Some results on surjectivity of augmented differential operators and on a conjecture by Trèves

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some parts joint work with L. Frerick (Trier University)

Bonet, Domański (2006): Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ and $\Omega \subset \mathbb{R}^d$ open such that

$$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

is surjective.

Does it follow that

$$P^+(D) : \mathscr{D}'(\Omega \times \mathbb{R}) \to \mathscr{D}'(\Omega \times \mathbb{R})$$

is surjective? (here $P^+(x_1, \ldots, x_{d+1}) := P(x_1, \ldots, x_d)$)

Well-known: Ω convex $\Rightarrow P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ surjective for arbitrary $P \neq 0$.

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Theorem (Vogt, 1983)

If P is elliptic then the kernel of $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ has $(P\Omega)$.

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Let Ω be P-konvex for supports. Then $\Omega \times \mathbb{R}$ is P^+-convex for supports.

Unfortunately: Analogous implication for convexity for singular supports is not true in general!

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Ω P-convex for singular supports <u>and</u> supports

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$$\mathrm{dist}(\mathrm{sing\,supp}\,\mu, (\Omega \times \mathbb{R})^c) = \mathrm{dist}(\mathrm{sing\,supp}\,P^+(-D)\mu, (\Omega \times \mathbb{R})^c)$$

for each $\mu \in \mathscr{E}'(\Omega \times \mathbb{R})$?

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Always: $\text{sing supp}\,P^+(-D)\mu\subset\text{sing supp}\,\mu$.

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And $P^+(-D)\mu \in C^\infty((\Omega \times \mathbb{R}) \cap (\operatorname{sing\,supp} P^+(-D)\mu)^c)$.

Given non-constant $Q \in \mathbb{C}[X_1, \dots, X_{d+1}]$ and $U \subset \mathbb{R}^{d+1}$ open.

Which additional properties (on U, P etc.) ensure for $\mu \in \mathscr{E}'(U)$

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Hörmander's theory of continuation of regularity!

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Theorem (Hörmander)

Let $U_1 \subset U_2 \subset \mathbb{R}^{d+1}$ open and convex, $Q \in \mathbb{C}[X_1, \ldots, X_{d+1}]$ non-constant. Then the following are equivalent:

i) $\mu \in C^{\infty}(U_2)$ for each $\mu \in \mathscr{D}'(U_2)$ satisfying $\mu_{|U_1} \in C^{\infty}(U_1)$ and $Q(-D)\mu \in C^{\infty}(U_2)$.

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Notation: $\sigma_Q(y) := \sigma_Q(span\{y\})$.
Q hypoelliptic $\Leftrightarrow$ $\sigma_Q(y) = 1 \,\forall\, y \neq 0$.

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Lemma

For a hyperplane $H = \{x; \langle x, N \rangle = \alpha\} \subset \mathbb{R}^{d+1}$ we have

$$\sigma_{P^+}(H^\perp) = 0 \Leftrightarrow \sigma_P(N') = 0 \text{ or } P_m(N') = 0.$$

(Here P_m denotes the principal part of P.)

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(Here P_m denotes the principal part of P.)

If P is hypoelliptic: $\sigma_{P+}(H^\perp) = 0 \Leftrightarrow P_m(N') = 0$.

Theorem

If $\Omega \times \mathbb{R}$ is P^+-convex for singular supports then the boundary distance $dist(\cdot, \Omega^c) : \Omega \to \mathbb{R}$ satisfies the minimum principle in every hyperplane $H = \{x \in \mathbb{R}^d; \langle x, N \rangle = \alpha\}$ with $P_m(N)\sigma_P(N) = 0$, i.e.

$$\forall\, K \subset H \cap \Omega \text{ compact} : dist(K, \Omega^c) = dist(\partial_H K, \Omega^c).$$

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Example

$P(x_1, x_2) = ix_1 + x_2^2$ is hypoelliptic
$\Rightarrow \Omega := \{(x_1, x_2); x_1 > 0, x_1^2 + x_2^2 > 1\}$ P-convex for singular supports.
$\Omega \times \mathbb{R}$ is NOT P^+-convex for singular supports: boundary distance does not satisfy minimum principle in $H = \{(2, x_2); x_2 \in \mathbb{R}\}$.

Well-known theorem

Assume Ω has a C^1-boundary such that

$$\forall x \in \partial\Omega : P_m(N(x)) \neq 0,$$

where $N(x)$ denotes the (exterior) normal at x. Then Ω is P-convex for supports.

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Analogously

Theorem

Assume Ω has a C^1-boundary.

i) If $\sigma_P(N(x)) \neq 0$ for every $x \in \partial\Omega$ then Ω is P-convex for singular supports.

ii) If $P_m(N(x))\sigma_P(N(x)) \neq 0$ for every $x \in \partial\Omega$ then $\Omega \times \mathbb{R}$ is P^+-convex for singular supports.

Corollary

If $P \in \mathbb{C}[X_1, \ldots, X_d]$ is hypoelliptic and Ω has a C^1-boundary such that $P_m(N(x)) \neq 0$ for every $x \in \partial\Omega$ then

$$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

as well as

$$P^+(D) : \mathscr{D}'(\Omega \times \mathbb{R}) \to \mathscr{D}'(\Omega \times \mathbb{R})$$

are surjective.

Let $\emptyset \neq \Gamma \subset \mathbb{R}^d$ be an open convex cone and

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d; \forall\, x \in \Gamma : \langle x, \xi \rangle \geq 0\}$$

its dual cone.

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Γ° is a closed convex proper cone.

Conversely: Every closed convex proper cone is the dual cone of a unique open convex cone Γ.

Theorem (Exterior Cone Conditions)

Let Ω be connected.

i) Ω is P-convex for supports if for every $x \in \partial\Omega$ there is an open convex cone $\Gamma \subset \mathbb{R}^d$ such that

$$(x + \Gamma^\circ) \cap \Omega = \emptyset \text{ and } P_m(y) \neq 0 \, \forall y \in \Gamma \backslash \{0\}.$$

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iii) $\Omega \times \mathbb{R}$ is P^+-convex for singular supports if for every $x \in \partial\Omega$ there is an open convex cone $\Gamma \subset \mathbb{R}^d$ such that

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Theorem (Hörmander)

Let $\Omega \subset \mathbb{R}^2$ be open and connected, $P \neq 0$. Then the following are equivalent.

i) Ω is P-convex for supports

ii) The boundary distance $dist(\cdot, \Omega^c)$ satisfies the minimum principle in every characterstic hyperplane.

iii) The intersection of Ω with every characteristic hyperplane is convex.

iv) For every $x \in \partial\Omega$ there is a closed convex proper cone Γ° such that $(x + \Gamma^\circ) \cap \Omega = \emptyset$ and no characteristic hyperplane intersects $x + \Gamma^\circ$ only in x.

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iv') For every $x \in \partial\Omega$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap \Omega = \emptyset$ and $P_m(y) \neq 0$ for every $y \in \Gamma \backslash \{0\}$.

Similar result for P-convexity for singular supports:

Theorem

Let $\Omega \subset \mathbb{R}^2$ be open and connected, $P \neq 0$ such that $\{y \in S^1; \sigma_P(y) = 0\}$ is finite. Then the following are equivalent.

i) Ω is P-convex for singular supports.

ii) The boundary distance $dist(\cdot, \Omega^c)$ satisfies the minimum principle in every hyperplane $H = \{x; \langle x, N\rangle = \alpha\}$ with $\sigma_P(N) = 0$.

iii) The intersection of Ω with every hyperplane as in ii) is convex.

iv) For every $x \in \partial\Omega$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap \Omega = \emptyset$ and $\sigma_p(y) \neq 0$ for every $y \in \Gamma\backslash\{0\}$.

Given $P \in \mathbb{C}[X_1, X_2]$. When is $\{y \in S^1; \sigma_P(y) = 0\}$ finite?

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Remark

In case of $d > 2$ the above inclusion is not true in general!
Take $P(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2$ and $y = (0, 0, 1)$. Then $\sigma_P(y) = 0$ but $P(y) = -1$.

Recall:

Theorem (Hörmander)

$\Omega \subset \mathbb{R}^2$ open, connected, $P \neq 0$. Tfae.

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iv') For every $x \in \partial\Omega$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap \Omega = \emptyset$ and $P_m(y) \neq 0$ for every $y \in \Gamma\backslash\{0\}$.

Theorem

$\Omega \subset \mathbb{R}^2$ open, connected, $P \neq 0$. Tfae.

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iv) For every $x \in \partial\Omega$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap \Omega = \emptyset$ and $\sigma_p(y) \neq 0$ for every $y \in \Gamma\backslash\{0\}$.

Corollary

Let $P \in \mathbb{C}[X_1, X_2]$ and $\Omega \subset \mathbb{R}^2$ be open. If Ω is P-convex for supports then Ω is automatically P-convex for singular supports.

This proves in the affirmative a conjecture by Trèves that, contrary to arbitrary dimension, in case of $d = 2$ surjectivity of $P(D)$ on $C^\infty(\Omega)$ implies surjectivity of $P(D)$ on $\mathscr{D}'(\Omega)$.

Corollary

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Theorem

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i) $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective.

ii) $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective.

iii) $P^+(D) : \mathscr{D}'(\Omega \times \mathbb{R}) \to \mathscr{D}'(\Omega \times \mathbb{R})$ is surjective.

iv) $\Omega \times \mathbb{R}$ is P^+-convex for singular supports.

For arbitrary dimension d only the following result.

Theorem

Let $\Omega_0 \subset \mathbb{R}^d$ be open and convex, $(x_n)_{n\in\mathbb{N}}$ a sequence in Ω_0, $\Gamma_1^\circ, \Gamma_2^\circ, \ldots$ a sequence of closed convex proper cones. Let Ω be the interior of $\Omega_0 \cap \bigcap_{n\in\mathbb{N}}(x_n + \Gamma_n^\circ)^c$.

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If there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that

$$B_{\varepsilon_n}(x_n) \cap (x_n + \Gamma_n^\circ)^c \subset \Omega$$

for each $n \in \mathbb{N}$, then the following are equivalent for $P \in \mathbb{C}[X_1, \ldots, X_d]$.

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