Surjectivity of augmented differential operators

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Bonet, Domański (2006): Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ and $X \subseteq \mathbb{R}^d$ be open such that

$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$$

is surjective.
Does it follows that

$$P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$$

is surjective? (Here $P^+(x_1, \ldots, x_d, x_{d+1}) := P(x_1, \ldots, x_d)$.)

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Recall: $P(x) = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$ with $a_\alpha \neq 0$ for some $|\alpha| = m$ is *elliptic* iff $P_m(x) := \sum_{|\alpha|=m} a_\alpha x^\alpha \neq 0 \,\forall\, x \in \mathbb{R}^d \backslash \{0\}$.
Examples: $P(D) = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}), P(D) = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$.

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$\Rightarrow$ $\mathscr{D}'_P(X)$ is a nuclear Fréchet space, hence $\mathscr{D}'_P(X)$ has $(P\Omega)$ iff it has (Ω).

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X convex $\Rightarrow$ $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective for any $P \neq 0$
$\Rightarrow$ "Yes" for any $P \neq 0$ and every convex X

$P(D):\mathscr{D}'(X)\rightarrow\mathscr{D}'(X)$ surjective $\Leftrightarrow P(D):C^\infty(X)\rightarrow C^\infty(X)$,
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Malgrange (1956): $P(D):C^\infty(X)\to C^\infty(X)$ surjective $\Leftrightarrow$ X is P-convex for supports.

X is P-convex for supports $\Leftrightarrow$ for every $\mu\in\mathscr{E}'(X)$:

$$\operatorname{dist}(X^c,\operatorname{supp}\mu)=\operatorname{dist}(X^c,\operatorname{supp}\check{P}(D)\mu),$$

where $\check{P}(x)=P(-x)$.

$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective $\Leftrightarrow P(D) : C^\infty(X) \to C^\infty(X)$,
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Malgrange (1956): $P(D) : C^\infty(X) \to C^\infty(X)$ surjective $\Leftrightarrow$ X is P-convex for supports.

Hörmander (1962): $P(D) : \mathscr{D}'(X)/C^\infty(X) \to \mathscr{D}'(X)/C^\infty(X)$ surjective $\Leftrightarrow$ X is P-convex for singular supports.

X is P-convex for singular supports $\Leftrightarrow$ for every $\mu \in \mathscr{E}'(X)$:

$$\text{dist}(X^c, \text{sing supp}\,\mu) = \text{dist}(X^c, \text{sing supp}\,\check{P}(D)\mu)$$

Reformulation of the problem of Bonet and Domański:

Let X be P-convex for supports as well as P-convex for singular supports.
Is $X \times \mathbb{R}$ P^+-convex for supports and P^+-convex for singular supports?

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Proposition

If X is P-convex for supports then $X \times \mathbb{R}$ is P^+-convex for supports.

Reformulation of the problem of Bonet and Domański:

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Is $X \times \mathbb{R}$ P^+-convex for singular supports?

X P-convex for singular supports $\Leftrightarrow$

$$\forall\, \mu \in \mathscr{E}'(X):\ \mathrm{dist}(X^c, \operatorname{sing\,supp} \mu) = \mathrm{dist}(X^c, \operatorname{sing\,supp} \check{P}(D)\mu).$$

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Let E be any fundamental solution for $\check{P}(D)$, i.e. $\check{P}(D)E = \delta$.

$$\forall\, \mu \in \mathscr{E}'(X) :\ \mu = \mu * \delta = \mu * \check{P}(D)E = \check{P}(D)\mu * E$$

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$$\forall\, \mu \in \mathscr{E}'(X):\ \mu = \mu * \delta = \mu * \check{P}(D)E = \check{P}(D)\mu * E$$

$$\Rightarrow \forall\, \mu \in \mathscr{E}'(X):\ \operatorname{sing\,supp} \mu \subseteq \operatorname{sing\,supp} \check{P}(D)\mu + \operatorname{sing\,supp} E$$

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$$\Rightarrow \forall\, \mu \in \mathscr{E}'(X): \ \text{sing supp}\, \mu \subseteq \text{sing supp}\, \check{P}(D)\mu + \text{sing supp}\, E$$

For a specific fundamental solution $E(\check{P})$ the location of its singular support is well understood by means of *localizations at infinity*:

Let $Q \in \mathbb{C}[X_1, \dots, X_d]$, $t \geq 1$ and $V \subseteq \mathbb{R}^d$ subspace.

For $\xi \in \mathbb{R}^d$ set $Q_\xi(x) := Q(x + \xi)$ and

$$\tilde{Q}_V(\xi, t) := \sup_{x \in V, |x| \leq t} |Q(x + \xi)| (= \sup_{x \in V, |x| \leq t} |Q_\xi(x)| = (\tilde{Q}_\xi)_V(0, t)),$$

as well as

$$\tilde{Q}(\xi, t) := \tilde{Q}_{\mathbb{R}^d}(\xi, t).$$

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The set of limits of the normalized polynomials

$$x \to \frac{\check{P}_\xi(x)}{\tilde{\check{P}}_\xi(0, 1)}$$

as ξ tends to infinity is denoted by $L(\check{P})$.

Non-zero multiples of elements of $L(\check{P})$ are called *localizations of* $\check{P}$ *at infinity*.

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Define

$$\Lambda(Q) := \{\eta \in \mathbb{R}^d;\ \forall\, x \in \mathbb{R}^d, t \in \mathbb{R} :\ Q(x + t\eta) = Q(x)\}$$

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- $\Lambda(Q)$ subspace and $\Lambda(Q) = \Lambda(\check{Q})$
- Q constant $\Leftrightarrow\ \Lambda(Q)^{\perp} = \{0\}$

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Hörmander:

- sing supp $E(\check{P}) \subseteq \overline{\bigcup_{Q \in L(P)} \Lambda(Q)^{\perp}}$
- $\dim \Lambda(Q)^{\perp} \leq d - 1$

For $\mu \in \mathscr{E}'(\Omega)$

$$\begin{aligned}\text{sing supp}\,\mu &\subseteq \text{sing supp}\,\check{P}(D)\mu + \text{sing supp}\,E(\check{P}) \\ &\subseteq \text{sing supp}\,\check{P}(D)\mu + \overline{\bigcup_{Q \in L(P)} \Lambda(Q)^{\perp}}\end{aligned}$$

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Non-constant $Q \in L(P)$ and their $\Lambda(Q)^{\perp}$s are responsible if

$$\exists\, \mu \in \mathscr{E}'(X) :\ \text{dist}(X^c, \text{sing supp}\,\mu) < \text{dist}(X^c, \text{sing supp}\,\check{P}(D)\mu).$$

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$Q \in L(P)$ non-constant $\Rightarrow \lim_{t\to\infty} \tilde{Q}(0,t) = \infty$ while
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$$\begin{aligned} 0 &= \inf_{t\geq 1} \frac{\tilde{Q}_{\Lambda(Q)}(0,t)}{\tilde{Q}(0,t)} = \inf_{t\geq 1} \lim_{n\to\infty} \frac{\tilde{P}_{\Lambda(Q)}(\xi_n,t)}{\tilde{P}(\xi_n,t)} \\ &\geq \inf_{t\geq 1} \liminf_{\xi\to\infty} \frac{\tilde{P}_{\Lambda(Q)}(\xi,t)}{\tilde{P}(\xi,t)}. \end{aligned}$$

Hörmander: For $V \subset \mathbb{R}^d$ subspace define

$$\sigma_P(V) = \inf_{t \geq 1} \liminf_{\xi \to \infty} \frac{\tilde{P}_V(\xi, t)}{\tilde{P}(\xi, t)}.$$

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Abbreviation: $\forall\, y \in \mathbb{R}^d \backslash \{0\} : \; \sigma_P(y) = \sigma_P(\mathrm{span}\{y\})$

Let $\emptyset \neq \Gamma \subset \mathbb{R}^d$ be an open convex cone and

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d; \forall\, x \in \Gamma : \langle x, \xi \rangle \geq 0\}$$

its dual cone.

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Conversely: Every closed proper convex cone C is the dual cone of a unique open convex cone Γ.

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From now on always $\Gamma^\circ \notin \{\emptyset, \{0\}\}$!

$\Rightarrow \Gamma \neq \mathbb{R}^d$ and $0 \notin \Gamma$!

Theorem

Let $\Gamma^\circ \subset \mathbb{R}^d$ be a closed proper convex cone, $X := \mathbb{R}^d \backslash \Gamma^\circ$ and let P be a polynomial.
X is P-convex for singular supports if and only if $\sigma_P(x) \neq 0$ for all $x \in \Gamma$.

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X is P-convex for singular supports if and only if $\sigma_P(x) \neq 0$ for all $x \in \Gamma$.

When is $(\mathbb{R}^d \backslash \Gamma^\circ) \times \mathbb{R}$ P^+-convex for singular supports?

For a subspace $V \subseteq \mathbb{R}^d$ let

$$\sigma_P^0(V) := \inf_{t \geq 1, \xi \in \mathbb{R}^d} \frac{\tilde{P}_V(\xi, t)}{\tilde{P}(\xi, t)}.$$

Again: $\forall\, y \in \mathbb{R}^d \backslash \{0\} : \sigma_P^0(y) := \sigma_P^0(\text{span}\{y\})$.

For a subspace $V \subseteq \mathbb{R}^d$ let

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Notation: For $x = (x_1, \ldots, x_d, x_{d+1}) \in \mathbb{R}^{d+1}$ let $x' := (x_1, \ldots, x_d) \in \mathbb{R}^d$. For $W \subseteq \mathbb{R}^{d+1}$ set $W' := \{x'; \, x \in W\}$.

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Lemma (L. Frerick, K.)

For a subspace $W \subseteq \mathbb{R}^{d+1}$ the following are equivalent.

i) $\sigma_{P+}(W) = 0$,

ii) $\sigma_P^0(W') = 0$.

Theorem

Let $\Gamma^\circ \subset \mathbb{R}^d$ be a closed proper convex cone, $X := \mathbb{R}^d \backslash \Gamma^\circ$ and let P be with principal part P_m.

i) X is P-convex for singular supports if and only if $\sigma_P(x) \neq 0$ for all $x \in \Gamma$.

ii) $X \times \mathbb{R}$ is P^+-convex for singular supports if and only if $\sigma_P^0(x) \neq 0$ for all $x \in \Gamma$.

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ii) $X \times \mathbb{R}$ is P^+-convex for singular supports if and only if $\sigma_P^0(x) \neq 0$ for all $x \in \Gamma$.

iii) X is P-convex for supports if and only if $P_m(x) \neq 0$ for all $x \in \Gamma$.

Theorem

Let $X \subseteq \mathbb{R}^d$ be open and connected and let P be with principal part P_m.

i) X is P-convex for singular supports if for every $x_0 \in \partial X$ there is a closed proper convex cone Γ° such that $(x_0 + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_P(y) \neq 0$ for every $y \in \Gamma$.

ii) $X \times \mathbb{R}$ is P^+-convex for singular supports if for every $x_0 \in \partial X$ there is a closed proper convex cone Γ° such that $(x_0 + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_P^0(y) \neq 0$ for every $y \in \Gamma$.

iii) X is P-convex for supports if for every $x_0 \in \partial X$ there is a closed proper convex cone Γ° such that $(x_0 + \Gamma^\circ) \cap X = \emptyset$ and $P_m(y) \neq 0$ for every $y \in \Gamma$.

Lemma

Let P have principal part P_m and let $V \subseteq \mathbb{R}^d$ be a subspace.

i) $\sigma_P^0(V) \leq \sigma_P(V)$ and $\forall\, k \in \mathbb{N}:\ \sigma_{P^k}^0(V) = (\sigma_P^0(V))^k$.

ii) $\sigma_P^0(V) \leq \sigma_{P_m}^0(V)$.

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Hörmander: $\exists$ polynomial $R, \deg R \leq 6 :\ P(x) := A^4(x) + R(x)$ is hypoelliptic, i.e. $\forall\, y \in \mathbb{R}^d \backslash \{0\} : \sigma_P(y) \neq 0$

Theorem

For every $d \geq 3$ there are $X \subset \mathbb{R}^d$ open and a polynomial P such that

$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$$

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By hypoellipticity and Bonet, Domański:
$\{f \in C^\infty(X) : P(D)f = 0\}$ does not have (Ω) for such P.

What about $d = 2$?

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Theorem

Let $X \subseteq \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]$. Then the following are equivalent.

i) $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective.

ii) $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective.

For $d \geq 3$ there are some positive results, too.

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Part i) of the following theorem is a well-known classical result.

Theorem

Let $X \subseteq \mathbb{R}^d$ be open with C^1-boundary, P with principal part P_m. Denote by $N(x)$ the exterior normal vector at $x \in \partial X$.

i) If $P_m(N(x)) \neq 0$ for every $x \in \partial X$ then X is P-convex for supports.

ii) If $\sigma_P(N(x)) \neq 0$ for every $x \in \partial X$ then X is P-convex for singular supports.

iii) If $\sigma_P^0(N(x)) \neq 0$ for every $x \in \partial X$ then $X \times \mathbb{R}$ is P^+-convex for singular supports.

Corollary

Let $X \subseteq \mathbb{R}^d$ be open with C^1-boundary, P with principal part P_m.

i) If P is semi-elliptic then $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective if $P_m(N(x)) \neq 0$ for every $x \in \partial X$.

ii) If P is homogeneous or of principal type then $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective if $P_m(N(x))\sigma_P(N(x)) \neq 0$ for every $x \in \partial X$.

Theorem

Let $\Gamma^\circ \subset \mathbb{R}^d$ be a closed proper convex cone, $X := \mathbb{R}^d \backslash \Gamma^\circ$ and P homogeneous, semi-elliptic, or of principal type. Then the following are equivalent.

i) $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective.

ii) $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective.

iii) $P_m(x)\sigma_P(x) \neq 0$ for all $x \in \Gamma$.

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