An approximation theorem of Runge type for kernels of certain non-elliptic partial differential operators

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Introduction

Runge's Approximation Theorem

For $X_1 \subseteq X_2 \subseteq \mathbb{C}$ open the following are equivalent.

i) For every $g \in \mathscr{H}(X_1)$, for every compact $K \subseteq X_1$, and for every $\varepsilon > 0$ there is $f \in \mathscr{H}(X_2)$ such that

$$\varepsilon > \sup_{z \in K} |f(z) - g(z)| =: \|f - g\|_{0,K},$$

i.e. $r : \mathscr{H}(X_2) \to \mathscr{H}(X_1), f \mapsto f_{|X_1}$ has dense range when $\mathscr{H}(X_1)$ is equipped with the compact-open topology.

ii) X_2 does not contain a compact connected component of $\mathbb{C} \backslash X_1$.

For $X \subseteq \mathbb{C} = \mathbb{R}^2$ we have $\mathscr{H}(X) = \{f \in C^\infty(X);\ \frac{1}{2}\big(\partial_1 + i\partial_2\big)f = 0\}$.

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$X \subseteq \mathbb{R}^d$ open, $\mathscr{E}(X) := C^\infty(X)$ equipped with its natural locally convex topology which is generated by the seminorms

$$\forall\, K \subseteq X \text{ compact}, l \in \mathbb{N}_0 :\ \|f\|_{l,K} := \sup_{|\alpha| \leq l} \sup_{x \in K} |\partial^\alpha f(x)|\ (f \in C^\infty(X))$$

and which makes $\mathscr{E}(X)$ a Fréchet space.

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$$\mathscr{E}_P(X) := \{f \in C^\infty(X);\ P(\partial)f = 0 \text{ in } X\}.$$

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Then $\mathscr{E}_P(X) = \mathscr{D}'_P(X)$ as locally convex spaces and therefore: topology of $\mathscr{E}_P(X)$ is generated by the seminorms $\{\|\cdot\|_{0,K};\ K \subseteq X \text{ compact}\}$.

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$P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$ *elliptic* $:\Leftrightarrow\ \forall\, \xi \in \mathbb{R}^d\backslash\{0\} : 0 \neq P_m(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha$

P elliptic $\Rightarrow$ P hypoelliptic

Lax-Malgrange Theorem ([4], [5])

For $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ open and P elliptic the following are equivalent.

i) The restriction map $r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1}$ has dense range.

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Objective: Given P non-constant, find conditions ensuring that the restriction map

$$r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1}$$

resp.

$$r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$$

has dense range.

A general approximation theorem for kernels of differential operators

Given $P \neq 0, X \subseteq \mathbb{R}^d$ open. Recall that the following are equivalent

i) $P(\partial) : \mathscr{E}(X) \to \mathscr{E}(X)$ is surjective.

ii) X is P-convex for supports, i.e.

$$\forall\, u \in \mathscr{E}'(X) :\ \mathrm{dist}(\mathrm{supp}\, \check{P}(\partial)u, \mathbb{R}^d \backslash X) = \mathrm{dist}(\mathrm{supp}\, u, \mathbb{R}^d \backslash X)$$

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If P is elliptic, every open $X \subseteq \mathbb{R}^d$ is P-convex for supports.

$f : X \to \mathbb{R}$ *satisfies the minimum principle in a closed subset* H *of* $\mathbb{R}^d$ if for every compact set $K \subseteq H \cap X$ we have

$$\inf_{x \in K} f(x) = \inf_{\partial_H K} f(x),$$

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We set $d_X : X \to \mathbb{R}, x \mapsto \operatorname{dist}(x, X^c)$, the *boundary distance of* X.
If X is P-convex for supports then d_X satisfies minimum principle in every characteristic hyperplane H for P, i.e.

$$H = \{\xi \in \mathbb{R}^d; \langle N, \xi \rangle = \beta\}, \beta \in \mathbb{R}, N \in \mathbb{R}^d, |N| = 1, P_m(N) = 0,$$

where $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$.

Theorem ([2, Theorem 4])

Given P non-constant, $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ open, X_2 P-convex for supports. Tfae.

i) X_1 is P-convex for supports and $r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$ has dense range.

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For elliptic P the above yields Lax-Malgrange:

- $\exists C$ compact connected component of $\mathbb{R}^d \backslash X_1, C \subseteq X_2$
  $\Rightarrow \exists \psi \in \mathscr{D}(X_1 \cup C)$; $\psi = 1$ in neighborhood of C so for $\zeta \in \mathbb{C}^d, \check{P}(\zeta) = 0$
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By hypothesis, every connected component of $\mathbb{R}^d \backslash X_1$ intersects $\partial_\infty X_2$ (boundary of X_2 in the one-point compactification of $\mathbb{R}^d$) and $\varphi = 0$ in a neighborhood of $\partial_\infty X_2$

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Hence, a representation of $\mathscr{E}_P(X_j)'$, resp. $\mathscr{D}'_P(X_j)'$, will be useful for the proof.

Representation of $\mathscr{E}_P(X)'$ for X being P-convex for supports due to Grothendieck (see [1]): E be a fixed fundamental solution for $\check{P}(\partial)$, $K \subseteq \mathbb{R}^d$ compact

- $u \in \mathscr{D}'_{\check{P}}(\mathbb{R}^d \backslash K)$ *regular at infinity w.r.t.* E iff for one (then every) $\psi \in \mathscr{E}(\mathbb{R}^d)$ with $\operatorname{supp} \psi \cap K = \emptyset$ and $\operatorname{supp}(1-\psi)$ compact:

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Then

$$\Phi_X : R_{\check{P}}(X^c)_{/\sim} \to \mathscr{E}_P(X)', \langle \Phi_X([u]_\sim), f\rangle := \langle \check{P}(\partial)(\psi u), f\rangle$$

is a well-defined (topological) isomorphism where additionally $\operatorname{supp}(1-\psi) \subseteq X$

Theorem ([2, Theorem 4])

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i) X_1 is P-convex for supports and $r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$ has dense range.

ii) X_1 is P-convex for supports and $r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1}$ has dense range.

iii) For every $u \in \mathscr{E}'(X_2)$ with $\operatorname{supp} \check{P}(\partial)u \subseteq X_1$ it holds $\operatorname{supp} u \subseteq X_1$.

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$P(\partial) : \mathscr{E}(X_1) \to \mathscr{E}(X_1)$ has dense range. By iii) and $P(\partial)^t = \check{P}(\partial)$ it follows that $P(\partial)^t(\mathscr{E}'(X_1))$ is closed in $\mathscr{E}'(X_1)$. By the Closed Range Theorem for Fréchet spaces $P(\partial)(\mathscr{E}(X_1))$ is closed in $\mathscr{E}(X_1)$.

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$\exists\, L \subseteq X_2$ compact : $\psi u \in \mathscr{E}'(L), \check{P}(\partial)(\psi u) \in \mathscr{E}'(X_1) \stackrel{iii)}{\Rightarrow} \psi u \in \mathscr{E}'(X_1)$ ☺

A Runge type approximation theorem for certain non-elliptic differential operators

We consider the class of differential operators $P(\partial)$ for which

$$\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} \text{ is a one-dimensional subspace of } \mathbb{R}^d$$

which contains e.g.

- $P(\partial) = i\frac{\partial}{\partial t} + \Delta_x$ with $(x,t) = (x_1, \ldots, x_{d-1}, t) \in \mathbb{R}^d$ (time-dependent free Schrödinger operator),
- $P(\partial) = \frac{\partial}{\partial t} - Q(\partial_x)$ with $(x,t) = (x_1, \ldots, x_{d-1}, t) \in \mathbb{R}^d$ and elliptic $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ of degree ≥ 2 (non-degenerate parabolic operators).

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Theorem ([3, Corollary 5])

Let P be such that $\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}$ is a one-dimensional subspace, let $X \subseteq \mathbb{R}^d$ be open. Tfae

i) X is P-convex for supports.

ii) d_X satisfies the minimum principle in every characteristic hyperplane.

Runge type approximation theorem ([2, Theorem 1])

Let P be such that $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\}$ is a one-dimensional subspace, let $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ be open and P-convex for supports.
Assume that no intersection of $\mathbb{R}^d \backslash X_1$ with a characteristic hyperplane for P has a compact connected component contained in X_2. Then, both restriction maps

$$r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1} \text{ and } r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$$

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We shall show $x_0 \notin \operatorname{supp} \varphi$ which will imply $\operatorname{supp} \varphi \subseteq L_1 \cup L_2$, where $L_1 := \{x \in X_1; \operatorname{dist}(x, X_1^c) \geq \operatorname{dist}(K, X_1^c)\} \cap \operatorname{conv}(K), L_2 := (X_2 \backslash X_1) \cap \operatorname{supp} \varphi$.

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Assume that no intersection of $\mathbb{R}^d \backslash X_1$ with a characteristic hyperplane for P has a compact connected component contained in X_2. Then, both restriction maps

$$r_{\mathscr{E}} : \mathscr{E}_P(X_2) \to \mathscr{E}_P(X_1), f \mapsto f_{|X_1} \text{ and } r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$$

have dense range.

Sketch of the proof: Claim follows if

$$\forall\, \varphi \in \mathscr{D}(X_2) : \big(\operatorname{supp} \check{P}(\partial)\varphi \subseteq X_1 \Rightarrow \operatorname{supp} \varphi \subseteq X_1\big)$$

Fix $\varphi \in \mathscr{D}(X_2)$ with $K := \operatorname{supp} \check{P}(\partial)\varphi \subseteq X_1$. Fix

$$x_0 \in \{x \in X_1; \operatorname{dist}(x, X_1^c) < \operatorname{dist}(K, X_1^c)\}.$$

We shall show $x_0 \notin \operatorname{supp} \varphi$ which will imply $\operatorname{supp} \varphi \subseteq L_1 \cup L_2$, where $L_1 := \{x \in X_1; \operatorname{dist}(x, X_1^c) \geq \operatorname{dist}(K, X_1^c)\} \cap \operatorname{conv}(K), L_2 := (X_2 \backslash X_1) \cap \operatorname{supp} \varphi$.
Thus, $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in \mathscr{D}(X_1), \varphi_2 \in \mathscr{D}(X_2 \backslash X_1)$.

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$$\Rightarrow \varphi_2 = 0 \Rightarrow \varphi = \varphi_1 \in \mathscr{D}(X_1).$$

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$\alpha 1)$ $\alpha(0) = x_0$,

$\alpha 2)$ $\alpha([0, T]) \cap K = \emptyset$,

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$$\exists\, \varepsilon > 0 :\ \varphi_{|B(\alpha(T),\varepsilon)} = 0, \check{P}(\partial)\varphi_{|\alpha([0,T])+B(0,\varepsilon)} = 0.$$

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$\alpha 4)$ and a result due to Hörmander now imply $0 = \varphi_{|B(\alpha(0),\varepsilon)} = \varphi_{|B(x_0,\varepsilon)}$. ☺

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Corollary ([2, Corollary 3])

Let $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ be elliptic of degree ≥ 2, $P(\partial) = \frac{\partial}{\partial x_d} - Q(\partial)$.
Moreover, let $Y_1 \subseteq Y_2 \subseteq \mathbb{R}^{d-1}$, $I \subseteq \mathbb{R}$ be open.
Then $Y_1 \times I$ and $Y_2 \times I$ are P-convex for supports.
Additionally, if Y_2 does not contain any compact connected component of $\mathbb{R}^{d-1} \backslash Y_1$ then $r_{\mathscr{E}} : \mathscr{E}_P(Y_2 \times I) \to \mathscr{E}_P(Y_1 \times I)$ has dense range.

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