On Banach spaces of vector-valued random variables and their duals motivated by risk measures

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Introduction

Risk measures from mathematical finance:
$Y =$ accumulated investment into a portfolio, i.e. $\mathbb{R}$-valued r.v. on probability space $(\Omega, \mathcal{F}, P)$

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portfolio usually composed of individual components $\Rightarrow$ desirable to measure not only the risk of accumulated portfolio but of its components $\Rightarrow$ replace $\mathbb{R}$-valued r.v. Y by $\mathbb{R}^d$-valued r.v., more general by Banach space valued r.v. Y

$(\Omega, \mathcal{F}, P)$ probability space, $Y : \Omega \to \mathbb{R}$ r.v.

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$X = (X, \|\cdot\|)$ Banach space with dual $(X^*, \|\cdot\|^*)$;
$Y : \Omega \to X, Z : \Omega \to X^*$ P-measurable, by Hardy-Littlewood-rearrangement inequality

$$\mathbb{E}|\langle Z, Y\rangle| \leq \mathbb{E}(\|Z\|^*\|Y\|) \leq \int_0^1 F^{-1}_{\|Z\|^*}(u) F^{-1}_{\|Y\|}(u)du.$$

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With $\sigma:=F^{-1}_{\|Z\|^*}$ we have for real Banach space X

$$\rho_Z(Y):=\sup\{\mathbb{E}(\langle Z,Y'\rangle);Y'\sim Y\}\leq\int_0^1\sigma(u)F^{-1}_{\|Y\|}(u)du$$

(*maximal correlation risk measure in direction Z*)

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General assumption on $(\Omega,\mathcal{F},P)$:
$\emptyset\neq\mathscr{U}(0,1):=\{[0,1]\text{-valued uniform r.v. on } (\Omega,\mathcal{F},P)\}$

The Banach spaces $L^p_\sigma(P, X)$

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$$\|Y\|^p_{\sigma,p} := \sup_{U \in \mathscr{U}(0,1)} \mathbb{E}(\sigma(U)\|Y\|^p).$$

Moreover,
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For $\sigma = 1$ one obtains the classical Bochner-Lebesgue spaces $L^p(P,X)$.

Theorem

i) $(L^p_\sigma(P, X), \|\cdot\|_{\sigma,p})$ is a Banach space which embeds contractively into $L^p(P, X)$.

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iii) If X is real, Z P-measurable X^*-valued with $\mathbb{E}\|Z\|^* = 1$ and $\sigma := F^{-1}_{\|Z\|^*}$, then

$$\rho_Z : L^p_\sigma(P,X) \to \mathbb{R}, \rho_Z(Y) := \sup\{\mathbb{E}\langle Z, Y'\rangle; Y \sim Y'\}$$

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iv) For $1 \leq p < p' < \infty$ we have $L^{p'}_\sigma(P,X) \subseteq L^p_\sigma(P,X)$ and $\|Y\|_{\sigma,p'} \leq \|Y\|_{\sigma,p}$ for each $Y \in L^{p'}_\sigma(P,X)$.

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v) $L^\infty(P,X)$ embeds contractively into $L^p_\sigma(P,X)$ and simple functions are dense in $L^p_\sigma(P,X)$.

Theorem

Let $X \neq \{0\}$.

a) Tfae

i) $\forall\, p \in [1,\infty)$: $L^p_\sigma(P,X)$ and $L^p(P,X)$ are isomorphic as Banach spaces.

ii) $\exists\, p \in [1,\infty)$: $L^p_\sigma(P,X) = L^p(P,X)$ as sets.

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b) Tfae

   i) $L^p_\sigma(P,X)$ is a Hilbert space.

   ii) X is a Hilbert space, $p = 2$, and $\sigma = 1$ on $(0,1)$.

The dual of $L^p_\sigma(P, X)$

For $X=\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ let $L^p_\sigma := L^p_\sigma(P) := L^p_\sigma(P,\mathbb{K})$ and $L^0(P) := \{Z : (\Omega,\mathcal{F},P)\to(\mathbb{K},|\cdot|);\ Z$ measurable$\}$.

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Definition

Let

$$L^p_\sigma(P)^\times := \{Z \in L^0(P);\ \forall\, Y \in L^p_\sigma : ZY \in L^1(P)\}$$

be the Köthe dual of L^p_σ.

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Taking $Y = \mathbb{1}_{\{Z \neq 0\}} \frac{\overline{Z}}{|Z|}$ for $Z \in L^p_\sigma(P)^\times$ it follows from $L^\infty(P) \subseteq L^p_\sigma(P)$ that $L^p_\sigma(P)^\times \subseteq L^1(P)$.

Proposition

For every $Z\in L^p_\sigma(P)^\times$

$$\varphi_Z: L^p_\sigma(P)\rightarrow\mathbb{K}, \varphi_Z(Y)=\mathbb{E}(ZY)$$

belongs to the dual $L^p_\sigma(P)^*$ of $L^p_\sigma(P)$ and

$$\Phi: L^p_\sigma(P)^\times\rightarrow L^p_\sigma(P)^*, Z\mapsto\varphi_Z$$

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$$|\mathbb{E}(ZY)| \leq \int_0^1 F^{-1}_{|Z|}(u)F^{-1}_{|Y|}(u)du$$

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$$\begin{aligned}|\mathbb{E}(ZY)| &\leq \lim_{n\to\infty}\sum_{j=1}^{m_n} c_{j,n}\int_0^1 F^{-1}_{|Z|}(u)\mathbb{1}_{(\alpha_{j,n},1]}(u)du\\ &\overset{!}{\leq} \|Z\|^*_{\sigma,1}\lim_{n\to\infty}\sum_{j=1}^{m_n} c_{j,n}\int_0^1 \sigma(u)\mathbb{1}_{(\alpha_{j,n},1]}(u)du\end{aligned}$$

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Finally, for $q\in(1,\infty)$

$$|Z|^*_{\sigma,q}:=\inf\{\|Z'\|_{\sigma,q};\ Z'\ {}_\sigma\!\succcurlyeq Z\}$$

and for $q\in(1,\infty]$ let

$$L^*_{\sigma,q}(P):=\{Z\in L^0(P);\ |Z|^*_{\sigma,q}<\infty\}.$$

Proposition

For $p\in[1,\infty)$ with conjugate exponent q, $L^*_{\sigma,q}(P)\subseteq L^p_\sigma(P)^\times$ and

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PROOF:

By the rearrangement inequality

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$p=1: F^{-1}_{|Y|}=\lim_{n\to\infty}\sum_{j=1}^{m_n}c_{j,n}1\!\!1_{(\alpha_{j,n},1]}$, Monotone Conv. Thm.

$$\int_0^1 F^{-1}_{|Z|}(u)F^{-1}_{|Y|}(u)du\leq |Z|^*_{\sigma,\infty}\int_0^1\sigma(u)F^{-1}_{|Y|}(u)du=|Z|^*_{\sigma,\infty}\|Y\|_{\sigma,1}.$$

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$p>1$: $Z'\in L^q_\sigma(P)$ with $Z'\ {}_\sigma\!\succcurlyeq Z$, $F^{-1}_{|Y|}=\lim_{n\to\infty}\sum_{j=1}^{m_n}c_{j,n}\mathbb{1}_{(\alpha_{j,n},1]}$, Monotone Conv. Thm., Hölder ineq.

$$\int_0^1 F^{-1}_{|Z|}(u)F^{-1}_{|Y|}(u)du\ \leq\ \int_0^1\sigma(u)F^{-1}_{|Z'|}(u)F^{-1}_{|Y|}(u)du\leq\|Z'\|_{\sigma,q}\|Y\|_{\sigma,p}.$$

$\square$

Lemma

$\forall\, Z\in L^0(P),\,\alpha\in[0,1)\,\exists\, E_\alpha\in\mathcal{F}:\; P(E_\alpha)=1-\alpha$ and $\int_\alpha^1 F_{|Z|}^{-1}(u)du=\mathbb{E}(|Z|\mathbb{1}_{E_\alpha})$.

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Fix $Z\in L^1_\sigma(P)^\times$

$$\Rightarrow\quad \int_\alpha^1 F_{|Z|}^{-1}(u)du=|\mathbb{E}\big(Z\mathbb{1}_{\{Z\neq 0\}}\frac{\bar Z}{|Z|}\mathbb{1}_{E_\alpha}\big)|=|\varphi_Z\big(\mathbb{1}_{\{Z\neq 0\}}\frac{\bar Z}{|Z|}\mathbb{1}_{E_\alpha}\big)|$$

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$$\leq\|\varphi_Z\|^*_{\sigma,1}\ \|\mathbb{1}_{\{Z\neq 0\}}\frac{\bar{Z}}{|Z|}\mathbb{1}_{E_\alpha}\|_{\sigma,1}\leq\|\varphi_Z\|^*_{\sigma,1}\ \|\mathbb{1}_{E_\alpha}\|_{\sigma,1}$$

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$\Rightarrow Z\in L^*_{\sigma,\infty}(P)$ and $|Z|^*_{\sigma,\infty}\leq\|\varphi_Z\|^*_{\sigma,1}$

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Corollary

For $p\in(1,\infty)$ the Banach space $L^p_\sigma(P)$ is reflexive.

Recall: $\mu:\mathcal{F}\to X^*$ vector measure $:\Leftrightarrow$
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Straightforward: $\{\varphi:\mathcal{S}(X)\to\mathbb{K};\ \varphi$ linear and $\|\cdot\|_\infty$-continuous$\}$ and $\{\mu:\mathcal{F}\to X^*;\ \mu$ vector measure of bounded variation$\}$ isomorphic via

$$\Phi:\varphi\mapsto\big(\langle\mu_\varphi(E),x\rangle:=\varphi(x1\!\!1_E)\big)$$

where
$\mathcal{S}(X)=\{Y:\Omega\to X;\ Y(\Omega)$ finite,$\forall\, x\in X:Y^{-1}(\{x\})\in\mathcal{F}\}$

Definition

Let $p \in [1,\infty)$ and $q \in (1,\infty]$.

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Theorem

For $p \in [1, \infty)$ with conjugate q the space $(L^*_{\sigma,q}(P, X^*), |\cdot|^*_{\sigma,q})$ is a Banach space and

$$\Psi : (L^*_{\sigma,q}(P, X^*), |\cdot|^*_{\sigma,q}) \to (L^p_\sigma(P, X)^*, \|\cdot\|^*_{\sigma,p}), \mu \mapsto \left(Y \mapsto \int_\Omega Y d\mu\right)$$

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In general: The dual of $L^p_\sigma(P,X)$ is isometrically isomorphic to a Banach space of X^*-valued vector measures.
When is it isometrically isomorphic to Banach space of X^*-valued random variables?

Definition

Set $L^{q\,*}_\sigma(P,X^*) := \{Z:\Omega\to X^*;\ Z\ P\text{-meas.},\ \|Z\|\in L^*_{\sigma,q}(P)\}$ for $q\in(1,\infty]$, and for $Z\in L^{q\,*}_\sigma(P,X^*)$ let $|Z|^{q,*}_\sigma := |\,\|Z\|\,|^*_{\sigma,q}$.
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$$\iota:(L^{q\,*}_\sigma(P,X^*),|\cdot|^{q,*}_\sigma)\to L^p_\sigma(P,X)^*, Z\mapsto \Big(Y\mapsto\mathbb{E}(\langle Z,Y\rangle)\Big)$$

is an isometry.

Recall: A Banach space X has the Radon-Nikodým property (RNP) with respect to $(\Omega,\mathcal{F},P)$ $:\Leftrightarrow$

$$\forall\,\mu:\mathcal{F}\to X \text{ v.m. of bdd var.}: \Big(\forall\,E\in\mathcal{F}, P(E)=0: \\ \mu(E)=0\Rightarrow\exists\,Y\in L^1(P,X)\,\forall\,E\in\mathcal{F}:\ \mu(E)=\int_E Y\,dP\Big)$$

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is an isomorphism if and only if X^* has the RNP w.r.t. $(\Omega,\mathcal{F},P)$.