectivity for $P(\partial) : \mathscr{E}(X) \to \mathscr{E}(X)$.

2.4 Theorem (Surjectivity criterion)

Let $(E, \mathscr{P})$ be a Fréchet space, $(F, \mathscr{Q})$ a Fréchet-Schwartz space, and let $A \in L(E, F)$ have dense image$^{(1)}$. Then A is surjective if and only if

$$\forall\, p \in \mathscr{P}\, \exists\, q \in \mathscr{Q}\, \forall\, u \in F' : (p^*(A^t(u)) < \infty \Rightarrow q^*(u) < \infty).$$

Proof: see L. Frerick, J. Müller, J. Wengenroth. Prescribed derivatives of holomorphic functions. *Complex Var. Theory Appl.* 48(2):165-173, 2003.

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Another abstract result which we will apply deals with bilinear forms.

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Let $(E, \mathscr{P}), (F, \mathscr{Q})$ be lcs and $b : E \times F \to \mathbb{K}$ be bilinear. Tfae:

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Obviously, continuity implies separate continuity. The converse is not true in general.
However, as an application of the Banach-Steinhaus Theorem, one can show the following.

2.6 Theorem

Let E be a Fréchet space and F be a semi-metrizable lcs. If $b : E \times F \to \mathbb{K}$ is bilinear and separately continuous, then b is continuous.

Proof: see e.g. W. Rudin: Functional Analysis, McGraw-Hill, Theorem 2.17

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Let E be a Fréchet space and F be a semi-metrizable lcs. If $b : E \times F \to \mathbb{K}$ is bilinear and separately continuous, then b is continuous.

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We close this section by introducing a lcs the dual of which will play a central role:

$\mathscr{D}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d);\ \operatorname{supp}\varphi \text{ compact}\}$, where

$$\operatorname{supp}\varphi := \overline{\{x \in \mathbb{R}^d;\ \varphi(x) \neq 0\}}$$

denotes the support of φ.

For $M \subseteq \mathbb{R}^d$ $\mathscr{D}(M) := \{\varphi \in \mathscr{D}(\mathbb{R}^d);\ \operatorname{supp}\varphi \subseteq M \text{ (compact)}\}$ is a subspace of $\mathscr{E}(\mathbb{R}^d)$.

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$\Rightarrow \forall\, K \Subset \mathbb{R}^d:\ (\mathscr{D}(K), \mathscr{P}_{\infty,c})$ is a Fréchet space with fundamental system of seminorms $\{\|\cdot\|_{l,K};\ l \in \mathbb{N}_0\}$

For $X \subseteq \mathbb{R}^d$ open $\mathscr{D}(X)$ is not a closed subspace of $\mathscr{E}(X)$ and thus not complete.

Therefore, we define

$$\mathscr{P}_{\mathscr{D}} := \{p : \mathscr{D}(X) \to [0, \infty) \text{ seminorm}; \forall\, K \subseteq X : \, p_{|\mathscr{D}(K)} \text{ is continuous}\}$$

which is a directed family of seminorms (exercise!) with $\mathscr{P}_{\infty,c} \subsetneq \mathscr{P}_{\mathscr{D}}$ (exercise!), thus, $(\mathscr{D}(X), \mathscr{P}_{\mathscr{D}})$ is separated. It can be shown that $(\mathscr{D}(X), \mathscr{P}_{\mathscr{D}})$ is complete.

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From now on, $\mathscr{D}(X)$ denotes the lcs $(\mathscr{D}(X), \mathscr{P}_{\mathscr{D}})$

2.7 Theorem

Let $X \subseteq \mathbb{R}^d$ be open, E be a lcs.

i) For each $K \Subset X$ the inclusion $i_K : (\mathscr{D}(K), \mathscr{P}_{\infty,c}) \hookrightarrow \mathscr{D}(X)$ is continuous.

ii) A linear map $A : \mathscr{D}(X) \to E$ is continuous if and only if

$$\forall\, K \Subset X : A \circ i_K : (\mathscr{D}(K), \mathscr{P}_{\infty,c}) \to E$$

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iii) For $P \in \mathbb{C}[X_1, \ldots, X_d], h \in C^\infty(X)$ the maps $P(\partial) : \mathscr{D}(X) \to \mathscr{D}(X)$ and $m_h : \mathscr{D}(X) \to \mathscr{D}(X), \varphi \mapsto h\varphi$ are linear and continuous.

iv) A linear $u : \mathscr{D}(X) \to \mathbb{C}$ is continuous if and only if

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The dual $\mathscr{D}'(X)$ of $\mathscr{D}(X)$ is the space of distributions on X.

3. Distributions and differential operators

Recall: for $X \subseteq \mathbb{R}^d$ open a distribution on X is a linear $u : \mathscr{D}(X) \to \mathbb{C}$ such that

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Instead of $u(\varphi)$ we may write $\langle u, \varphi \rangle$.

Examples:

a) For $f \in L^1_{\text{loc}}(X)$

$$u_f : \mathscr{D}(X) \to \mathbb{C}, \varphi \mapsto \int_X f(x)\varphi(x)dx$$

is a well-defined linear mapping, $\forall\, K \Subset X, \varphi \in \mathscr{D}(K)$:

$$|u_f(\varphi)| \leq \int_K |f(x)\varphi(x)|dx \leq \int_K |f(x)|dx\, \|\varphi\|_{0,K},$$

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Recall the "Fundamental lemma of calculus of variations":

$$\forall\, f \in L^1_{\mathrm{loc}}(X) : \left(\forall\, \varphi \in \mathscr{D}(X) : \int_X f(x)\varphi(x)dx = 0 \Rightarrow f = 0\right)$$

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$\Rightarrow$ the linear mapping $L^1_{\text{loc}}(X) \to \mathscr{D}'(X), f \mapsto u_f$ is one-to-one
$\Rightarrow$ we can write f instead of the distribution u_f

Examples continued:

b) For every regular/complex measure μ on the Borel-σ-algebra over X

$$u_\mu : \mathscr{D}(X) \to \mathbb{C}, \varphi \mapsto \int_X \varphi(x) d\mu(x)$$

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Again, $\mu \mapsto u_\mu$ is one-to-one, so we write μ instead of u_μ.

If for $u \in \mathscr{D}'(X)$ there is $l \in \mathbb{N}_0$ such that

$$(*) \quad \forall\, K \Subset X \,\exists\, C > 0 \,\forall\, \varphi \in \mathscr{D}(K) : \; |\langle u, \varphi \rangle| \le C \|\varphi\|_{l,K}$$

u is of finite order and

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Each $f \in L^1_{\mathrm{loc}}(X)$ and every regular/complex measure μ are distributions of order 0.

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(exercise! use the Fundamental lemma of calculus of variations)
For $u \in \mathscr{D}'(X)$ we have

- $\operatorname{supp} u$ is a closed subset of X (by definition)
- $X \backslash \operatorname{supp} u$ is the largest open subset of X where u vanishes,

$$\forall \varphi \in \mathscr{D}(X) : (\operatorname{supp} \varphi \cap \operatorname{supp} u = \emptyset \Rightarrow \langle u, \varphi \rangle = 0)$$

(this requires an argument which involves partitions of unity)

3.1 Proposition

Let $X \subseteq \mathbb{R}^d$ be open.

i) The inclusion $i : \mathscr{D}(X) \hookrightarrow \mathscr{E}(X)$ is linear, continuous, and has dense range.

ii) The transpose $i^t : \mathscr{E}'(X) \to \mathscr{D}'(X)$ is injective and given by $i^t(u) = u_{|\mathscr{D}(X)}$ for all $u \in \mathscr{E}'(X)$.

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Proof: (advanced) exercise:
To show i) use that $\forall\, K \Subset X\, \exists\, \varphi \in \mathscr{D}(X) :\, \varphi = 1$ in a neighborhood of K.
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By proposition 3.1 ii) we can (and will) interpret $\mathscr{E}'(X)$ as a subspace of $\mathscr{D}'(X)$.

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so $m_\chi : \mathscr{E}(X) \to (\mathscr{D}(\operatorname{supp} \chi), \mathscr{P}_{\infty,c})$ continuous $\Rightarrow m_\chi : \mathscr{E}(X) \to \mathscr{D}(X)$ is continuous
$\mu := u \circ m_\chi \in \mathscr{E}'(X)$ and $\forall \varphi \in \mathscr{D}(X)$:

$$\mu(\varphi) = u(\chi\varphi) = u(\chi\varphi) + u((1-\chi)\varphi) = u(\chi\varphi + (1-\chi)\varphi) = u(\varphi),$$

where we used $\operatorname{supp} u \cap \operatorname{supp}((1-\chi)\varphi) = \emptyset$, so that $u((1-\chi)\varphi) = 0$. $\square$

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If $P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$, repeated use of integration by parts gives for $f \in C^m(X) (\subseteq L^1_{\text{loc}}(X))$

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\langle P(\partial)^t(f), \varphi \rangle &= \sum_{|\alpha| \le m} a_\alpha \int_X f(x)\partial^\alpha \varphi(x)dx \\
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For arbitrary $u \in \mathscr{D}'(X)$ we <u>define</u> $hu := m_h^t(u)$ and $P(\partial)u := P(-\partial)^t(u)$.

We can now consider differential equations for distributions: given $f \in \mathscr{D}(X)$ - or f in any subspace of $\mathscr{D}(X)$ like, e.g. $C^\infty(X), L^1_{\mathsf{loc}}(X)$ - is there $u \in \mathscr{D}'(X)$ such that $P(\partial)u = f$, i.e.

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For $h \in \mathscr{E}(X)$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ the following hold.

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Proof: exercise

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Assume that for every $f \in \mathscr{E}(X)$ there is $u \in \mathscr{D}'(X)$ such that $P(\partial)u = f$. Then

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$\Rightarrow$ $\operatorname{supp} \varphi \subseteq K'$ $\square$

Given $X \subseteq \mathbb{R}^d$ open, $P \in \mathbb{C}[X_1, \ldots, X_d]$. X is called P-convex for supports :⇔

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For X and P are equivalent:

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iv) $\forall\, u \in \mathscr{E}'(X) :\ \operatorname{dist}(\operatorname{supp} P(-\partial)u, X^c) = \operatorname{dist}(\operatorname{supp} u, X^c)$.

Proof: see e.g. L. Hörmander: The Analysis of Partial Differential Operators II, Springer, Theorem 10.6.3.

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$$(\|P(\partial)^t u\|_{l,K}^* < \infty \Rightarrow \|u\|_{l',K'}^* < \infty)$$

This is the condition from the Surjectivity criterion 2.4.
All we have to show: $P(\partial) \in L(\mathscr{E}(X))$ has dense image

For $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and $f \in L^1_{\mathsf{loc}}(\mathbb{R}^d)$

$$f * \varphi : \mathbb{R}^d \to \mathbb{C}, f * \varphi(x) = \int_{\mathbb{R}^d} f(y)\varphi(x-y)dy$$

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Instead of $\langle u, \varphi_x \rangle$ it is more convenient to write $\langle u_y, \varphi(x-y) \rangle$.

Example:
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3.6 Theorem

The mapping $\mathscr{D}'(\mathbb{R}^d) \times \mathscr{D}(\mathbb{R}^d) \to \mathscr{E}(\mathbb{R}^d), (u, \varphi) \mapsto u * \varphi$ is well-defined and bilinear and $\forall\, \alpha \in \mathbb{N}_0^d :\ \partial^\alpha (u * \varphi) = \partial^\alpha u * \varphi = u * \partial^\alpha \varphi$.
Thus, $P(\partial)(u * \varphi) = P(\partial) u * \varphi = u * P(\partial) \varphi$.

Proof: see e.g. W. Rudin: Functional Analysis, McGraw-Hill, Theorem 6.30.

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For $P \in \mathbb{C}[X_1, \ldots, X_d]$ a distribution $E \in \mathscr{D}'(\mathbb{R}^d)$ is a fundamental solution for $P(\partial) :\Leftrightarrow P(\partial)E = \delta_0$.

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If E is a fundamental solution for $P(\partial)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$ then

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i.e. $E * \varphi$ is a C^∞-solution u of $P(\partial)u = \varphi$.

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For every open $X \subseteq \mathbb{R}^d$ and every $P \in \mathbb{C}[X_1, \ldots, X_d] \backslash \{0\}$ the differential operator $P(\partial) : \mathscr{E}(X) \to \mathscr{E}(X)$ has dense image.

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Combining the previous results we obtain:

3.9 Theorem (Malgrange, 1955)

For open $X \subseteq \mathbb{R}^d$ and $P \in \mathbb{C}[X_1, \ldots, X_d] \backslash \{0\}$ tfae:

i) $P(\partial) : C^\infty(X) \to C^\infty(X)$ is surjective.

ii) $\forall\, f \in C^\infty(X)\, \exists\, u \in \mathscr{D}'(X) :\ P(\partial)u = f$.

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It can be shown that the above are also equivalent to

iv) $\forall\, v \in \mathscr{D}'(X), \operatorname{ord} v < \infty\, \exists\, u \in \mathscr{D}'(X), \operatorname{ord} u < \infty :\ P(\partial)u = v$.

v) $\forall\, f \in W^{p,m}_{\text{loc}}(X) \exists\, u \in W^{p,m+\deg P}_{\text{loc}}(X) : P(\partial)u = f$ $(1 \leq p \leq \infty, m \in \mathbb{N}_0$ arbitrary).

(see Hörmander: ALPDO II, Section 10.6)

Geometric conditions for/characterization of P-convexity for supports?

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Every open $X \subseteq \mathbb{R}^d$ is P-convex for supports iff P is elliptic, i.e. if $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ then

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For $d = 2$ P-convexity for supports is completely characterized.

4. Conditions for P-convexity for supports

X P-convex for supports $\Leftrightarrow$

$$\forall\, u \in \mathscr{E}'(X) : \ \mathsf{dist}(\mathsf{supp}\, P(-\partial)u, X^c) = \mathsf{dist}(\mathsf{supp}\, u, X^c)$$

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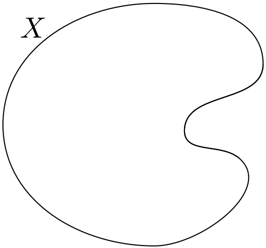
What can we say about the location of $\mathsf{supp}\, u$ if we know $\mathsf{supp}\, P(-\partial)u$?



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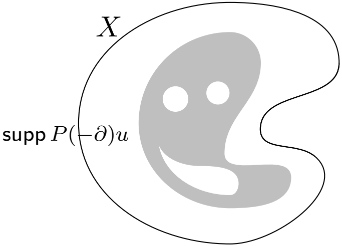

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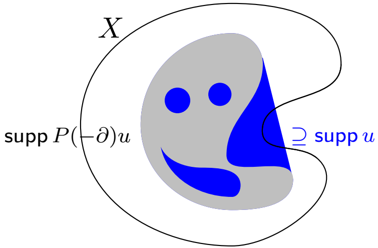

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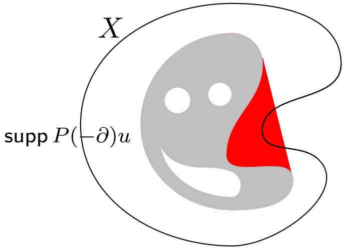

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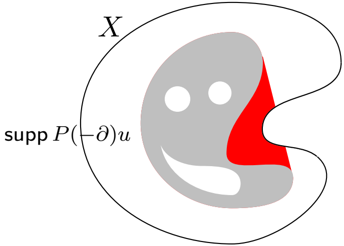
What can we say about the location of supp u if we know supp $P(-\partial)u$?



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A hyperplane $H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$ ($N \in \mathbb{R}^d$, $\|N\| = 1, \alpha \in \mathbb{R}$) is characteristic for P if $P_m(N) = 0$ (P_m principal part of P).

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$$f_{\gamma,N} : \mathbb{R}^d \to \mathbb{R}, f_{\alpha,N}(x) := g(\gamma - \langle N, x \rangle),$$

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For $P(\xi) = P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ it follows

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Thus, for every characteristic hyperplane H of a homogeneous polynomial P there is $f \in C^\infty(\mathbb{R}^d)$ with $P(-\partial) f = 0$ and support equal to a half space bounded by H.

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Thus, for every characteristic hyperplane H of a homogeneous polynomial P there is $f \in C^\infty(\mathbb{R}^d)$ with $P(-\partial) f = 0$ and support equal to a half space bounded by H. More general we have:

4.1 Theorem (Hörmander, see ALPDO I, Theorem 8.6.7)

Let $H = \{x \in \mathbb{R}^d;\ \langle N, x\rangle = \gamma\}$ be a characteristic hyperplane for $P(-\partial)$. Then there is $f \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f = \{x \in \mathbb{R}^d;\ \langle x, N\rangle \leq \gamma\}$ and $P(-\partial)f = 0$.

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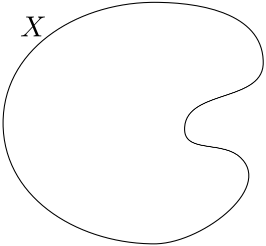

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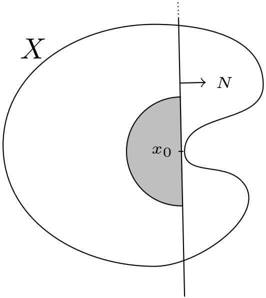

f as above for $\gamma = \langle N, x_0\rangle$
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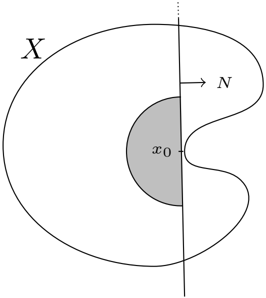

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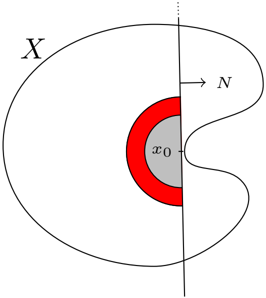

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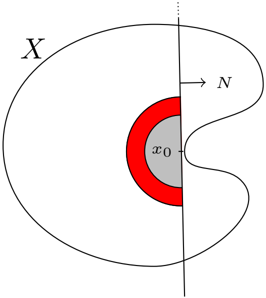

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$\Rightarrow$ X not P-convex for supports

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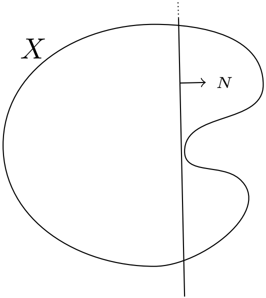

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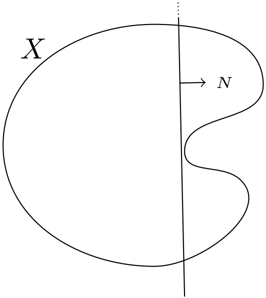

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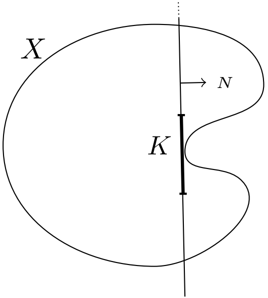

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Complementary to 4.1 we have the following result:

4.3 Holmgren's uniqueness theorem (see Hörmander, ALPDO I, Theorem 8.6.5)

If $u \in \mathscr{D}'(X)$ satisfies $P(-\partial)u = 0$ and H is a hyperplane which is not characteristic for P and is such that u vanishes on one side of H, then u vanishes in a neighborhood of $X \cap H$.

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Combining 4.1 and 4.3 gives:

4.4 Corollary (Hörmander, see ALPDO I, Theorem 8.6.8)

Let $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ be open and convex. Tfae:

i) $\forall\, v \in \mathscr{D}'(X_2), P(-\partial)v = 0 : \big(v_{|X_1} = 0 \Rightarrow v = 0\big)$

ii) Every characteristic hyperplane for P which intersects X_2 already intersects X_1.

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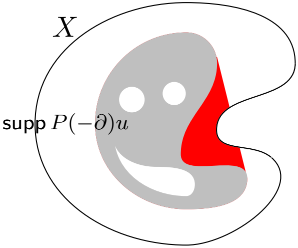

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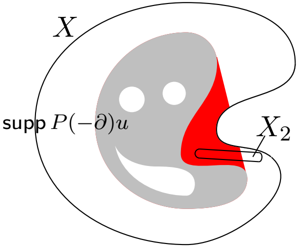

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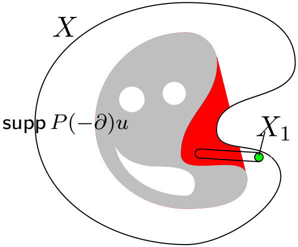

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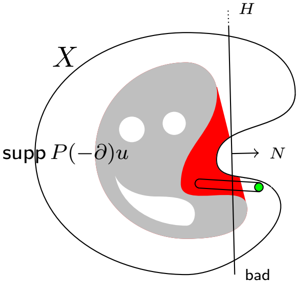

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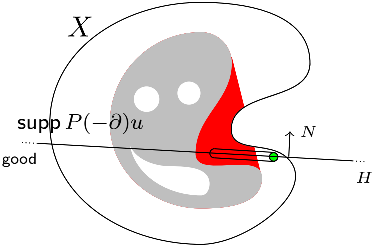

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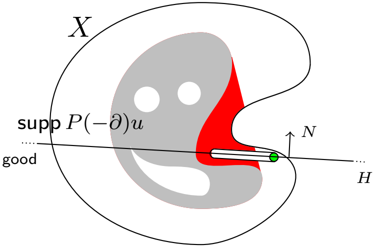

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Let P have principal part P_m, let $\{0\} \neq W \subseteq \mathbb{R}^d$ be a subspace such that d_X satisfies the minimum principle in $x + W$ for every $x \in \mathbb{R}^d$.
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4.6 Corollary

Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ and let $X \subseteq \mathbb{R}^d$ be open.

i) If P is elliptic X is P-convex for supports.

ii) If $\{x;\, P_m(x) = 0\}$ is a one-dimensional subspace then X is P-convex for supports iff d_X satisfies the minimum principle in $x + W$ for every $x \in \mathbb{R}^d$ with $W = \{x;\, P_m(x) = 0\}^\perp$.

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ii) applicable to $-i\partial_t - \Delta_x$ and parabolic operators, e.g. $\partial_t - \Delta_x$.

We close this section by giving the characterisation of P-convexity for supports in case of $d = 2$.

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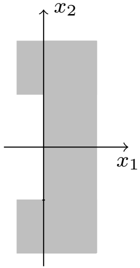

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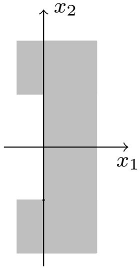

5. Surjectivity of $P(\partial)$ on $\mathscr{D}'(X)$

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Exercise: Let $A : E \to E$ be linear on vector space E, let $F \subseteq E$ be a subspace with $F \subseteq A(F)$. Then A is surjective iff

- $F \subseteq A(E)$
- the linear map on the quotient space E/F induced by A

  $$\tilde{A} : E/F \to E/F, x + F \mapsto Ax + F$$

  is surjective

Given open $V \subseteq X \subseteq \mathbb{R}^d$ and $u \in \mathscr{D}'(X)$, we say that u is smooth in V :$\Leftrightarrow$ $u_{|V} \in \mathscr{E}(V)$, i.e.

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For $u \in \mathscr{D}'(X)$ we have

- sing supp u is a closed subset of X (by definition)
- $X \setminus$sing supp u is the largest open subset of X where u is smooth

5.1 Theorem (Hörmander, 1962, see: ALPDO II, Section 10.7)

For open $X \subseteq \mathbb{R}^d$ we have $\mathscr{D}'(X)/C^\infty(X) = P(D)(\mathscr{D}'(X)/C^\infty(X))$ iff X P-convex for singular supports, i.e.

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For open $X \subseteq \mathbb{R}^d$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ tfae:

i) $P(\partial) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective.

ii) X is strongly P-convex.

Geometric conditions for/characterizing P-convexity for singular supports?

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$$\forall\, X \subseteq \mathbb{R}^d \text{ open}, u \in \mathscr{D}'(X) : \operatorname{sing\,supp} P(\partial)u = \operatorname{sing\,supp} u$$

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Every convex open X is P-convex for singular support for each $P \neq 0$.
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For $d = 2$ P-convexity for singular supports is completely characterized.

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