Surjectivity of linear partial differential operators on spaces of scalar-valued and vector-valued distributions

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February 23, 2016

For general $P \in \mathbb{C}[X_1, \ldots, X_d]$ set

$$P(D) := P(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_d}).$$

(E.g. $\Delta = P_L(D)$ for $P_L(\xi) = -\sum_{j=1}^{d} \xi_j^2$ (Laplace operator)

$\Delta_x - \frac{\partial}{\partial t} = P_H(D)$ for $P_H(\xi_1, \ldots, \xi_d) = i\xi_d - \sum_{j=1}^{d-1} \xi_j^2$ (Heat operator)

$\Delta_x + i\frac{\partial}{\partial t} = P_S(D)$ for $P_S(\xi_1, \ldots, \xi_d) = -\xi_d - \sum_{j=1}^{d-1} \xi_j^2$ (Schrödinger operator)

$\Delta_x - \frac{\partial^2}{\partial t^2} = P_W(D)$ for $P_W(\xi_1, \ldots, \xi_d) = \xi_d^2 - \sum_{j=1}^{d-1} \xi_j^2$ (Wave operator).)

For $X \subseteq \mathbb{R}^d$ open and connected, f given solve $P(D)u = f$ in X.

Let $P \in \mathbb{C}[X_1, \ldots, X_d] \backslash \{0\}$ and let $X \subseteq \mathbb{R}^d$ be open and connected.

i) When is $P(D) : C^\infty(X) \to C^\infty(X)$ surjective?

ii) When is $C^\infty(X) \subseteq P(D)(\mathscr{D}'(X))$?

iii) When is $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective?

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Answers will depend on combined properties of P and X.

Theorem (Malgrange, 1956)

Tfae:

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i) $P(D) : C^\infty(X) \to C^\infty(X)$ is surjective.

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iii) X is P-convex for supports, i.e.

$$\forall\, u \in \mathscr{E}'(X) : \operatorname{dist}(\operatorname{supp} \check{P}(D)u, X^c) = \operatorname{dist}(\operatorname{supp} u, X^c).$$

Here $\check{P}(\xi) := P(-\xi)$ and $\operatorname{supp} u$ is the support of u, i.e. the complement in X of the largest open subset Y of X, such that $u(\varphi) = 0$ for all $\varphi \in \mathscr{D}(Y)$.

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Theorem of supports: $\forall\, u \in \mathscr{E}'(X) : \overline{\mathrm{conv}(\mathrm{supp}\,\check{P}(D)u)} = \overline{\mathrm{conv}(\mathrm{supp}\,u)}$
$\Rightarrow$ convex sets are P-convex for supports for every $P \neq 0$

Geometrical conditions for/characterization of P-convexity for supports?

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For $d = 2$ P-convexity for supports is completely characterized.

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Theorem (Hörmander, 1962)

$\mathscr{D}'(X)/C^\infty(X) = P(D)(\mathscr{D}'(X)/C^\infty(X))$ iff X P-convex for singular supports, i.e.

$$\forall\, u \in \mathscr{E}'(X) : \operatorname{dist}(\operatorname{sing\,supp} \check{P}(D)u, X^c) = \operatorname{dist}(\operatorname{sing\,supp} u, X^c).$$

Here $\operatorname{sing\,supp} u$ is the singular support of u, i.e. the complement in X of the largest open subset Y of X where $u(\varphi) = \int_Y \varphi(x) f(x)\, dx$ with $f \in C^\infty(Y)$ for all $\varphi \in \mathscr{D}(Y)$.

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X strongly P-convex $:\Leftrightarrow$ X P-convex for supports and singular supports

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For $d = 2$ P-convexity for singular supports is completely characterized.

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E be a complete locally convex space (of functions or distributions describing the kind of parameter dependence, e.g. $H(K)$ for $K \subseteq \mathbb{C}^n$ compact).

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Examples: $H(K)$ for $K \subseteq \mathbb{C}^n$ compact, $\mathscr{S}', \mathscr{E}'_{2\pi}(\mathbb{R}^n), \mathscr{D}'(K)$ for $K \subseteq \mathbb{R}^n$ compact

Theorem (Bonet, Domański 2006)

Let E be as above and let $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ be surjective. If $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective, where $P^+(\xi_1, \ldots, \xi_d, \xi_{d+1}) := P(\xi_1, \ldots, \xi_d)$, then $P(D) : \mathscr{D}'(X, E) \to \mathscr{D}'(X, E)$ is surjective.

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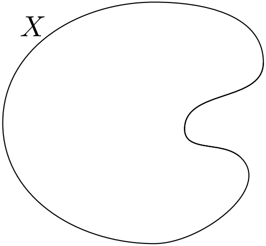

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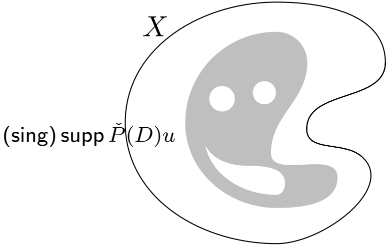

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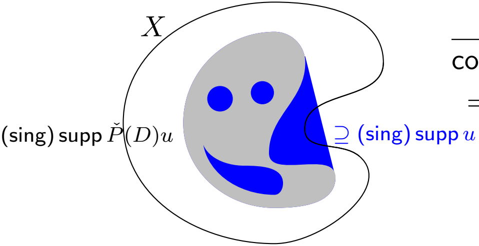

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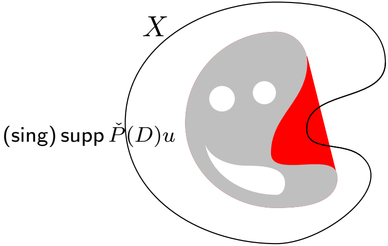

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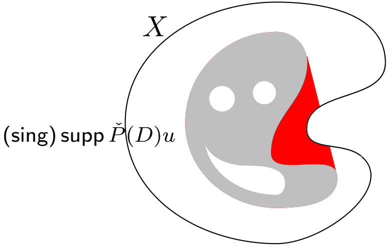
What can we say about the location of (sing) supp u if we know (sing) supp $\check{P}(D)u$?



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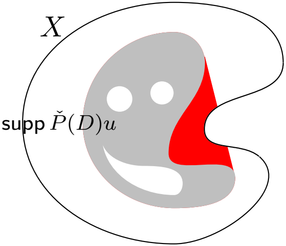

Consequence of a Theorem due to Holmgren

Let $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ be open and convex. Tfae:

i) $\forall\, v \in \mathscr{D}'(X_2), \check{P}(D)v = 0 : \big(v_{|X_1} = 0 \Rightarrow v = 0\big)$

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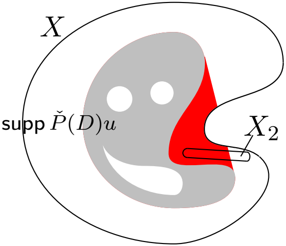

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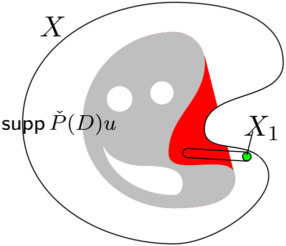

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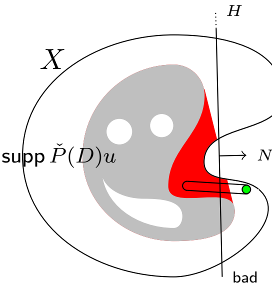

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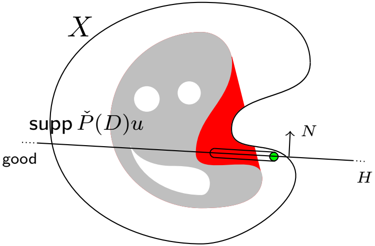

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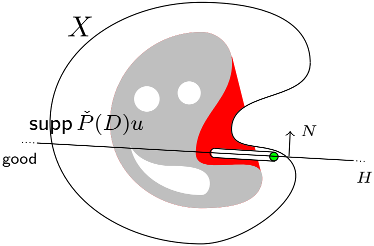

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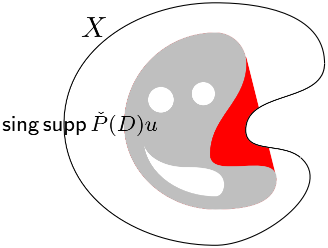

X
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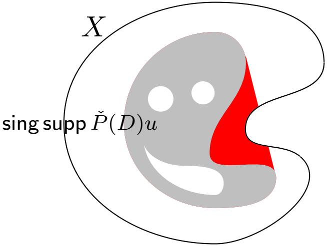

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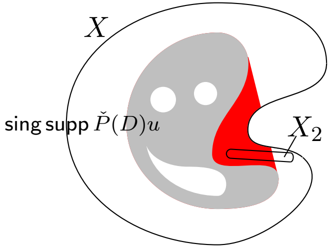

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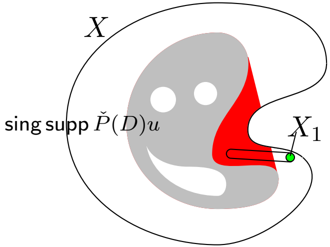

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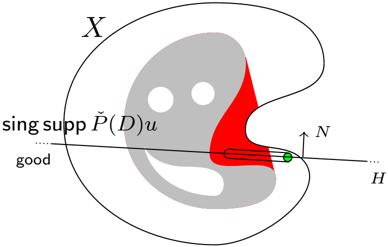

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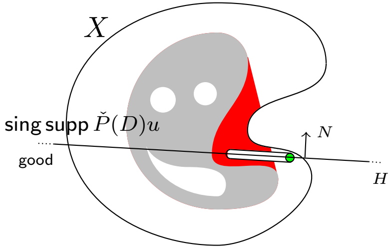

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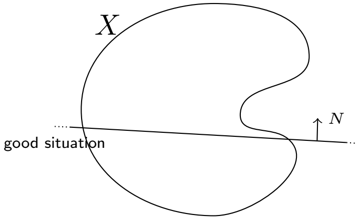

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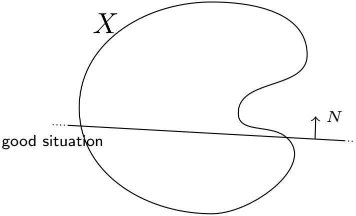

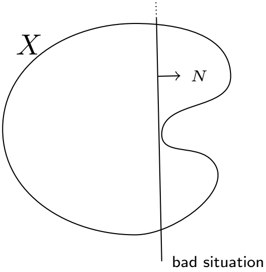

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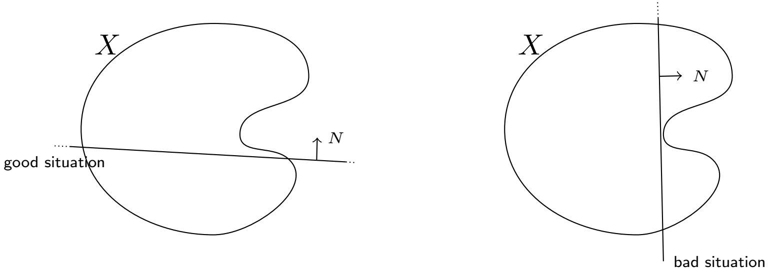

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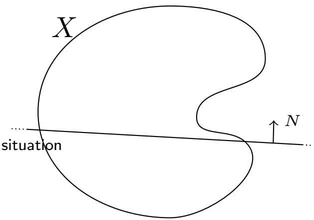

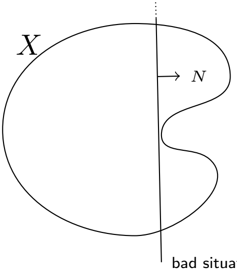

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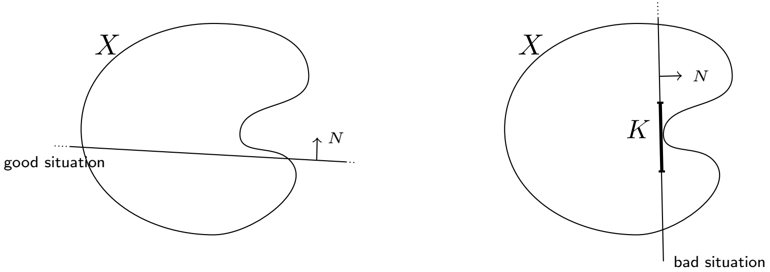

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Corollary 1

If $\{x;\, P_m(x) = 0\}$ is a one-dimensional subspace then X is P-convex for supports iff d_X satisfies the minimum principle in $x + W$ for every $x \in \mathbb{R}^d$ with $W = \{x;\, P_m(x) = 0\}^\perp$.

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Applicable to $P_S(D) = \Delta_x + i\frac{\partial}{\partial t}$ and parabolic operators, e.g. $P_H(D) = \Delta_x - \frac{\partial}{\partial t}$.

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Corollary 2

If $\{x;\, \sigma_P(x) = 0\} = W^\perp$ for some subspace $W \subseteq \mathbb{R}^d$ with $\sigma_P(W^\perp) = 0$ then X is P-convex for singular supports iff d_X satisfies the minimum principle in $x + W$ for every $x \in \mathbb{R}^d$.

Let $\emptyset \neq \Gamma \subset \mathbb{R}^d$ be an open convex cone and

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d; \forall\, x \in \Gamma : \langle x, \xi \rangle \geq 0\}$$

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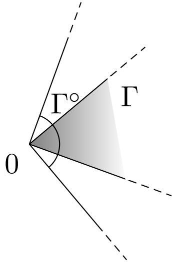

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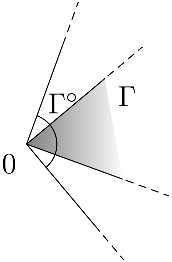

Γ° is a closed, proper, convex cone

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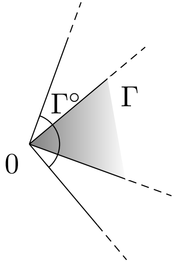

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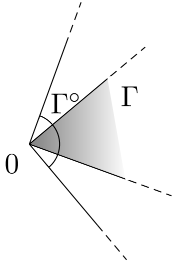

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From now on always $\emptyset \neq \Gamma \neq \mathbb{R}^d \Rightarrow 0 \notin \Gamma$ and $\Gamma^\circ \notin \{\mathbb{R}^d, \{0\}\}$

Corollary 3

Let $\Gamma^\circ \subset \mathbb{R}^d$ be a closed proper convex cone, $X := \mathbb{R}^d \backslash \Gamma^\circ$ and let P be with principal part P_m.

i) X is P-convex for supports if and only if $P_m(x) \neq 0$ for all $x \in \Gamma$.

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Hörmander: $\exists$ polynomial $R, \deg R \leq 6 :\ P(x) := A^4(x) + R(x)$ is hypoelliptic, i.e. $\forall\, y \in \mathbb{R}^d \backslash \{0\} :\ \sigma_P(y) \neq 0$

Theorem 2

For every $d \geq 3$ there are $X \subset \mathbb{R}^d$ open and a polynomial P such that

$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$$

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$d \geq 3$ is essential here! It can be shown that for $d = 2$ surjectivity of $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ implies surjectivity of $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$.

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Let $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ be surjective, P_m be the principal part of P. Then $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective in the following cases.

i) P is semi-elliptic with $\{x;\, P_m(x) = 0\}$ being a one-dimensional subspace. This holds in particular for parabolic P, like e.g. the heat operator $P(D) = \Delta_x - \frac{\partial}{\partial t}$.

ii) P acts along a subspace W and is elliptic as a polynomial on W.

iii) P factorises into linear factors.

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- Hörmander: X P-convex for supports $\Leftrightarrow$ d_X satisfies the minimum principle in $x + W$ for all $x \in \mathbb{R}^d$

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Hence, X P-convex for supports iff X P-convex for singular supports.

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Hence, X P-convex for supports iff X P-convex for singular supports. P^+ acts along a subspace and is elliptic there, too

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