Surjectivity of augmented linear partial differential operators

Thomas Kalmes
TU Chemnitz

First Joint International Meeting
RSME-SCM-SEMA-SIMAI-UMI,
Bilbao,
June 30 - July 4, 2014

Introduction

For general $P \in \mathbb{C}[X_1, \ldots, X_d]$ set

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E.g. $\Delta = P_L(D)$ for $P_L(\xi) = -\sum_{j=1}^{d} \xi_j^2$ (Laplace operator)

$\Delta_x - \frac{\partial}{\partial t} = P_H(D)$ for $P_H(\xi_1, \dots, \xi_d) = i\xi_d - \sum_{j=1}^{d-1} \xi_j^2$ (Heat operator)

$\Delta_x + i\frac{\partial}{\partial t} = P_S(D)$ for $P_S(\xi_1, \dots, \xi_d) = -\xi_d - \sum_{j=1}^{d-1} \xi_j^2$ (Schrödinger operator)

$\Delta_x - \frac{\partial^2}{\partial t^2} = P_W(D)$ for $P_W(\xi_1, \dots, \xi_d) = \xi_d^2 - \sum_{j=1}^{d-1} \xi_j^2$ (Wave operator)

Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ and let $X \subseteq \mathbb{R}^d$ be open.

i) When is $P(D) : C^\infty(X) \to C^\infty(X)$ surjective?

ii) When is $C^\infty(X) \subseteq P(D)(\mathscr{D}'(X))$?

iii) When is $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective?

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Answers will depend on combined properties of P and X.

Theorem (Malgrange, 1956)

For $X \subseteq \mathbb{R}^d$ open, $P \in \mathbb{C}[X_1, \dots, X_d]$ tfae:

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Here iii) means

$$\forall\, u \in \mathscr{E}'(X) : \mathrm{dist}(\mathrm{supp}\, u, X^c) = \mathrm{dist}(\mathrm{supp}\, \check{P}(D)u, X^c)$$

where $\check{P}(\xi) := P(-\xi)$ and $\mathrm{supp}\, u$ is the support of u, i.e. the complement in X of the largest open subset Y of X, such that $u(\varphi) = 0$ for all $\varphi \in \mathscr{D}(Y)$.

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Theorem (Hörmander, 1962)

For $X \subseteq \mathbb{R}^d$ open and $P \in \mathbb{C}[X_1, \ldots, X_d]$ it holds that $\mathscr{D}'(X)/C^\infty(X) = P(D)(\mathscr{D}'(X)/C^\infty(X))$ iff X P-convex for singular supports

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Theorem (Hörmander, 1962)

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$$\forall\, u \in \mathscr{E}'(X) : \operatorname{dist}(\operatorname{sing\,supp} u, X^c) = \operatorname{dist}(\operatorname{sing\,supp} \check{P}(D)u, X^c)$$

Here $\operatorname{sing\,supp} u$ is the singular support of u, i.e. the complement in X of the largest open subset Y of X where $u(\varphi) = \int_Y \varphi(x) f(x)\, dx$ with $f \in C^\infty(Y)$ for all $\varphi \in \mathscr{D}(Y)$.

Let $X_j \subseteq \mathbb{R}^{d_j}$ be open, $P_j \in \mathbb{C}[X_1, \ldots, X_{d_j}]$, $j = 1, 2$,

$$P_1 \otimes P_2 : \mathbb{R}^{d_1+d_2} \to \mathbb{C}, P_1 \otimes P_2(x, y) := P_1(x)P_2(y)$$

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Natural question:

$$P_j(D) : \mathscr{D}'(X_j) \to \mathscr{D}'(X_j) \text{ surjective}, j = 1, 2,$$
$$\overset{?}{\Rightarrow} P_1 \otimes P_2(D) : \mathscr{D}'(X_1 \times X_2) \to \mathscr{D}'(X_1 \times X_2) \text{ surjective}$$

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Using a result of Valdivia and Vogt (independent): $P_1 \otimes P_2(D)$ surjective on $\mathscr{D}'(X_1 \times X_2)$ iff

$$P_1^+(D) \text{ surjective on } \mathscr{D}'(X_1 \times \mathbb{R})$$

and

$$P_2^+(D) \text{ surjective on } \mathscr{D}'(X_2 \times \mathbb{R}),$$

where

$$P_1^+(\xi_1, \ldots, \xi_{d_1+1}) := P_1(\xi_1, \ldots, \xi_{d_1}) \text{ and analogously } P_2^+.$$

Bonet, Domański (2006): Does $P(D)(\mathscr{D}'(X)) = \mathscr{D}'(X)$ always imply $P^+(D)(\mathscr{D}'(X \times \mathbb{R})) = \mathscr{D}'(X \times \mathbb{R})$?

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Theorem (Vogt, 1983)

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If X is P-convex for supports then $X \times \mathbb{R}$ is P^+-convex for supports.

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So the problem reduces to: X P-convex for supports and singular supports $\overset{?}{\Rightarrow}$ $X \times \mathbb{R}$ P^+-convex for singular supports

Theorem (K. 2012)

i) Let $P \in \mathbb{C}[X_1, X_2]$ and $X \subseteq \mathbb{R}^2$ be open. Then surjectivity of $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ implies surjectivity of the augmented operator $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$.

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ii) For every $d \geq 3$ there is a hypoelliptic $P \in \mathbb{C}[X_1, \ldots, X_d]$ and $X \subseteq \mathbb{R}^d$ open such that $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective but $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is not.

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When does P-convexity for supports and singular supports of $X \subseteq \mathbb{R}^d$ imply P^+-convexity for singular supports of $X \times \mathbb{R}$?

General conditions for P-convexity for (singular) supports

X P-convex for singular supports $\Leftrightarrow$

$$\forall\, u \in \mathscr{E}'(X) : \mathrm{dist}(\mathrm{sing\,supp}\, u, X^c) = \mathrm{dist}(\mathrm{sing\,supp}\, \check{P}(D)u, X^c)$$

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What can be said about the location of $\operatorname{sing\,supp} u$ if we know $\operatorname{sing\,supp} \check{P}(D)u$?

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Problem of "Continuation of regularity" from $\check{P}(D)u$ to u.

For $\xi \in \mathbb{R}^d, t \geq 1$ and a subspace $V \subseteq \mathbb{R}^d$ define

$$\tilde{P}_V(\xi,t) := \sup_{x \in V, |x| \leq t} |P(x+\xi)|, \ \tilde{P}(\xi,t) := \tilde{P}_{\mathbb{R}^d}(\xi,t),$$

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In order to determine P^+-convexity for singular supports of $X \times \mathbb{R}$ by properties of P and X the functional σ_{P^+} should be modified!

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We set $d_X : X \to \mathbb{R}, x \mapsto \mathrm{dist}(x, X^c)$, the *boundary distance of* X.

Theorem 1

Let P be a polynomial with principal part P_m, $W \subseteq \mathbb{R}^d$ be a subspace, and $X \subseteq \mathbb{R}^d$ open such that d_X satisfies the minimum principle in $x + W$ for every $x \in \mathbb{R}^d$.

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i) If $\{x \in \mathbb{R}^d; \sigma_P(x) = 0\} \subseteq W^\perp$ then X is P-convex for singular supports.

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iii) If $\{x \in \mathbb{R}^d; P_m(x) = 0\} \subseteq W^\perp$ then X is P-convex for supports.

Theorem 2

Let P have principal part P_m and let $X \subseteq \mathbb{R}^d$ be open.

i) If $\{x;\, P_m(x) = 0\}$ is a one-dimensional subspace then X is P-convex for supports iff d_X satisfies the minimum principle in $x + W$ for every $x \in \mathbb{R}^d$ where $W = \{x;\, P_m(x) = 0\}^\perp$.

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i) applicable to $P_H(\xi_1, \ldots, \xi_d) = i\xi_d - \sum_{j=1}^{d-1} \xi_j^2$, i) and ii) applicable to $P_S(\xi_1, \ldots, \xi_d) = -\xi_d - \sum_{j=1}^{d-1} \xi_j^2$

Surjectivity of some augmented partial differential operators

Theorem 3

Let P have principal part P_m and let $X \subseteq \mathbb{R}^d$ be open such that $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective. Then the augmented operator $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective if

i) P acts along a subspace W and is elliptic as a polynomial on W.

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ii) P is a semi-elliptic polynomial with $\{x;\, P_m(x) = 0\}$ being a one-dimensional subspace (e.g. $P_H(\xi) = i\xi_d - \sum_{j=1}^{d-1} \xi_j^2$).

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Let P have principal part P_m and let $X \subseteq \mathbb{R}^d$ be open such that $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective. Then the augmented operator $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective if

i) P acts along a subspace W and is elliptic as a polynomial on W.

ii) P is a semi-elliptic polynomial with $\{x;\, P_m(x) = 0\}$ being a one-dimensional subspace (e.g. $P_H(\xi) = i\xi_d - \sum_{j=1}^{d-1} \xi_j^2$).

iii) $P(\xi) = -\xi_d - \sum_{j=1}^{d-1} \xi_j^2$.

Sketch of the proof of i): $\{x; \sigma_P(x) = 0\} = \{x; \sigma_P^0(x) = 0\} = W^\perp$
Hörmander: X P-convex for supports $\Leftrightarrow$ d_X satisfies minimum principle in $x + W$ for all x
Theorem 1 $\Rightarrow$ $X \times \mathbb{R}$ P^+-convex for singular supports. $\square$

References

- J. Bonet, P. Domański. Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences. *J. Funct. Anal.* 230 (2006), 329-381.
- L. Frerick, T. Kalmes. Some results on surjectivity of augmented semi-elliptic differential operators. *Math. Ann.* 347 (2010), 81-94.
- L. Hörmander. *The Analysis of Linear Partial Differential Operators I and II*, Springer-Verlag, Berlin, 1983.
- T. Kalmes. Some results on surjectivity of augmented differential operators. *J. Math. Anal. Appl.* 386 (2012), 125-134.
- T. Kalmes. The augmented operator of a surjective partial differential operator with constant coefficients needs not to be surjective. *Bull. London Math. Soc.* 44 (2012), 610-614.
- D. Vogt. On the solvability of $P(D)f = g$ for vector valued functions. *RIMS Kokyoroku* 508 (1983), 168-181.