On a conjecture of Trèves and its generalization to ultradistributions

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Let $\Omega \subset \mathbb{R}^d$ be open, $P(D) = P(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_d})$ where $P \in \mathbb{C}[X_1, \ldots, X_d]$.

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Ω is P-convex for supports $\Leftrightarrow$ for every $\mu \in \mathscr{E}'(\Omega)$ one has

$$\mathrm{dist}(\Omega^c, \mathrm{supp}\, \mu) = \mathrm{dist}(\Omega^c, \mathrm{supp}\, \check{P}(D)\mu)$$

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Hörmander (1962): $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective if and only if Ω is P-convex for supports as well as P-convex for singular supports.

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Ω is P-convex for singular supports $\Leftrightarrow$ for every $\mu \in \mathscr{E}'(\Omega)$ one has

$$\mathrm{dist}(\Omega^c, \operatorname{sing\,supp} \mu) = \mathrm{dist}(\Omega^c, \operatorname{sing\,supp} \check{P}(D)\mu)$$

Natural question:

$P(D)$ surjective on $C^\infty(\Omega) \overset{?}{\Rightarrow} P(D)$ surjective on $\mathscr{D}'(\Omega)$

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General answer: No, for $d \geq 3$ there are $\Omega \subset \mathbb{R}^d$ and P such that

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Trèves' conjecture (1966): Yes, for $d = 2$.

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If E is a fundamental solution of $\check{P}(D)$ then

$$\mu = \check{P}(D)\mu * E,$$

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$$\operatorname{sing\,supp} \mu \subset \operatorname{sing\,supp} \check{P}(D)\mu + \operatorname{sing\,supp} E.$$

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There is one fundamental solution of $E(\check{P})$ for which the location of $\operatorname{sing\,supp} E(\check{P})$ is well understood by means of *localizations at infinity*:

Let $Q \in \mathbb{C}[X_1, \ldots, X_d]$, $t \geq 1$ and $V \subset \mathbb{R}^d$ subspace.

Set for $\xi \in \mathbb{R}^d$

$$\tilde{Q}_V(\xi, t) := \sup_{x \in V, |x| \leq t} |Q(x + \xi)|,$$

as well as

$$\tilde{Q}(\xi, t) := \tilde{Q}_{\mathbb{R}^d}(\xi, t).$$

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The set of limits of the normalized polynomials

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For $N \in S^{d-1}$ the set of limits of the normalized polynomials

$$x \rightarrow \frac{\check{P}_\xi(x)}{\tilde{\check{P}}_\xi(0, 1)}$$

as ξ tends to infinity and $\xi / |\xi| \rightarrow N$ is denoted by $L_N(\check{P})$, $L(\check{P}) = \cup_{N \in S^{d-1}} L_N(\check{P})$.

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Define

$$\Lambda(Q) := \{\eta \in \mathbb{R}^d;\ \forall\, x \in \mathbb{R}^d, t \in \mathbb{R} :\ Q(x + t\eta) = Q(x)\}$$

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Hörmander:

- sing supp $E(\check{P}) \subset \overline{\bigcup_{Q \in L(P)} \Lambda(Q)^{\perp}}$
- $N \in \Lambda(Q)$ for all $Q \in L_N(P)$

For $\mu \in \mathscr{E}'(\Omega)$

$$\begin{aligned}
\operatorname{sing\,supp} \mu &\subset \operatorname{sing\,supp} \check{P}(D)\mu + \operatorname{sing\,supp} E(\check{P}) \\
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$Q \in L(P)$ non-constant $\Rightarrow \lim_{t\to\infty} \tilde{Q}(0,t) = \infty$ while $\tilde{Q}_{\Lambda(Q)}(0,t) = \sup_{x \in \Lambda(Q), |x| \leq t} |Q(x+0)| = |Q(0)|$

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For suitable $(\xi_n)_{n\in\mathbb{N}}$ tending to infinity:

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Hörmander: For $\{0\} \neq V \subset \mathbb{R}^d$ subspace define

$$\sigma_P(V) = \inf_{t \geq 1} \liminf_{\xi \to \infty} \frac{\tilde{P}_V(\xi, t)}{\tilde{P}(\xi, t)}.$$

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Abbreviation: $\sigma_P(y) = \sigma_P(\text{span}\{y\}) \quad \Rightarrow \sigma_P(y) = \sigma_P(\lambda y) \, \forall \lambda \neq 0$

Let $\emptyset \neq \Gamma \subset \mathbb{R}^d$ be an open convex cone and

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d; \forall\, x \in \Gamma : \langle x, \xi \rangle \geq 0\}$$

its dual cone.

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Γ° is a closed proper convex cone.

Conversely: Every closed proper convex cone C is the dual cone of a unique open convex cone Γ.

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From now on always $\Gamma^\circ \notin \{\emptyset, \{0\}\}$!

$\Rightarrow \Gamma \neq \mathbb{R}^d$, in particular $0 \notin \Gamma$!

Theorem (Exterior cone condition)

Let $\Omega \subset \mathbb{R}^d$ be open and connected. Then Ω is P-convex for singular supports if for every $x_0 \in \partial\Omega$ there is a closed proper convex cone Γ° such that $(x_0 + \Gamma^\circ) \cap \Omega = \emptyset$ and $\sigma_P(y) \neq 0$ for every $y \in \Gamma$.

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We will show that in case of $d = 2$ and Ω being P-convex for supports and connected the above sufficient condition is always satisfied!

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Let P be of degree m with principal part P_m.

i) Let $N \in S^{d-1}$ and $Q \in L_N(P)$. If $P_m(N) \neq 0$ then Q is constant.

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ii) If P is non-elliptic then for every subspace $\{0\} \neq V \subset \mathbb{R}^d$

$$\sigma_P(V) = \inf_{t \geq 1} \inf_{\substack{N \in S^{d-1}, \\ P_m(N)=0}} \inf_{Q \in L_N(P)} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

One advantage of $\mathbb{R}^2$ over $\mathbb{R}^d$, d arbitrary:

Theorem (Hörmander)

Let $\Omega \subset \mathbb{R}^2$ be open and connected, P a polynomial of degree m with principal part P_m. The following are equivalent:

i) Ω is P-convex for supports.

ii) For every $x_0 \in \partial\Omega$ there is a closed proper convex cone Γ° with $(x_0 + \Gamma^\circ) \cap \Omega = \emptyset$, such that no hyperplane $H = \{x;\ \langle x, N\rangle = \alpha\}$ with $P_m(N) = 0$ intersects $x_0 + \Gamma^\circ$ only in x_0.

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Another advantage of $\mathbb{R}^2$ over $\mathbb{R}^d$, d arbitrary:

Lemma 2

Let $P \in \mathbb{C}[X_1, X_2]$ be of degree m with principal part P_m. Then

$$\{y \in S^1;\ \sigma_P(y) = 0\} \subseteq \{y \in S^1;\ P_m(y) = 0\}.$$

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Let $Q \in L(P)$ non-constant $\underset{\text{Lemma 1}}{\Rightarrow}$ $\exists 1 \leq j \leq l : Q \in L_{N_j}(P)$.

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Let $Q \in L(P)$ non-constant $\underset{\text{Lemma 1}}{\Rightarrow}$ $\exists 1 \leq j \leq l : Q \in L_{N_j}(P)$.

Show that for $y \in S^1$

$$\frac{\tilde{Q}_{\mathrm{span}\{y\}}(0,t)}{\tilde{Q}(0,t)} \geq C\,|\langle y, x_j\rangle|^m,$$

where $C > 0$ depends only on m.

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For $1 \leq j \leq l$ choose $x_j \in S^1$ with $x_j \perp N_j$.
With Lemma 1 it follows for $y \in S^1$:

$$\begin{aligned} \sigma_P(y) &= \inf_{t \geq 1} \inf_{\substack{N \in S^{d-1},\\ P_m(N)=0}} \inf_{Q \in L_N(P)} \frac{\tilde{Q}_{\mathrm{span}\{y\}}(0,t)}{\tilde{Q}(0,t)} \\ &= \inf_{t \geq 1} \min_{1 \leq j \leq l} \inf_{Q \in L_{N_j}(P)} \frac{\tilde{Q}_{\mathrm{span}\{y\}}(0,t)}{\tilde{Q}(0,t)} \geq C \min_{1 \leq j \leq l} |\langle y, x_j \rangle|^m \end{aligned}$$

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Sketch of the proof: W.l.o.g. let P be non-elliptic.
Let $\{N \in S^1;\ P_m(N) = 0\} = \{N_1, \ldots, N_l\}$.
For $1 \leq j \leq l$ choose $x_j \in S^1$ with $x_j \perp N_j$.
With Lemma 1 it follows for $y \in S^1$:

$$\sigma_P(y) \geq C \min_{1 \leq j \leq l} |\langle y, x_j \rangle|^m,$$

Another advantage of $\mathbb{R}^2$ over $\mathbb{R}^d$, d arbitrary:

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so if $0 = \sigma_P(y)$ we have $y \perp x_j$ hence $y \in \{N_j, -N_j\}$ for some $1 \leq j \leq l$. $\square$

Solution of Trèves' conjecture

Let $P \in \mathbb{C}[X_1, X_2]$ and $\Omega \subset \mathbb{R}^2$ open. The following are equivalent:

i) $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective.

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Let $\omega : [0,\infty) \to [0,\infty)$ be a non-quasianalytic weight function, i.e. ω is continuous, increasing and satisfies

(α) $\exists\, K \geq 1\, \forall\, t \geq 0 : \ \omega(2t) \leq K(1+\omega(t))$,

(β) $\int_0^\infty \frac{\omega(t)}{1+t^2}dt < \infty$,

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$$\begin{aligned}\mathscr{D}_{(\omega)}(K) &= \{f \in C^\infty(\mathbb{R}^d); \text{supp}\, f \subset K \text{ and} \\ &\qquad \int_{\mathbb{R}^d} |\hat{f}(x)| \exp(\lambda\omega(x))\, dx < \infty \text{ for all } \lambda \geq 1\}\end{aligned}$$

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$$\mathscr{D}_{(\omega)}(\Omega) = \bigcup_{K \Subset \Omega} \mathscr{D}_{(\omega)}(K)$$

with its natural (LF)-space topology.

$\mathscr{D}'_{(\omega)}(\Omega)$ are the *ultradistributions of Beurling type* over Ω.

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The associated local space

$$\mathscr{E}_{(\omega)}(\Omega) = \{u \in \mathscr{D}'_{(\omega)}(\Omega);\ \varphi u \in \mathscr{D}_{(\omega)}(\Omega) \text{ for all } \varphi \in \mathscr{D}_{(\omega)}(\Omega)\}$$

is the space of *ultradifferentiable functions of Beurling type* over Ω.

Theorem (Bonet, Galbis, Meise; Frerick, Wengenroth)

Let $\Omega \subset \mathbb{R}^d$ be open, ω non-quasianalytic weight function and $P \in \mathbb{C}[X_1, \ldots, X_d]$. The following are equivalent.

i) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ is surjective.

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In case of $d = 2$ is P-convexity for (ω)-singular supports implied by P-convexity for supports?

Langenbruch: For $\{0\} \neq V \subset \mathbb{R}^d$ subspace let

$$\sigma_{P,(\omega)}(V) := \inf_{t \geq 1} \liminf_{\xi \to \infty} \frac{\tilde{P}_V(\xi, t\omega(\xi))}{\tilde{P}(\xi, t\omega(\xi))}.$$

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Theorem (Exterior cone condition for weight functions)

Let $\Omega \subset \mathbb{R}^d$ be open and connected, ω a non-quasianalytic weight function. Then X is P-convex for (ω)-singular supports if for every $x_0 \in \partial\Omega$ there is a closed proper convex cone Γ° such that $(x_0 + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_{P,(\omega)}(y) \neq 0$ for every $y \in \Gamma$.

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Lemma 3

Let $P \in \mathbb{C}[X_1, X_2]$ be of degree m with principal part P_m. Then

$$\{y \in S^1;\, \sigma_{P,(\omega)}(y) = 0\} \subseteq \{y \in S^1;\, P_m(y) = 0\}$$

for any weight function ω.

Main Theorem

Let $P \in \mathbb{C}[X_1, X_2]$ and $\Omega \subset \mathbb{R}^2$ open. The following are equivalent.

i) $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective.

ii) $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective.

iii) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ is surjective for every non-quasianalytic weight function ω.

iv) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ is surjective for some non-quasianalytic weight function ω.

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