On surjectivity of partial differential operators with a single characteristic direction and on Runge pairs for such operators

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Analysis Meeting

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1. Surjectivity of differential operators

2. An approximation theorem of Runge type

3. The linear topological invariant (Ω) for kernels

Surjectivity of differential operators

For $P \in \mathbb{C}[X_1, \ldots, X_d]$ set

$$P(\partial) := P(\partial_1, \ldots, \partial_d) := P(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}).$$

(E.g. $\Delta = P_L(\partial)$ for $P_L(\xi) = \sum_{j=1}^{d} \xi_j^2$ (Laplace operator)

$\frac{\partial}{\partial t} - \Delta_x = P_H(\partial)$ for $P_H(\xi_1, \ldots, \xi_d) = \xi_d - \sum_{j=1}^{d-1} \xi_j^2$ (Heat operator)

$i\frac{\partial}{\partial t} + \Delta_x = P_S(\partial)$ for $P_S(\xi_1, \ldots, \xi_d) = i\xi_d + \sum_{j=1}^{d-1} \xi_j^2$ (Schrödinger operator)

$\frac{\partial^2}{\partial t^2} - \Delta_x = P_W(\partial)$ for $P_W(\xi_1, \ldots, \xi_d) = \xi_d^2 - \sum_{j=1}^{d-1} \xi_j^2$ (Wave operator)

$\frac{1}{2}\big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\big) = \partial_{\bar{z}}$ for $P(\xi_1, \xi_2) = \frac{1}{2}\big(\xi_1 + i\xi_2\big)$ (Cauchy-Riemann operator).)

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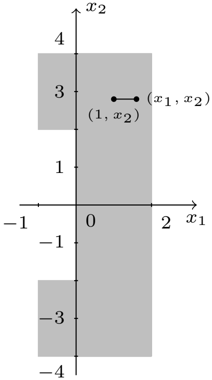
Is this possible for every f from a fixed space of functions/distributions?
"Solution" in which sense; classical, distributional?

Example: $X = \Big((0,2)\times(-4,4)\Big) \cup \Big((-1,1)\times(-4,-2)\Big) \cup \Big((-1,1)\times(2,4)\Big)$

$P_1(\xi_1,\xi_2) = \xi_1 \;\Rightarrow P_1(\partial) = \partial_1$;

given $f \in C^\infty(X) \;\Rightarrow$
$u(x_1,x_2) := \int_1^{x_1} f(t,x_2)\,dt \;\in C^\infty(X)$
satisfies $\partial_1 u = f$



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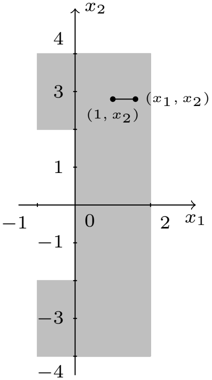
$\Rightarrow P_1(\partial) : C^\infty(X) \to C^\infty(X)$ surjective



Example: $X = \Big((0,2)\times(-4,4)\Big) \cup \Big((-1,1)\times(-4,-2)\Big) \cup \Big((-1,1)\times(2,4)\Big)$

$P_2(\xi_1,\xi_2) = \xi_2 \;\Rightarrow P_2(\partial) = \partial_2$;

choose $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 0$ for $t \notin [-1,1]$ and $\int_{-1}^{1} \eta(t)\,dt > 0$; set

$$f(x_1,x_2) = \begin{cases} \frac{\eta(x_2)}{x_1}, & \text{if } x_1 > 0 \\ 0, & \text{if } x_1 \leq 0 \end{cases}$$

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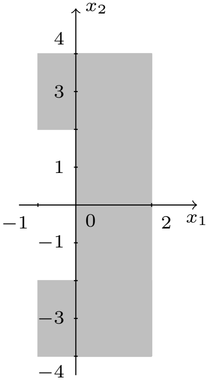

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assume $\exists\, u \in C^1(X) : \partial_2 u = f$;

for $x_1 \in (0,2)$ we then have

$u(x_1,3) - u(x_1,-3) = \int_{-3}^3 \partial_2 u(x_1,t)dt$

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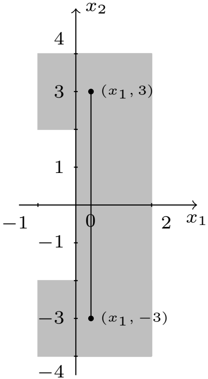

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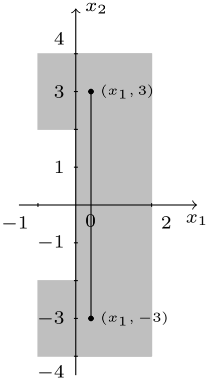
$\Rightarrow C^\infty(X) \not\subseteq P(\partial)(C^1(X))$



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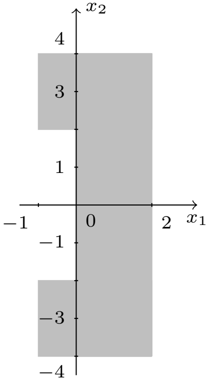
For $P_1(\xi_1, \xi_2) = \xi_1$ resp. $P_2(\xi_1, \xi_2) = \xi_2$ is

$P_1(\partial) : C^\infty(X) \to C^\infty(X)$ surjective,

$P_2(\partial) : C^\infty(X) \to C^\infty(X)$ not surjective.

Is it possible to "see" this without calculation? What about $P_2(\partial)$ if we allow for more general solutions of $P_2(\partial)u = f, f \in C^\infty(X)$, than $u \in C^1(X)$?



Let $P \in \mathbb{C}[X_1, \ldots, X_d]\backslash\{0\}$ and let $X \subseteq \mathbb{R}^d$ be open.

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Some thoughts on i): Equip $C^\infty(X)$ with topology generated by the seminorms

$$\| \cdot \|_{l,K} : C^\infty(X) \to [0, \infty), f \mapsto \max_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq l} \max_{x \in K} |\partial^\alpha f(x)| \, (l \in \mathbb{N}_0, K \Subset X)$$

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$C^\infty(X)' = \mathscr{E}'(X)$ and $P(\partial)^t = \check{P}(\partial)$, where $\check{P}(\xi) = P(-\xi)$.
$P(\partial) : C^\infty(X) \to C^\infty(X)$ has always dense range

Theorem (Malgrange, 1956)

Tfae:

i) $P(\partial): C^\infty(X) \to C^\infty(X)$ is surjective.

ii) X is P-convex for supports, i.e.

$$\forall\, K \Subset X\, \exists\, \tilde{K} \Subset X\, \forall\, u \in \mathscr{E}'(X): \big(\operatorname{supp} \check{P}(\partial)u \subseteq K \Rightarrow \operatorname{supp} u \subseteq \tilde{K}.\big)$$

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$$\Leftrightarrow \quad \forall u \in \mathscr{E}'(X) : \operatorname{dist}(\mathbb{R}^d \backslash X, \operatorname{supp} u) \geq \operatorname{dist}(\mathbb{R}^d \backslash X, \operatorname{supp} \check{P}(\partial)u)$$

$$\Leftrightarrow \quad \forall \varphi \in \mathscr{D}(X) : \operatorname{dist}(\mathbb{R}^d \backslash X, \operatorname{supp} \varphi) \geq \operatorname{dist}(\mathbb{R}^d \backslash X, \operatorname{supp} \check{P}(\partial)\varphi)$$

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iv) $P(\partial) : \mathscr{D}'_F(X) \to \mathscr{D}'_F(X)$ is surjective.

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Theorem of supports: $\forall\, u \in \mathscr{E}'(X):\ \operatorname{conv}\big(\operatorname{supp} u\big) = \operatorname{conv}\big(\operatorname{supp} \check{P}(\partial)u\big)$
$\Rightarrow$ every convex X is P-convex for supports (Recall: $P \neq 0$!)

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$$\forall\, \xi \in \mathbb{R}^d \backslash \{0\}; 0 \neq P_m(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha \, (\textit{principal part of } P)$$

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If P acts along a subspace of $\mathbb{R}^d$ and is elliptic there, then P-convexity for supports is completely characterized (Nakane, 1979).

For polynomials with principal part $P_2(\xi) = \xi_d^2 - \sum_{j=1}^{d-1} \xi_j^2$ P-convexity for supports is completely characterized (Persson, 1981).

For P of real principal type there are characterizations if

- X is bounded and ∂X is analytic (Tintarev, 1988)
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For $d = 2$ P-convexity for supports is completely characterized (Hörmander, 1971).

Necessary condition for P-convexity for supports:

$f : X \to \mathbb{R}$ *satisfies the minimum principle in a (fixed) closed subset* F *of* $\mathbb{R}^d$ if for every compact set $K \subseteq F \cap X$ we have

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We set $d_X : X \to \mathbb{R}, x \mapsto \mathrm{dist}(x, \mathbb{R}^d \backslash X)$, the *boundary distance of* X.

X is P-convex for supports $\Rightarrow d_X$ satisfies the minimum principle in every characteristic hyperplane H for P, i.e. in

$$H = x + \big(\mathrm{span}\{N\}\big)^{\perp} \; (x \in \mathbb{R}^d, N \in \mathbb{R}^d, |N| = 1, P_m(N) = 0).$$

$\mathrm{span}\{N\}$ *characteristic direction of* P

Necessary condition for P-convexity for supports:
$f : X \to \mathbb{R}$ *satisfies the minimum principle in a (fixed) closed subset* F *of* $\mathbb{R}^d$ if for every compact set $K \subseteq F \cap X$ we have

$$\inf_{x \in K} f(x) = \inf_{\partial_F K} f(x),$$

where $\partial_F K$ denotes the boundary of K in F.

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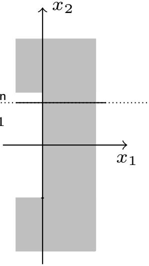
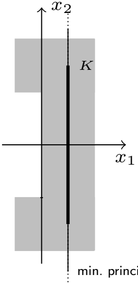
min. principle satisfied in
char. for $P(\partial) = \partial_1$





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Theorem [6, Theorem 1]

Given P with principal part P_m such that $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\} \subseteq W$ for a subspace $W \subsetneq \mathbb{R}^d$. Moreover, let $X \subseteq \mathbb{R}^d$ be open such that d_X satisfies the minimum principle in $x + W^\perp$ for every $x \in \mathbb{R}^d$. Then X is P-convex for supports.

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$\alpha 1)$ $\alpha(0) = x$,

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By $\alpha 4) :\, \forall\, y \in \mathbb{R}^d, N \in \mathbb{R}^d, |N| = 1, P_m(N) = 0 :$

$$\Big(H_{y,N} \cap \big(\alpha([0, T]) + B(0, \varepsilon)\big) \neq \emptyset \Rightarrow H_{y,N} \cap B(\alpha(T), \varepsilon) \neq \emptyset\Big).$$

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$u_{|B(\alpha(T),\varepsilon)} = 0$ and a UCP now imply $u_{|B(\alpha(0),\varepsilon)} = 0$, thus by $\alpha 1)\, x \notin \operatorname{supp} u$ ☺

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Corollary [6, Corollary 5]

i) Given P with principal part P_m, $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\} = \text{span}\{N\}, |N| = 1$. Then X is P-convex for supports if and only if d_X satisfies the minimum principle in $x + \big(\text{span}\{N\}\big)^\perp$ for every $x \in \mathbb{R}^d$.

ii) Let $p \in \mathbb{N}_0, d \in \mathbb{N}, d \geq 2$, and let $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ be elliptic with $\deg(Q) =: m \geq p + 1$. Moreover, let $Y \subseteq \mathbb{R}^{d-1}, I \subseteq \mathbb{R}$ be open. The operator

$$\frac{\partial^p}{\partial t^p} - Q(\partial_y) : C^\infty(Y \times I) \to C^\infty(Y \times I)$$

is surjective (coefficients of Q_m real, $p = 1$: non-degenerate parabolic operator; more general, p odd: p-parabolic operator).

An approximation theorem of Runge type

Runge's Approximation Theorem

For $X_1 \subseteq X_2 \subseteq \mathbb{C}$ open the following are equivalent.

i) For every $g \in \mathscr{H}(X_1)$, for every compact $K \subseteq X_1$, and for every $\varepsilon > 0$ there is $f \in \mathscr{H}(X_2)$ such that

$$\varepsilon > \sup_{z \in K} |f(z) - g(z)| = \|f - g\|_{0,K},$$

i.e. $r : \mathscr{H}(X_2) \to \mathscr{H}(X_1), f \mapsto f_{|X_1}$ has dense range when $\mathscr{H}(X_1)$ is equipped with the compact-open topology (topology of local uniform convergence); (X_1, X_2) is a *Runge pair*.

ii) For every compact connected component C of $\mathbb{C} \backslash X_1$ it holds $C \nsubseteq X_2$.

For $P \in \mathbb{C}[X_1, \ldots, X_d] \backslash \{0\}$ set $C_P^\infty(X) := \{f \in C^\infty(X);\, P(\partial)f = 0 \text{ in } X\}$.

$\Rightarrow C_P^\infty(X)$ is a closed subspace of $C^\infty(X)$ thus a Fréchet space.

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Set $\mathscr{D}'_P(X) := \{u \in \mathscr{D}'(X);\, P(\partial)u = 0\}$.
P *hypoelliptic* $:\Leftrightarrow \forall\, X \subseteq \mathbb{R}^d$ open : $\mathscr{D}'_P(X) = C_P^\infty(X)$

P elliptic $\Rightarrow$ P hypoelliptic

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We equip $\mathscr{D}'(X)$ with the strong dual topology and endow $\mathscr{D}'_P(X)$ with the subspace topology.

P hypoelliptic $\Rightarrow C_P^\infty(X) = \mathscr{D}'_P(X)$ as locally convex spaces and therefore: topology of $C_P^\infty(X)$ is generated by the seminorms $\{\|\cdot\|_{0,K};\ K \Subset X\}$, i.e. it is the compact-open topology (topology of local uniform convergence).

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$d = 2, P(\xi_1, \xi_2) = \frac{1}{2}(\xi_1 + i\xi_2) \Rightarrow C_P^\infty(X) = \mathscr{H}(X), X \subseteq \mathbb{C} = \mathbb{R}^2$ open

Lax-Malgrange Theorem

For $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ open and P elliptic the following are equivalent.

i) The restriction map $r_{C^\infty} : C_P^\infty(X_2) \to C_P^\infty(X_1), f \mapsto f_{|X_1}$ has dense range, i.e. (X_1, X_2) is a Runge pair for $P(\partial)$.

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Consider the class of differential operators $P(\partial)$ for which

$$\exists\, N \in \mathbb{R}^d, |N| = 1 : \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} = \mathrm{span}\{N\}$$

which contains e.g. $P(\partial) = \frac{\partial^p}{\partial t^p} - Q(\partial_y)$ for $x = (y,t) = (y_1, \ldots, y_{d-1}, t) \in \mathbb{R}^d$ where $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ is elliptic with $\deg(Q) =: m \geq p+1$ (heat operator, time dependent free Schrödinger operator, etc.).

When have

$$r_{C^\infty} : C_P^\infty(X_2) \to C_P^\infty(X_1), f \mapsto f_{|X_1}, \text{ resp. } r_{\mathscr{D}'} : \mathscr{D}_P'(X_2) \to \mathscr{D}_P'(X_1), u \mapsto u_{|X_1},$$

dense range?

Theorem [7, Theorem 4]

Given P non-constant, $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ open, X_2 P-convex for supports. Tfae.

i) X_1 is P-convex for supports and $r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$ has dense range.

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For elliptic P the above yields Lax-Malgrange:

- $\exists C$ compact connected component of $\mathbb{R}^d \backslash X_1, C \subseteq X_2$
  $\Rightarrow \exists \psi \in \mathscr{D}(X_1 \cup C)$; $\psi = 1$ in neighborhood of C so for $\zeta \in \mathbb{C}^d, \check{P}(\zeta) = 0$
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  Given $\varphi \in \mathscr{D}(X_2)$ with $\operatorname{supp} \check{P}(\partial)\varphi \subseteq X_1 \Rightarrow \varphi$ real analytic in $\mathbb{R}^d \backslash X_1$
  By hypothesis, every connected component of $\mathbb{R}^d \backslash X_1$ intersects $\partial_\infty X_2$
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For elliptic P the above yields Lax-Malgrange:

- $\exists\, C$ compact connected component of $\mathbb{R}^d \backslash X_1, C \subseteq X_2$ $\Rightarrow\ \exists \psi \in \mathscr{D}(X_1 \cup C)$; $\psi = 1$ in neighborhood of C so for $\zeta \in \mathbb{C}^d, \check{P}(\zeta) = 0$ $\operatorname{supp} \check{P}(\partial)(e^{\langle \zeta, \cdot \rangle}\psi) \subseteq X_1$. Thus, iv) does not hold for $\varphi := e^{\langle \zeta, \cdot \rangle}\psi$
- Assume, no compact connected component of $\mathbb{R}^d \backslash X_1$ is contained in X_2. Given $\varphi \in \mathscr{D}(X_2)$ with $\operatorname{supp} \check{P}(\partial)\varphi \subseteq X_1 \Rightarrow \varphi$ real analytic in $\mathbb{R}^d \backslash X_1$ By hypothesis, every connected component of $\mathbb{R}^d \backslash X_1$ intersects $\partial_\infty X_2$ (boundary of X_2 in the one-point compactification of $\mathbb{R}^d$) and $\varphi = 0$ in a neighborhood of $\partial_\infty X_2 \Rightarrow \varphi_{|\mathbb{R}^d \backslash X_1} = 0$, i.e. iv) holds.

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With Hahn-Banach Theorem:

r_{C^∞}, resp. $r_{\mathscr{D}'}$, has dense range iff $r^t_{C^\infty}$, resp. $r^t_{\mathscr{D}'}$, is injective.

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Hence, a representation of $C^\infty_P(X_j)'$, resp. $\mathscr{D}'_P(X_j)'$, will be useful for the proof.

Representation of $C_P^\infty(X)'$ for X being P-convex for supports due to Grothendieck: Fix fundamental solution E for $\check{P}(\partial)$.

- For $K \Subset \mathbb{R}^d$ we call $u \in \mathscr{D}'_{\check{P}}(\mathbb{R}^d \backslash K)$ *regular at infinity w.r.t.* E iff for one (then every) $\psi \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \psi \cap K = \emptyset$ and $\operatorname{supp}(1-\psi)$ compact:
$$E * \check{P}(\partial)(\psi u) = \psi u.$$

$$R_{\check{P}}(\mathbb{R}^d \backslash K) := \{u \in \mathscr{D}'_{\check{P}}(\mathbb{R}^d \backslash K);\, u \text{ regular w.r.t. } E\}$$

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- $R_{\check{P}}(X^c) := \cup_{K \Subset X} R_{\check{P}}(\mathbb{R}^d \backslash K)$. For $u, v \in R_{\check{P}}(X^c)$ we define
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$$\Rightarrow \Phi_X : R_{\check{P}}(X^c)_{/\sim} \to C_P^\infty(X)', \langle \Phi_X([u]_\sim), f\rangle := \langle \check{P}(\partial)(\psi u), f\rangle$$
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Example (Köthe, 1953):
$P(\partial) = \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2), X = B(0,1) \subseteq \mathbb{R}^2 = \mathbb{C}, E(z) = \frac{1}{\pi z}$.
$\Rightarrow R_{\check{P}}(B(0,1)^c) = \{u \in \mathscr{H}(\mathbb{C} \backslash B(0,1));\, \lim_{|z|\to\infty} u(z) = 0\}$
$$\forall\, f, u : \langle \Phi([u]_\sim), f\rangle = -\int_{\mathbb{C}} \partial_{\bar{z}}(\psi u)(z) f(z) dz = \frac{1}{2\pi i} \int_{|z|=1-\frac{\varepsilon}{2}} u(z) f(z)\, dz.$$
where $\varepsilon \in (0,1)$ is such that u has a holomorphic representative on $\mathbb{C} \backslash B[0, 1-\varepsilon]$.

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$P(\partial) : C^\infty(X_1) \to C^\infty(X_1)$ has dense range. By iii) and $P(\partial)^t = \check{P}(\partial)$ it follows that $P(\partial)^t(\mathscr{E}'(X_1))$ is closed in $\mathscr{E}'(X_1)$. By the Closed Range Theorem for Fréchet spaces $P(\partial)(C^\infty(X_1))$ is closed in $C^\infty(X_1)$.

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where $\psi \in C^\infty(\mathbb{R}^d), \operatorname{supp}\psi \cap K = \emptyset, \operatorname{supp}(1-\psi) \subseteq X_1$ compact. $\Phi_{X_2}([u]_\sim) = 0 \Rightarrow \exists\, L \Subset X_2 : \psi u \in \mathscr{E}'(L)$

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Let $K \Subset X_1$, $u \in R_{\check{P}}(\mathbb{R}^d \backslash K)$ with $r^t_{C^\infty}(\Phi_{X_1}([u]_\sim)) = 0$, i.e.

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where $\psi \in C^\infty(\mathbb{R}^d), \operatorname{supp} \psi \cap K = \emptyset, \operatorname{supp}(1-\psi) \subseteq X_1$ compact. $\Phi_{X_2}([u]_\sim) = 0 \Rightarrow \exists\, L \Subset X_2 : \psi u \in \mathscr{E}'(L), \check{P}(\partial)(\psi u) \in \mathscr{E}'(X_1) \overset{iii)}{\Rightarrow} \psi u \in \mathscr{E}'(X_1)$ ☺

Theorem [7, Theorem 1]

Given P with $\exists\, N \in \mathbb{R}^d, |N| = 1 :\ \{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\} = \text{span}\{N\}$ and let $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ be open and P-convex for supports.
Assume

$$\forall\, x \in \mathbb{R}^d : \big(C \text{ compact connected component of } (\mathbb{R}^d \backslash X_1) \cap H_x \Rightarrow C \nsubseteq X_2\big),$$

where $H_x = x + \big(\text{span}\{N\}\big)^\perp$. Then, both restriction maps $r_{C^\infty} : C_P^\infty(X_2) \to C_P^\infty(X_1), f \mapsto f_{|X_1}$ and $r_{\mathscr{D}'} : \mathscr{D}'_P(X_2) \to \mathscr{D}'_P(X_1), u \mapsto u_{|X_1}$ have dense range.

Theorem [7, Theorem 1]

Given P with $\exists\, N \in \mathbb{R}^d, |N| = 1 : \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} = \mathrm{span}\{N\}$ and let $X_1 \subseteq X_2 \subseteq \mathbb{R}^d$ be open and P-convex for supports.
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have dense range.

Corollary [7, Corollary 3]

Let $p \in \mathbb{N}_0, d \in \mathbb{N}, d \geq 2$, and let $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ be elliptic of degree $\geq p + 1$. Moreover, let $Y_1 \subseteq Y_2 \subseteq \mathbb{R}^{d-1}, I_1 \subseteq I_2 \subseteq \mathbb{R}$ be open such that Y_2 does not contain a compact connected component of $\mathbb{R}^{d-1} \backslash Y_1$. Then, with $P(\partial) = \frac{\partial^p}{\partial t^p} - Q(\partial_y)$ both restriction maps

$$r_{C^\infty} : C_P^\infty(Y_2 \times I_2) \to C_P^\infty(Y_1 \times I_1) \text{ and } r_{\mathscr{D}'} : \mathscr{D}_P'(Y_2 \times I_2) \to \mathscr{D}_P'(Y_1 \times I_1)$$

have dense range.

The linear topological invariant (Ω) for kernels

Let $P(\partial) : C^\infty(X) \to C^\infty(X)$ be surjective.
Given a locally convex space F, is $P(\partial) : C^\infty(X, F) \to C^\infty(X, F)$ surjective?

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$C_P^\infty(X)$ has (Ω) if and only if

$\exists\, T : s \to C_P^\infty(X)$ linear, continuous, surjective (Vogt, Wagner 1980)

(Thus: $C_P^\infty(X)$ has $(\Omega) \Rightarrow C_P^\infty(X)$ has a (absolute) Schauder basis)

E Fréchet space with a fundamental sequence of seminorms $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$

(E.g. $E = \mathscr{H}(X)$, $\|\cdot\|_k := \|\cdot\|_{0,K_k}$, for a compact exhaustion $(K_k)_{k\in\mathbb{N}}$ of X).

For $u \in E', k \in \mathbb{N}$, set $\|u\|_k^* := \sup_{f\in E, \|f\|_k\leq 1} |\langle u, f\rangle|$, *dual seminorm to* $\|\cdot\|_k$.

E has (Ω) :$\Leftrightarrow$

$$\forall\, k \in \mathbb{N}\, \exists\, l \geq k\, \forall\, n \geq l\, \exists \lambda \in (0,1), C > 0 :\ \|\cdot\|_l^* \leq C \|\cdot\|_k^{*\,\lambda}\, \|\cdot\|_n^{*\,1-\lambda}.$$

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Example: $P(\xi_1, \xi_2) = \frac{1}{2}\big(\partial_1 + i\partial_2\big) \Rightarrow C_P^\infty(B(0,1)) = \mathscr{H}(B(0,1))$, fundamental sequence of seminorms $\|f\|_k := \sup_{|z|\leq 1-\frac{1}{k+1}} |f(z)|, k \in \mathbb{N}$.

Grothendieck-Köthe duality:

$$\mathscr{H}(B(0,1))' \cong \{u \in \mathscr{H}(\mathbb{C}\backslash B(0,1));\ \lim_{|z|\to\infty} u(z) = 0\}$$

as well as

$$\{u \in \mathscr{H}(B(0,1))';\ \|u\|_k^* < \infty\} \cong \{u \in \mathscr{H}(\mathbb{C}\backslash B(0, 1 - \frac{1}{k+1}));\ \lim_{|z|\to\infty} u(z) = 0\}$$

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with $\|u\|_k^* = \sup_{|z|=1-1/(k+1)} |u(z)|$. Hadamard's Three Circles Theorem applied to $z \mapsto \tilde{u}(z) := u(\frac{1}{z})$ (holomorphic in a neighborhood of $B[0, 1+\frac{1}{k}]$) gives

$$\|u\|_l^* = \sup_{|z|=1+\frac{1}{l}} |\tilde{u}(z)| \leq \Big(\sup_{|z|=1+\frac{1}{k}} |\tilde{u}(z)|\Big)^\lambda \Big(\sup_{|z|=1+\frac{1}{n}} |\tilde{u}(z)|\Big)^{1-\lambda} = \|u\|_k^{*\,\lambda} \|u\|_n^{*\,1-\lambda}$$

for $k \leq l \leq n$ with $\lambda = \frac{\ln(1+1/l)-\ln(1+1/n)}{\ln(1+1/k)-\ln(1+1/n)}$, so $\mathscr{H}(B(0,1))$ has (Ω)

$C_P^\infty(X)$ has (Ω) if

- P is elliptic, X arbitrary (Vogt, 1983)
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For $\alpha, \mathbf{m} \in \mathbb{N}_0^d$ define $|\alpha : \mathbf{m}| := \sum_{j=1}^d \alpha_j/m_j$; P is called *semi-elliptic* if it is possible to write

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P elliptic $\Rightarrow$ P semi-elliptic $\Rightarrow$ P hypoelliptic

Examples: $P(\xi) = \xi_d - \sum_{j=1}^{d-1} \xi_j^2$; more general $P(\xi) = \xi_d^p - Q(\xi_1, \ldots, \xi_{d-1})$, with $p \in \mathbb{N}$, p odd, $Q \in \mathbb{C}[X_1, \ldots, X_{d-1}]$ elliptic, $\deg(Q) = m \geq p + 1$, coefficients of Q_m real.

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P semi-elliptic $\Rightarrow$ $\{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}$ is a subspace of $\mathbb{R}^d$.

Theorem [6, Theorem 18]

Let P be semi-elliptic with principal part P_m and let $X \subseteq \mathbb{R}^d$ be open.
If d_X satisfies the minimum principle in $x + \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}^\perp$ for every $x \in \mathbb{R}^d$ then X is P-convex for supports and $C_P^\infty(X)$ has (Ω).

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Corollary

Let P be semi-elliptic such that $\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\} = \operatorname{span}\{N\}$ with $|N| = 1$.
Tfae

i) X is P-convex for supports.

ii) X is P-convex for supports and $C_P^\infty(X)$ has (Ω).

iii) $\forall\, x \in \mathbb{R}^d :\, d_X$ satisfies the minimum principle in $x + \big(\operatorname{span}\{N\}\big)^\perp$.

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