UNIVERSAL ZERO SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL OPERATORS

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Abstract. In this paper a generalized approach to several universality results is given by replacing holomorphic or harmonic functions by zero solutions of arbitrary linear partial differential operators. Instead of the approximation theorems of Runge and others, we use an approximation theorem of Hörmander.

1. Introduction

The first universality result in complex analysis is the famous theorem of G. D. Birkhoff [3], which slightly modified states as follows:

Theorem 1.1 (Birkhoff (1929)). *There exists an entire function u such that to every entire function f, every compact set $K \subset \mathbb{C}$ and to every $\varepsilon > 0$, there is a $p \in \mathbb{N}$ satisfying*

$$|u(z+p) - f(z)| < \varepsilon \text{ for all } z \in K.$$

We say that u has *universal translates*. Analogues were obtained by Seidel and Walsh [14] for non-Euclidean translates in the unit disk $\mathbb{D}$ in 1941, and for delations $u(p \cdot z)$, with $p \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, instead of translates by Zappa [16] in 1989. Using linear transformations $u(a\cdot z+b)$ Luh [11] generalized the concept of universality onto simply connected domains in 1979, and onto arbitrary domains later on with his colleagues. The residuality of the corresponding sets of universal functions was discovered by Duyos-Ruiz [5] in 1984.

A further generalization is due to Bernal and Montes [2] (1995), in which composition operators induced by a sequence (f_n) of conformal automorphisms on a general open $\Omega \subset \mathbb{C}$ were considered. Such a sequence is called *run-away* if for every compact subset K of Ω there is some $n \in \mathbb{N}$ with $K \cap f_n(K) = \emptyset$. In these terms, they stated the following result.

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Theorem 1.2 (Bernal, Montes (1995))**.** *Let $\Omega \subset \mathbb{C}$ be an open set that is not conformally equivalent to $\mathbb{C}^*$, and let (f_n) be a sequence of automorphisms of Ω. Then there exists a function u holomorphic on Ω for which the set $\{u \circ f_n : n \in \mathbb{N}\}$ is dense in $H(\Omega)$, the space of all holomorphic functions on Ω, if and only if (f_n) is a run-away sequence. In case of existence the set of such functions u is residual in $H(\Omega)$.*

The first universality result in harmonic analysis is due to Dzagnidze [6] (1969) and is the harmonic analogon of Birkhoff's Theorem.

Our aim is to give a generalized approach and proof of all these universality results. That is, instead of considering the special polynomials

$$P_1 : \mathbb{R}^2 \to \mathbb{C}, \xi \mapsto \frac{1}{2}(\xi_1 + i\xi_2) \qquad \text{and} \qquad P_2 : \mathbb{R}^N \to \mathbb{C}, \xi \mapsto \sum_{j=1}^{N} \xi_j^2$$

which give $P_1(\partial)f = \bar{\partial}f$, i.e. the Cauchy-Riemann operator, and $P_2(\partial)f = \Delta f$, the Laplacian, respectively, we consider arbitrary differential operators $P(\partial)$ and their kernels, where P is a non-constant polynomial on $\mathbb{R}^N$ with complex coefficients. We are interested in properties of sequences of diffeomorphisms $(f_m)_{m\in\mathbb{N}}$ of Ω such that there is an element u of the kernel of $P(\partial)$ such that $\{u \circ f_m; m \in \mathbb{N}\}$ is dense in the kernel.

As domain of definition of $P(\partial)$ we choose the Fréchet spaces $\bigcap_{j=1}^{\infty} B_{p_j,k_j}^{\text{loc}}(\Omega)$ introduced by Hörmander [10], see second section. As a special case, these Fréchet spaces include the space $C^\infty(\Omega)$ equipped with its standard Fréchet space topology, i.e. local uniform convergence of all partial derivatives, which we denote as usual by $\mathscr{E}(\Omega)$. Since the kernels of $P_1(\partial)$ and $P_2(\partial)$ considered as operators on $\mathscr{E}(\Omega)$ give the space of holomorphic functions $H(\Omega)$ on $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ and the space of harmonic functions $h(\Omega)$ on $\Omega \subset \mathbb{R}^N$, respectively, holomorphic as well as harmonic universal functions are covered within this framework. (Note that by a standard application of the Open Mapping Theorem for Fréchet spaces it follows that the topologies inherited by $\mathscr{E}(\Omega)$ are indeed the usual Fréchet space topologies on $H(\Omega)$ and $h(\Omega)$, respectively!)

The price we have to pay for this generality is that we lose special structures of the function spaces. Instead of the approximation theorems of Runge and others, we use a general approximation theorem due to Hörmander, cf. Theorem 4.2 which forces us to impose stronger geometrical conditions on Ω, namely we assume the components of Ω to be convex.

A similar approach in case of translations has been made by Calderón-Moreno and Müller [4]. They are using the famous Lax-Malgrange theorem which guarantees them less losings concerning the structure of the open sets Ω, but they are restricted to elliptic partial differential operators.

Finally, we also want to mention two recent and different directions that are related to the mentioned results. Gauthier and Pouryayevali [7] obtained universal subharmonic functions on $\mathbb{R}^N$ and universal plurisubharmonic functions on $\mathbb{C}^N$ in 2007. By universal they mean the universality property due to Birkhoff. Grosse-Erdmann and Mortini [9] worked on an analogon of Theorem 1.2 but for sequences (f_n) of eventually injective or arbitrary holomorphic self-maps of Ω.

The paper is organized as follows. In section 2 we recall some facts about the Fréchet spaces $\bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega)$. In section 3 we consider the kernel of $P(\partial)$ as a subspace of $\bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega)$ and introduce composition operators $u \mapsto u \circ f$ on these kernels. It turns out that in general f can only be chosen from a very small class of diffeomorphisms in order to ensure that the composition operator is well-defined. Section 4 contains a sufficient condition on the sequence of diffeomorphisms $(f_m)_{m\in\mathbb{N}}$ ensuring the existence of universal zero solutions of $P(\partial)$ as well as an analogous result to Theorem 1.2. Finally, section 5 deals with dense subspaces of universal zero-solutions.

Throughout the paper, we are using the following notations. The interior of a subset M of $\mathbb{R}^N$ is denoted by M°. The Fourier transform of a tempered distribution u is denoted by $\hat{u}$ or $\mathcal{F}(u)$ (where for $u \in L^1(\mathbb{R}^N)$ we set $\hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle} u(x)\,dx$). For a topological vector space E its topological dual E' is always equipped with the weak$*$-topology and by a diffeomorphism we always mean a C^∞-diffeomorphism. As usual, we denote by $\mathscr{S}'$ the space of tempered distributions over $\mathbb{R}^N$, $\mathscr{D}(\Omega)$ is the space of compactly supported C^∞-functions over Ω equipped with its usual inductive limit topology, $\mathscr{D}'(\Omega)$ its dual, i.e. the space of distributions over Ω, and for an arbitrary subset A of $\mathbb{R}^N$ $\mathscr{E}'(A)$ denotes the space of distributions on $\mathbb{R}^N$ having compact support in A. Besides that, we are using the standard notation from functional analysis, which appear in the same way f. i. in [12, 13].

2. The Fréchet space $\bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega)$

In this section we recall some facts about the spaces $\bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega)$ for $\Omega \subset \mathbb{R}^N$ open, introduced by Hörmander. As a reference see e.g. [10, Section 10.1]. A special case of this spaces is the Fréchet space $\mathscr{E}(\Omega)$, i.e. the space $C^\infty(\Omega)$ equipped with its natural topology, that is the topology induced by the seminorms

$$p_{K,m}(f) := \max_{x\in K,|\alpha|\leq m} |\partial^\alpha f(x)|, m \in \mathbb{N}_0, K \subset \Omega \text{ compact}.$$

More general, recall that $k : \mathbb{R}^N \to (0,\infty)$ is called a *tempered weight function* if there are constants $C > 0, m \in \mathbb{N}$ such that

$$\forall\, \xi, \eta \in \mathbb{R}^N : \quad k(\xi + \eta) \leq (1 + C|\xi|)^m k(\eta).$$

Typical examples of tempered weight functions are $k(\xi) = (1 + |\xi|^2)^{s/2}$, where s is an arbitrary real number, or $\tilde{P}(\xi) = (\sum_{|\alpha|\geq 0} |\partial^\alpha P(\xi)|^2)^{1/2}$, where $P \in \mathbb{C}[X_1, \ldots, X_N]$ is a polynomial (see [10, Example 10.1.3]).

For a tempered weight function k and $1 \leq p < \infty$ let

$$\begin{aligned} B_{p,k} := \{u \in \mathscr{S}'&;\ \hat{u} \text{ is a function and} \\ &\|u\|_{p,k} := ((2\pi)^{-N} \int_{\mathbb{R}^N} |k(\xi)\hat{u}(\xi)|^p \, d\xi)^{1/p} < \infty\}. \end{aligned}$$

Then $B_{p,k}$ together with the norm $\|\cdot\|_{p,k}$ is a Banach space, cf. [10, Theorem 10.1.7].

Moreover, for $1 \leq p < \infty$ and a tempered weight function k let

$$B_{p,k}^{\mathrm{loc}}(\Omega) := \{u \in \mathscr{D}'(\Omega);\ \forall\, \phi \in \mathscr{D}(\Omega) :\ \phi u \in B_{p,k}\}.$$

$B_{p,k}^{\mathrm{loc}}(\Omega)$ equipped with the family of seminorms $u \mapsto \|\phi u\|_{p,k}$, $\phi \in \mathscr{D}(\Omega)$, becomes a Fréchet space. For $p = 2$ and $k(\xi) = (1 + |\xi|^2)^{s/2}, s \in \mathbb{R}$, one obtains in this way the local Sobolev space $\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega)$ of order s.

Obviously, for any compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of Ω and $\phi_n \in \mathscr{D}(\Omega)$ satisfying $K_n \subset \{\phi_n = 1\}$ the topology of $B_{p,k}^{\mathrm{loc}}(\Omega)$ is generated by the sequence of seminorms $u \mapsto \|\phi_n u\|_{p,k}, n \in \mathbb{N}$. Furthermore, $\mathscr{E}(\Omega) \subset B_{p,k}^{\mathrm{loc}}(\Omega)$ and the inclusion $\mathscr{E}(\Omega) \hookrightarrow B_{p,k}^{\mathrm{loc}}(\Omega)$ is continuous and has dense range, cf. [10, Theorem 10.1.26 and Theorem 10.1.17].

Finally, for a sequence $(p_j)_{j\in\mathbb{N}} \in [1,\infty)^{\mathbb{N}}$ and a sequence $(k_j)_{j\in\mathbb{N}}$ of tempered weight functions let $\bigcap_{j=1}^\infty B_{p_j,k_j}^{\mathrm{loc}}(\Omega)$ be equipped with the family of seminorms $u \mapsto \|\varphi u\|_j := \|\varphi u\|_{p_j,k_j}$, $j \in \mathbb{N}, \varphi \in \mathscr{D}(\Omega)$. With these seminorms $\bigcap_{j=1}^\infty B_{p_j,k_j}^{\mathrm{loc}}(\Omega)$ is a Fréchet space whose topology is obviously generated by the increasing sequence of seminorms

$$q_n(u) := \max_{1\leq k,j\leq n} \|\phi_k u\|_j, n \in \mathbb{N},$$

with ϕ_k as above.

By the preceding remarks we have $\mathscr{E}(\Omega) \hookrightarrow \bigcap_{j=1}^\infty B_{p_j,k_j}^{\mathrm{loc}}(\Omega)$ continuously with dense range, cf. [10, Theorem 10.1.17]. Since polynomials are dense in $\mathscr{E}(\Omega)$, cf. [15, p. 160], it now follows immediately that polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ are dense in $\bigcap_{j=1}^\infty B_{p_j,k_j}^{\mathrm{loc}}(\Omega)$, so that $\bigcap_{j=1}^\infty B_{p_j,k_j}^{\mathrm{loc}}(\Omega)$ is a separable Fréchet space for each open subset $\Omega \subset \mathbb{R}^N$.

For the special case $k_j(\xi) = (1+|\xi|)^j$ and arbitrary $1 \leq p_j < \infty$ one obtains $\mathscr{E}(\Omega) = \bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega)$ as Fréchet space, cf. [10, Remark following Theorem 10.1.26].

3. The space of zero solutions

For a polynomial $P \in \mathbb{C}[X_1,\ldots,X_N]$ let $\tilde{P}(\xi) = (\sum_{|\alpha|\geq 0} |\partial^\alpha P(\xi)|^2)^{1/2}$. Then $\tilde{P}$ is a tempered weight function (see [10, Example 10.1.3]) and since real powers and products of tempered weight functions are again tempered weight functions, cf. [10, Theorem 10.1.4] $k/\tilde{P}$ is a tempered weight function whenever is k.

By [10, Theorem 10.1.22 and its proof] the mapping

$$P(D) : B^{\mathrm{loc}}_{p,k}(\Omega) \to B^{\mathrm{loc}}_{p,k/\tilde{P}}(\Omega), u \mapsto P(D)u$$

is continuous, where as usual $P(D)u = \sum_{|\alpha|\leq m} (-i)^{|\alpha|} a_\alpha \partial^\alpha u$ for $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$. Therefore,

$$P(D) : \bigcap_{j=1}^{\infty} B^{\mathrm{loc}}_{p_j,k_j}(\Omega) \to \bigcap_{j=1}^{\infty} B^{\mathrm{loc}}_{p_j,k_j/\tilde{P}}(\Omega),$$

is continuous, so that

$$\mathcal{N}_{P,(p_j,k_j)}(\Omega) := \{u \in \bigcap_{j=1}^{\infty} B^{\mathrm{loc}}_{p_j,k_j}(\Omega);\ P(D)u = 0\}$$

is a closed subspace of the separable Fréchet space $\bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega)$, hence a separable Fréchet space itself. When it is clear from the context, we omit the reference to the sequence of tempered weight functions $(k_j)_{j\in\mathbb{N}}$ and $(p_j)_{j\in\mathbb{N}}$ and simply write $\mathcal{N}_P(\Omega)$ instead of $\mathcal{N}_{P,(p_j,k_j)}(\Omega)$. For the special case $k_j(\xi) = (1+|\xi|)^j$, that is $\bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega) = \mathscr{E}(\Omega)$ we simply write $\mathscr{E}_P(\Omega)$ instead of $\mathcal{N}_{P,(p_j,k_j)}(\Omega)$, i.e. $\mathscr{E}_P(\Omega)$ is the vector space

$$\{u \in C^\infty(\Omega);\ P(D)u = 0\}$$

equipped with the topology induced by the seminorms

$$p_{K,m}(u) := \max_{x\in K, |\alpha|\leq m} |\partial^\alpha u(x)|, m \in \mathbb{N}_0, K \subset \Omega \text{ compact}.$$

Obviously, $\mathscr{E}_P(\Omega) \subset \mathcal{N}_{P,(p_j,k_j)}(\Omega)$ for every $(p_j,k_j)_{j\in\mathbb{N}}$. Note that for a hypoelliptic polynomial P one always has $\mathcal{N}_{P,(p_j,k_j)}(\Omega) \subset C^\infty(\Omega)$. Hence it follows from the continuity of the inclusion $\mathscr{E}(\Omega) \hookrightarrow \bigcap_{j=1}^\infty B^{\mathrm{loc}}_{p_j,k_j}(\Omega)$ and the Open Mapping Theorem that $\mathcal{N}_{P,(p_j,k_j)}(\Omega) = \mathscr{E}_P(\Omega)$ as Fréchet spaces, whenever P is hypoelliptic.

In the special case, when $N = 2$ and $P(D) = 1/2\,(\partial_1 + i\partial_2)$ we obtain again by the Open Mapping Theorem, that $\mathscr{E}_P(\Omega)$ is the space of holomorphic functions on Ω equipped with the compact-open topology.

We now introduce composition operators on $\mathcal{N}_{P,(p_j,k_j)}(\Omega)$. For two open subsets Ω_1 and Ω_2 of $\mathbb{R}^N$ and a diffeomorphism $f : \Omega_1 \to \Omega_2$ there is a unique continuous linear mapping $f^* : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_1)$ such that $f^*u = u \circ f$ if $u \in C(\Omega_2)$. For $\varphi \in \mathscr{D}(\Omega_1)$ one has $\langle f^*u, \varphi\rangle = \langle u, |\det Jf^{-1}|\,\varphi \circ f^{-1}\rangle$, where Jf^{-1} denotes the Jacobian of f^{-1}. $(g \circ f)^*u = f^*g^*u$ for a second diffeomorphism g from Ω_2 to Ω_3 and every distribution u on Ω_3 (see e.g. [10, Section 6.1]). We sometimes use the notation $u(f)$ or $u \circ f$ instead of f^*u.

Note that for $\phi \in \mathcal{E}(\Omega_2)$ one has

$$\langle f^*(\phi u), \varphi\rangle = \langle u, |\det Jf^{-1}|(\phi \circ f \circ f^{-1})(\varphi \circ f^{-1})\rangle = \langle (f^*\phi)(f^*u), \varphi\rangle$$

for every $\varphi \in \mathscr{D}(\Omega_1)$, i.e. $f^*(\phi u) = (f^*\phi)(f^*u)$.

A simple property of f^* is stated in the next proposition.

Proposition 3.1. *Let $f : \Omega_1 \to \Omega_2$ be a diffeomorphism. For $u \in \mathscr{D}'(\Omega_2)$ one has $\operatorname{supp} f^*u = f^{-1}(\operatorname{supp} u)$.*

Proof: Let V be an open superset of $\Omega_2 \backslash \operatorname{supp} u$, i.e. $\langle u, \varphi\rangle = 0$ for all $\varphi \in \mathscr{D}(V)$. If $\psi \in \mathscr{D}(f^{-1}(V))$, clearly $\psi \circ f^{-1} \in \mathscr{D}(V)$, hence $|\det Jf^{-1}|(\psi \circ f^{-1}) \in \mathscr{D}(V)$, so that

$$\langle f^*u, \psi\rangle = \langle u, |\det Jf^{-1}|(\psi \circ f^{-1})\rangle = 0.$$

Because $f^{-1}(V)$ is open it follows that $\operatorname{supp} f^*u \subset \Omega_1 \backslash f^{-1}(V)$. Since V was an arbitrary open subset of $\Omega_2 \backslash \operatorname{supp} u$ it follows that

$$\begin{aligned} \operatorname{supp} f^*u &\subset \bigcap_{V \subset \Omega_2 \backslash \operatorname{supp} u, V \text{open}} \Omega_1 \backslash f^{-1}(V) = \Omega_1 \backslash f^{-1}\Big(\bigcup_{V \subset \Omega_2 \backslash \operatorname{supp} u, V \text{open}} V\Big) \\ &= \Omega_1 \backslash f^{-1}(\Omega_2 \backslash \operatorname{supp} u) = f^{-1}(\operatorname{supp} u). \end{aligned}$$

Applying to the diffeomorphism f^{-1} what has been shown so far, we also see

$$\operatorname{supp} u = \operatorname{supp} (f^{-1})^* f^*u \subset f(\operatorname{supp} f^*u),$$

i.e. $f^{-1}(\operatorname{supp} u) \subset \operatorname{supp} f^*u$, too. $\square$

We are interested in such diffeomorphisms which respect the kernel of a given differential operator on $\bigcap_{j=1}^{\infty} B^{\text{loc}}_{p_j,k_j}(\Omega_2)$. This is expressed by the following notion.

Definition 3.2. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ be open, $(p_j)_{j\in\mathbb{N}} \in [1,\infty)^{\mathbb{N}}$, $(k_j)_{j\in\mathbb{N}}$ be a sequence of tempered weight functions, and let $f : \Omega_1 \to \Omega_2$ be a diffeomorphism. A polynomial P is called $(p_j, k_j)_{j\in\mathbb{N}}$-$f$*-invariant* (or simply f*-invariant*), if for every $u \in \mathscr{D}'(\Omega_2)$ one has $u \in \mathcal{N}_{P,(p_j,k_j)}(\Omega_2)$ if and only if $u \circ f \in \mathcal{N}_{P,(p_j,k_j)}(\Omega_1)$.

Remark 3.3. i) Obviously, P is f-invariant if and only if P is f^{-1}-invariant. Moreover, if $\Omega_1 = \Omega_2$, for given P the set of the diffeomorphisms f on Ω_1 for which P is $(p_j, k_j)_{j\in\mathbb{N}}$-$f$-invariant forms a group under composition.
ii) Since translations $\tau_b(x) = x + b$ commute with $P(D)$ and $\mathcal{F}(u \circ \tau_b)(\xi) = e^{i\langle b,\xi\rangle}\mathcal{F}(u)(\xi)$ it follows that P is τ_b-invariant for every $b \in \mathbb{R}^N$.
iii) Because $\mathscr{E}_P(\Omega_2) \subset \mathcal{N}_P(\Omega_2)$ it is necessary for the f-invariance of P that for $\varphi \in C^\infty(\Omega_2)$ one has $P(D)\varphi = 0$ if and only if $P(D)(\varphi \circ f) = 0$. Therefore, if $\mathcal{N}_{P,(p_j,k_j)}(\Omega_2) = \mathscr{E}_P(\Omega_2)$ (which holds in particular when $k_j(\xi) = (1 + |\xi|)^j, j \in \mathbb{N}$, or when P is hypoelliptic) the aforementioned necessary condition is also sufficient for $(p_j, k_j)_{j\in\mathbb{N}}$-$f$-invariance of P.
iv) For a given polynomial P the conditions on $f = (f_1, \cdots, f_N)$ ensuring that P is f-invariant can be quite restrictive. Since exponential solutions of $P(D)u = 0$, i.e. solutions of the form $u(x) = Q(x)\exp(i\langle\zeta, x\rangle)$ where Q is a polynomial and $\zeta \in \mathbb{C}^N$ is a root of P, always belong to $\mathcal{N}_P(\Omega)$, one can derive certain differential equations which have to be satisfied by f in order that P is f-invariant.

For example, if P is a polynomial of degree 2, $P(\zeta) = \sum_{j,k=1}^N a_{j,l}\zeta_j\zeta_k + \sum_{j=1}^N b_j\zeta_j + c$ and one defines for $1 \leq j, k \leq N$ the differential operators

$$B_j(\zeta, f) := \sum_{m=1}^N \zeta_m \partial_j f_m$$

and

$$A_{j,k}(\zeta, f) := \sum_{m,l=1}^N \zeta_m\zeta_l\partial_j f_m \partial_k f_l - i\sum_{m=1}^N \zeta_m \partial_j\partial_k f_m,$$

then f has to satisfy (by taking $Q \equiv 1$) the (non-linear!) differential equations

$$(\forall \zeta \in \{z \in \mathbb{C}^N;\, P(z) = 0\} : \sum_{j,k=1}^N a_{j,k}A_{j,k}(\zeta, f) + \sum_{j=1}^N b_j B_j(\zeta, f) + c = 0.$$

At the end of this section we give characterizations of those f such that certain important polynomials are f-invariant.

Proposition 3.4. *Let $f : \Omega_1 \to \Omega_2$ be a diffeomorphism. If P is a $(p_j, k_j)_{j\in\mathbb{N}}$-$f$-invariant polynomial then $f^* : \mathcal{N}_{P,(p_j,k_j)}(\Omega_2) \to \mathcal{N}_{P,(p_j,k_j)}(\Omega_1)$ is a topological isomorphism.*

Proof: By the f-invariance of P the mapping is well-defined and obviously linear. From the continuity of $\bigcap_{j=1}^{\infty} B^{\mathrm{loc}}_{p_j,k_j}(\Omega_2) \hookrightarrow \mathscr{D}'(\Omega_2)$ and $f^*: \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_1)$ the continuity of f^* follows immediately from the Closed Graph Theorem for Fréchet spaces. Since f^* is obviously one-to-one and onto the Open Mapping Theorem for Fréchet spaces gives the result. $\square$

Recall, that for $\phi \in \mathscr{E}(\Omega)$ and $u \in \mathscr{D}'(\Omega)$ one has $f^*(\phi u) = f^*(\phi) f^*(u)$. Moreover, recall that the topology on $\mathcal{N}_{P,(p_j,k_j)}(\Omega)$ is generated by the increasing sequence of seminorms $q_n(u) := \max_{1\le l,j\le n} \|\varphi_l u\|_j, n \in \mathbb{N}$, where $\|\varphi u\|_j = \|\varphi u\|_{p_j,k_j}$ and $(\varphi_n)_{n\in\mathbb{N}} \in \mathscr{D}(\Omega)^{\mathbb{N}}$ satisfies $\operatorname{supp}\varphi_n \subset K_{n+1} \subset \{\varphi_{n+1} = 1\}, n \in \mathbb{N}$, for a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of Ω.

Corollary 3.5. *Let $\Omega \subset \mathbb{R}^N$ be open, $\Omega_1 \subset \Omega$ open, and $f : \Omega \to \Omega_1$ a diffeomorphism. If P is a $(p_j, k_j)_{j\in\mathbb{N}}$-$f$-invariant polynomial then the mapping*

$$\mathcal{N}_{P,(p_j,k_j)}(\Omega) \to \mathcal{N}_{P,(p_j,k_j)}(\Omega), u \mapsto u_{|\Omega_1} \circ f$$

is linear and continuous. We denote it again by f^* and somtimes write $u \circ f$ instead of f^*u, too. With this notation we have supp $f^*u = f^{-1}(\operatorname{supp} u \cap \Omega_1)$.

Moreover,

$$\forall\, n \in \mathbb{N} : \; q_n(f^*u) \le \max_{1\le l\le n} q_{n+1}(f^*[(\varphi_l \circ f^{-1})u])$$

Proof: The continuity follows immediately from the obvious continuity of the restriction map $\mathcal{N}_P(\Omega) \to \mathcal{N}_P(\Omega_1), u \mapsto u_{|\Omega_1}$ and Proposition 3.4, while $\operatorname{supp} f^*u = f^{-1}(\operatorname{supp} u \cap \Omega_1)$ is a direct consequence of Proposition 3.1.

Finally, because of $\operatorname{supp}\varphi_l \subset K_{l+1} \subset \{\varphi_{l+1} = 1\}$ we have

$$\varphi_{l+1} f^*[(\varphi_l \circ f^{-1})u] = \varphi_{l+1} f^*(\varphi_l \circ f^{-1}) f^*(u) = \varphi_{l+1}\varphi_l f^*(u) = \varphi_l f^*(u),$$

so that

$$\begin{aligned} q_n(f^*u) &= \max_{1\le l,j\le n} \|\varphi_l f^*(u)\|_j = \max_{1\le l,j\le n} \|\varphi_{l+1} f^*[(\varphi_l \circ f^{-1})u]\|_j \\ &\le \max_{1\le l,j\le n} q_{n+1}(\varphi_{l+1} f^*[(\varphi_l \circ f^{-1})u]) = \max_{1\le l\le n} q_{n+1}(f^*[(\varphi_l \circ f^{-1})u]). \square \end{aligned}$$

We now give characterizations of those f such that certain important polynomials P are f-invariant. We are sure that these are known. Nevertheless, since we could not find a reference, we give the proofs for the sake of completeness. In order to formulate the next proposition more conveniently, we write $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$.

Proposition 3.6. a) *Let $N = 2$ and $P(\xi) = \frac{1}{2}(i\xi_1 - \xi_2)$, i.e. $P(D) = \bar{\partial}$ is the Cauchy-Riemann operator. Then P is f-invariant if and only if $\bar{\partial} f = 0$, i.e. f is holomorphic.*

b) *Let $P(\xi) = -|\xi|^2$, i.e. $P(D) = \triangle$ is the Laplacian. Then P is f-invariant if and only if the following conditions hold.*

i) *$\triangle f_j = 0$ for all $1 \leq j \leq N$.*

ii) *$|\nabla f_j| = |\nabla f_k|$ and $\langle \nabla f_j, \nabla f_k \rangle = 0$ for all $1 \leq j \neq k \leq N$,*

that is, the Jacobian of f is a multiple of an orthogonal matrix in each point.

c) *On $\mathbb{R}^{N+1}$ let $P(\xi) = |\xi'|^2 + i\xi_{N+1}$, i.e. $P(D)$ is the heat operator $H = \triangle_{x'} - \partial_{N+1}$, where the $(N+1)$-th variable is considered as time and $\triangle_{x'}$ denotes the Laplacian with respect to the first N variables.*

Then P is f-invariant if and only if

$$f(x) = (\alpha A x', \alpha^2 x_{N+1}) + b$$

where $\alpha \in \mathbb{R} \backslash \{0\}, b \in \mathbb{R}^{N+1}$, and $A \in \mathbb{R}^{N \times N}$ is an orthogonal matrix.

d) *On $\mathbb{R}^{N+1}$ let $P(\xi) = \xi_{N+1}^2 - |\xi'|^2$, i.e. $P(D)$ is the wave operator $\Box = \triangle_{x'} - \partial_{N+1}^2$, where again the $(N+1)$-th variable is considered as time and $\triangle_{x'}$ denotes the Laplacian with respect to the first N variables.*

If all tempered weight functions k_j are radial functions and one requires $\partial_k f_{N+1} = \partial_{N+1} f_k = 0$ for all $1 \leq k \leq N$ (i.e. via the transformation f the time-variable has no influence on the space-variables and vice versa), then P is f-invariant if and only if

$$f(x) = \alpha(A x', x_{N+1}) + b$$

where $\alpha \in \mathbb{R} \backslash \{0\}, b \in \mathbb{R}^{N+1}$ and $A \in \mathbb{R}^{N \times N}$ is an orthogonal matrix.

Proof: Since the polynomials in a)-c) are hypoelliptic, in these cases f-invariance of P means that for $u \in C^\infty(\Omega_2)$ one has $P(D)u = 0$ if and only if $P(D)(u \circ f) = 0$. Now a) follows after a short calculation from the Cauchy-Riemann equations.

In case of b), taking $Q(x) = x_j$ and $\zeta = 0$, the corresponding exponential solution $Q(x) \exp(i\langle \zeta, x \rangle)$ gives that $\triangle f_j = 0, 1 \leq j \leq N$, is necessary for P being f-invariant. With this, the necessary conditions (1) above turn into

$$\forall \zeta \in \{z \in \mathbb{C}^N;\, P(z) = 0\} : 0 = -\sum_{m,l=1}^{N} \zeta_m \zeta_l \langle \nabla f_m, \nabla f_l \rangle.$$

For $1 \leq j \neq k \leq N$ set $\zeta_j = 1, \zeta_k = i$ and $\zeta_l = 0$ for $l \notin \{j,k\}$. Then ζ is a root of P and hence

$$\forall\, 1 \leq j \neq k \leq N:\ 0 = |\nabla f_k|^2 - |\nabla f_j|^2 - 2i\langle \nabla f_j, \nabla f_k\rangle$$

is necessary, which gives necessity of conditions $i)$ and $ii)$.

On the other hand, a straight forward calculation gives for $u \in C^\infty(\Omega_2)$

$$\triangle(u \circ f) = \sum_{j=1}^{N} (\triangle f_j)((\partial_j u)\circ f) + \sum_{j,k=1}^{N} ((\partial_j\partial_k u)\circ f)\langle \nabla f_j, \nabla f_k\rangle.$$

Therefore, if f satisfies $i)$ and $ii)$ we have for all $u \in C^\infty(\Omega_2)$

$$\triangle(u\circ f) = |\nabla f_1|^2((\triangle u)\circ f).$$

Since f is a diffeomorphism, we have $|\nabla f_1|^2 \neq 0$ everywhere, so it follows that $\triangle u = 0$ if and only if $\triangle(u\circ f) = 0$, so that $i)$ and $ii)$ are also sufficient for the f-invariance of P.

To show necessity in $c)$, assume that P is f-invariant. Take $Q(x) = x_j$ for $1 \leq j \leq N$ and $\zeta = 0$. From the corresponding exponential solutions it follows that $Hf_j = 0$ for $1 \leq j \leq N$. Therefore, the necessary conditions (1) become

$$\begin{aligned}\forall\, \zeta \in \{z \in \mathbb{C}^N;\ P(z) = 0\} : 0 &= iB_{N+1}(\zeta, f) + \sum_{j=1}^{N} A_{j,j}(\zeta, f)\\ &= i\zeta_{N+1} H f_{N+1} + \sum_{l,m=1}^{N+1} \zeta_l\zeta_m \langle \nabla_{x'} f_l, \nabla_{x'} f_m\rangle\end{aligned}$$

where $\nabla_{x'} f_j := (\partial_1 f_j, \ldots, \partial_N f_j)$ denotes the gradient of f_j with respect to the space-variables x'.

Taking $\zeta_j = 1$ for a fixed $j \in \{1,\ldots,N\}$, $\zeta_{N+1} = i$ and $\zeta_l = 0$ for $l \notin \{j, N+1\}$ we get a root of P giving

$$0 = -Hf_{N+1} - |\nabla_{x'} f_{N+1}|^2 + |\nabla_{x'} f_j|^2 + 2i\langle \nabla_{x'} f_j, \nabla_{x'} f_{N+1}\rangle,$$

so that

$$\forall\, 1 \leq j \leq N : Hf_{N+1} + |\nabla_{x'} f_{N+1}|^2 = |\nabla_{x'} f_j|^2,\ \langle \nabla_{x'} f_j, \nabla_{x'} f_{N+1}\rangle = 0.$$

On the other hand, taking $\zeta_{N+1} = 1$, for fixed $j \in \{1,\ldots,N\}$ any square root $\zeta_j = \sqrt{i}$ of i, and $\zeta_l = 0$ for $l \notin \{j, N+1\}$ gives another root of P which yields

$$0 = iHf_{N+1} + |\nabla_{x'} f_{N+1}|^2 + i|\nabla_{x'} f_j|^2 + 2\sqrt{i}\langle \nabla_{x'} f_{N+1}, \nabla_{x'} f_j\rangle,$$

so that $|\nabla_{x'} f_{N+1}|^2 = 0$, i.e. f_{N+1} only depends on x_{N+1}.

From the two last sets of equations it follows that $|\nabla_{x'}f_j|^2 = Hf_{N+1} = \partial_{N+1}f_{N+1}$ for all $1 \leq j \leq N$. Because f_{N+1} only depends on x_{N+1} it follows from this, that $\triangle_{x'}f_j = 0$ for all $1 \leq j \leq N$, so that in addition $0 = Hf_j = \partial_{N+1}f_j$ for all $1 \leq j \leq N$. Therefore, f_j does not depent on x_{N+1} for $1 \leq j \leq N$. Since f_{N+1} does not depent on x' and $|\nabla_{x'}f_j|^2 = \partial_{N+1}f_{N+1}$ it follows that $\nabla_{x'}f_j$ as well as $\partial_{N+1}f_{N+1}$ are constant, i.e. f_j is an affine function of x' and f_{N+1} is an affine function of x_{N+1}.

This means that there are $\gamma, \beta \in \mathbb{R}$, $a \in \mathbb{R}^N$ and $B \in \mathbb{R}^{N\times N}$ such that

$$\forall x \in \Omega_1 : \, f(x', x_{N+1}) = (Bx' + a, \gamma x_{N+1} + \beta).$$

Since f is a diffeomorphism and $|\nabla_{x'}f_j|^2 = \partial_{N+1}f_{N+1}$ we have $\gamma > 0$ and B has to be invertible.

To see that B actually is a multiple of an orthogonal matrix, fix $1 \leq j \neq k \leq N$ and set $\zeta_j = 1, \zeta_k = i$ and $\zeta_l = 0$ for $l \notin \{j,k\}$. This gives another root of P yielding the equation

$$0 = |\nabla_{x'}f_j|^2 - |\nabla_{x'}f_k|^2 + 2i\langle\nabla_{x'}f_j, \nabla_{x'}f_k\rangle$$

so that

$$\forall\, 1 \leq j \neq k \leq N : |\nabla_{x'}f_j| = |\nabla_{x'}f_k|, \; \langle\nabla_{x'}f_j, \nabla_{x'}f_k\rangle = 0$$

which means, since $|\nabla_{x'}f_j|^2 = \partial_{N+1}f_{N+1} = \gamma > 0$, that $A := \frac{1}{\sqrt{\gamma}}B$ is orthogonal.

This shows that the condition on f stated in b) is necessary for P to be f-invariant.

To show its sufficiency as well, observe that by a straightforward calculation for f of the stated form one obtains

$$\forall\, u \in C^\infty(\Omega_2) : \; H(u \circ f) = \alpha^2((Hu) \circ f)$$

so that indeed $Hu = 0$ if and only if $H(u \circ f) = 0$.

In order to prove d) we assume that all tempered weight functions are radial and that $\partial_k f_{N+1} = \partial_{N+1}f_k = 0$ for all $1 \leq k \leq N$. To prove necessity of the condition assume that P is f-invariant. Taking $Q(x) = x_j$ and $\zeta = 0$ it follows with the corresponding exponential solution that $\Box f_j = 0$ for all $1 \leq j \leq N+1$. Since by assumption f_{N+1} is independent of x' it follows that $0 = \Box f_{N+1} = \partial_{N+1}^2 f_{N+1}$ so that f_{N+1} has to be an affine function of x_{N+1}, i.e. there are $\alpha, \beta \in \mathbb{R}$ such that $f_{N+1}(x', x_{N+1}) = \alpha x_{N+1} + \beta$.

With this, the necessary conditions (1) turn into

$$\forall\, \zeta \in \{z \in \mathbb{C}^N ; \, P(z) = 0\} : 0 = \sum_{l,m=1}^{N+1} \zeta_l\zeta_m(\langle\nabla_{x'}f_l, \nabla_{x'}f_m\rangle - \partial_{N+1}f_l\partial_{N+1}f_m).$$

For fixed $1 \leq j \neq k \leq N$ set $\zeta_j = 1, \zeta_k = i$ and $\zeta_l = 0$ for $l \notin \{j, k\}$ so that ζ is a root of P giving

$$0 = |\nabla_{x'} f_j|^2 - |\nabla_{x'} f_k|^2 + 2i \langle \nabla_{x'} f_j, \nabla_{x'} f_k \rangle$$

so that

$$\forall\, 1 \leq j \neq k \leq N : \, |\nabla_{x'} f_j| = |\nabla_{x'} f_k|, \, \langle \nabla_{x'} f_j, \nabla_{x'} f_k \rangle = 0.$$

On the other hand, for $1 \leq j \leq N$ fixed, let $\zeta_j = \zeta_{N+1} = 1$ and $\zeta_l = 0$ for $l \notin \{j, N+1\}$ so that ζ is a root of P yielding

$$\forall\, 1 \leq j \leq N : \, 0 = |\nabla_{x'} f_j|^2 - (\partial_{N+1} f_{N+1})^2 = |\nabla_{x'} f_j|^2 - \alpha^2.$$

In particular $\nabla_{x'} f_j$ is a constant function for all $1 \leq j \leq N$, independent of x_{N+1} by hypothesis, i.e. f_j is an affine function of x'. Because of $\alpha^2 = |\nabla_{x'} f_j|^2$ and $\langle \nabla_{x'} f_j, \nabla_{x'} f_k \rangle = 0$ for all $1 \leq j \neq k \leq N$ there are $b' \in \mathbb{R}^N$ and an orthogonal matrix $A \in \mathbb{R}^{N \times N}$ such that $(f_1, \ldots, f_N)(x', x_{N+1}) = \alpha A x' + b'$, proving necessity of the condition.

In order to prove its sufficiency, recall that for $b \in \mathbb{R}^{N+1}$ and invertible $B \in \mathbb{R}^{(N+1)\times(N+1)}$ one has $\mathcal{F}(u \circ B)(\xi) = |\det B^{-1}| \mathcal{F}(u)((B^t)^{-1}\xi)$, where B^t denotes the transpose of B, as well as $\mathcal{F}(u \circ \tau_b)(\xi) = e^{i\langle b, \xi \rangle} \mathcal{F}(u)(\xi)$.

Since k_j is a tempered weight function, there are $C > 0, m \in \mathbb{N}$ such that $k_j(\xi + \eta) \leq (1 + C|\xi|)^m k_j(\eta)$ for all $\xi, \eta \in \mathbb{R}^{N+1}$. From this follows

$$\forall\, \xi \in \mathbb{R}^{N+1} : \, k_j(0)(1 + C|\xi|)^{-m} \leq k_j(\xi) \leq k_j(0)(1 + C|\xi|)^m.$$

Since k_j is supposed to be a radial function and A is orthogonal, it follows that $k(\xi) = k(A\xi', \xi_{N+1})$. Using this we obtain for $\phi \in \mathscr{D}(\Omega_2)$

$$\begin{aligned}
& \int_{\mathbb{R}^{N+1}} |k_j(\xi) \mathcal{F}(\phi(u \circ f))(\xi)|^{p_j} \, d\xi \\
= & \int_{\mathbb{R}^{N+1}} |k_j(\xi) \alpha^{-(N+1)} \mathcal{F}((\phi \circ f^{-1})u)(\alpha^{-1}(A\xi', \xi_{N+1}))|^{p_j} \, d\xi \\
= & \int_{\mathbb{R}^{N+1}} |k_j(\xi) \mathcal{F}((\phi \circ f^{-1})u)(\xi)|^{p_j} (\frac{k(\alpha\xi)}{k(\xi)})^{p_j} \, d\xi
\end{aligned}$$

Since $k(\alpha\xi)/k(\xi) \leq \max\{1, |\alpha|\}$ and since f^{-1} is of the same form as f, it follows that $u \in \bigcap_{j=1}^\infty B_{p_j,k_j}^{\mathrm{loc}}(\Omega_2)$ if and only if $u \circ f \in \bigcap_{j=1}^\infty B_{p_j,k_j}^{\mathrm{loc}}(\Omega_1)$. Using again that f^{-1} is of the same form as f, it is straight forward to show that

$$\forall\, u \in \mathscr{D}'(\Omega) : \, \Box(u \circ f) = \alpha^2 (\Box u) \circ f,$$

too. This finally shows that P is indeed f-invariant. $\Box$

4. Universal zero solutions

In this section we give a sufficient condition for a sequence of diffeomorphisms $f_m : \Omega \to \Omega_m \subset \Omega, m \in \mathbb{N}$, such that there are (f_m^*)-universal elements in $\mathcal{N}_P(\Omega)$. We first introduce the following notion.

Definition 4.1. Let $\Omega_1 \subset \Omega$ be open subsets of $\mathbb{R}^N$, P be a non-constant polynomial, $(p_j)_{j\in\mathbb{N}} \in [1,\infty)^{\mathbb{N}}$, and $(k_j)_{j\in\mathbb{N}}$ a sequence of tempered weight functions. We say that Ω_1 is *P-approximable in* Ω if $\{u_{|\Omega_1};\, u \in \mathcal{N}_{P,(p_j,k_j)}(\Omega)\}$ is dense in $\mathcal{N}_{P,(p_j,k_j)}(\Omega_1)$. Again, if there is no danger of confusion we omit the reference to $(p_j)_{j\in\mathbb{N}} \in [1,\infty)^{\mathbb{N}}$ and $(k_j)_{j\in\mathbb{N}}$.

As is usually the case, the heart of our universality result is given by an approximation theorem. In our case it is the following theorem due to L. Hörmander. Recall that for an arbitrary subset A of $\mathbb{R}^N$ $\mathscr{E}'(A)$ denotes the space of distributions on $\mathbb{R}^N$ having compact support contained in A.

Theorem 4.2. [10, Theorem 10.5.2] *Let P be a non-constant polynomial, $\Omega_1 \subset \Omega$ open subsets of $\mathbb{R}^N$. Assume that for every $\mu \in \mathscr{E}'(\overline{\Omega})$ satisfying $\operatorname{supp} P(-D)\mu \subset \Omega_1$ it already follows that $\operatorname{supp} \mu \subset \Omega_1$. Then Ω_1 is P-approximable in Ω.*

In general, neither P-convexity for supports of Ω nor of Ω_1 is sufficient for Ω_1 to be P-approximable in Ω. For example, let Ω be any open subset of $\mathbb{R}^2$ containing the unit disk, $\Omega_1 = \{x \in \Omega;\, |x| > 1/2\}$, and let $P(D) = \frac{1}{2}(\partial_1 + i\partial_2)$. Then $\mathscr{E}_P(\Omega)$ consists of the holomorphic functions in Ω and $P(D)$ being elliptic, Ω as well as Ω_1 are P-convex for supports. However, $z = x_1 + ix_2 \mapsto 1/z$ obviously belongs to $\mathscr{E}_P(\Omega_1)$ but not to the closure of $\{u_{|\Omega_1};\, u \in \mathscr{E}_P(\Omega)\}$ in $\mathscr{E}_P(\Omega)$.

The next proposition gives a simple sufficient condition for P-approximability.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^N$ be open and P a non-constant polynomial.*

i) *If $\Omega_1 \subset \Omega$ has convex components then Ω_1 is P-approximable in Ω.*
ii) *Let $\Omega_m \subset \Omega$ be open subsets of Ω such that $\overline{\Omega_m} \cap \overline{\cup_{n\neq m}\Omega_n} = \emptyset$. Assume that for every m it holds that for all $u \in \mathscr{E}'(\bar{\Omega})$ already $u \in \mathscr{E}'(\Omega_m)$ if $\operatorname{supp} P(-D)u \subset \Omega_m$. Then $\cup_m\Omega_m$ is P-approximable in Ω.*

Proof: The proof of *i)* follows immediately from Theorem 4.2 and by applying to the different components of Ω_1 the fact that $ch\operatorname{supp} u = ch\operatorname{supp} P(-D)u$ where $ch\, K$ denotes the closed convex hull of a set K, cf. [10, Theorem 7.3.2].

ii) We show that the hypothesis of Theorem 4.2 is satisfied with $\cup_m\Omega_m$ in place of Ω_1. Let $u \in \mathscr{E}'(\bar{\Omega})$ with $\operatorname{supp} P(-D)u \subset \cup_m\Omega_m$. Set $f := P(-D)u$. Then, by compactness of $\operatorname{supp} f$ there is $r \in \mathbb{N}$ such that $\operatorname{supp} f \subset \cup_{l=1}^r \Omega_{m_l}$. Let $\varphi_1, \ldots, \varphi_r \in \mathscr{E}(\Omega)$ satisfy $\Omega_{m_l} \subset \{\varphi_l = 1\}$ and $\operatorname{supp}\varphi_l \cap \operatorname{supp}\varphi_k = \emptyset$ for all $1 \leq l \neq k \leq r$.

For $\phi \in \mathscr{D}(\Omega_{m_l})$ it follows that $D^\alpha\varphi_l = 0$ in a neighbourhood of $\operatorname{supp}\phi$ for all $\alpha \neq 0$, so that with Leibniz' formula

$$\langle f, \phi\rangle = \langle \varphi_l f, \phi\rangle = \langle u, P(D)(\varphi_l\phi)\rangle = \langle u, \varphi_l P(D)\phi\rangle = \langle P(-D)(\varphi_l u), \phi\rangle$$

for all $\phi \in \mathscr{D}(\Omega_{m_l})$. Since the equation $P(-D)v = f$ has at most one solution with compact support, cf. [10, Theorem 7.3.2], it follows that $u = \sum_{l=1}^r \varphi_l u$. Moreover, $\varphi_l u \in \mathscr{E}'(\overline{\Omega})$, $P(-D)(\varphi_l u) = \varphi_l f \in \mathscr{E}'(\Omega_{m_l})$ so that by hypothesis on Ω_{m_l} we have $\operatorname{supp}\varphi_l u \subset \Omega_{m_l}$. Hence $\operatorname{supp} u \subset \cup_m\Omega_m$. □

Theorem 4.4. *Let Ω be an open subset of $\mathbb{R}^N$ and let $f_m : \Omega \to \Omega_m$, $m \in \mathbb{N}$, be diffeomorphisms with $\Omega_m \subset \Omega$. Moreover, let P be a non-constant polynomial which is f_m-invariant for all $m \in \mathbb{N}$.*

If for every compact subset K of Ω there are $m \in \mathbb{N}$ and $U \subset \Omega$ open with $K \subset U$ such that $f_m(U)\cup U$ is P-approximable in Ω and $f_m(U)\cap U = \emptyset$ then

$$\mathcal{U} := \{u \in \mathcal{N}_P(\Omega);\ (u \circ f_m)_{m\in\mathbb{N}} \text{ is dense in } \mathcal{N}_P(\Omega)\}$$

is a dense G_δ-subset of $\mathcal{N}_P(\Omega)$.

Proof: Since $\mathcal{N}_P(\Omega)$ is a separable Fréchet space, by [8, Theorem 1] it suffices to show that for every pair of non-empty open subsets $V, W \subset \mathcal{N}_P(\Omega)$ there is $m \in \mathbb{N}$ with $f_m^*(V) \cap W \neq \emptyset$.

In order to do so, let $(K_n)_{n\in\mathbb{N}}$ be a compact exhaustion of Ω and for $n \in \mathbb{N}$ choose $\varphi_n \in \mathscr{D}(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi_n \subset K_{n+1}$, and $K_n \subset \{\varphi_n = 1\}$. As mentioned in section 2, the topology of $\mathcal{N}_P(\Omega)$ is generated by the increasing sequence of seminorms $q_n(u) = \max_{1\leq j,k\leq n} \|\varphi_k u\|_j$, $n \in \mathbb{N}$.

Let V, W be two non-empty open subsets of $\mathcal{N}_P(\Omega)$. Pick $v \in V, w \in W$. Then there is $n \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\{u \in \mathcal{N}_P(\Omega); q_n(u - v) < \varepsilon\} \subset V$$

$$\{u \in \mathcal{N}_P(\Omega); q_n(u - w) < \varepsilon\} \subset W.$$

By hypothesis there are $m \in \mathbb{N}$ and $U \subset \Omega$ open with $K_{n+2} \subset U$ such that $f_m(U) \cap U = \emptyset$ and $f_m(U) \cup U$ is P-approximable. Since f_m^* is continuous, there are $C \geq 1, n' \in \mathbb{N}$ such that

$$\forall\, u \in \mathcal{N}_P(\Omega) :\ q_{n+1}(f_m^* u) \leq C q_{n'}(u). \tag{2}$$

From the choice of U it follows that $\varphi_k \in \mathscr{D}(U)$ as well as $\varphi_k \circ f_m^{-1} \in \mathscr{D}(f_m(U))$ for $1 \leq k \leq n$. We define $\tilde{u} \in \mathscr{D}'(f_m(U) \cup U)$ via

$$\begin{aligned} \forall\, \phi \in \mathscr{D}(U) : \langle \tilde{u}, \phi \rangle &:= \langle v, \phi \rangle, \\ \forall\, \phi \in \mathscr{D}(f_m(U)) : \langle \tilde{u}, \phi \rangle &:= \langle w \circ f_m^{-1}, \phi \rangle. \end{aligned}$$

Note that $\tilde{u}$ is well-defined since $f_m(U) \cap U = \emptyset$. Because $v \in \mathcal{N}_P(\Omega)$ and $w \circ f_m^{-1} \in \mathcal{N}_P(\Omega_m)$ it follows that $\tilde{u} \in \mathcal{N}_P(U \cup f_m(U))$.

Because $\varphi_k \in \mathscr{D}(U)$ and $\varphi_k \circ f_m^{-1} \in \mathscr{D}(f_m(U))$, $1 \leq k \leq n$, it follows that q_n as well as $\max_{1 \leq k \leq n} q_{n'}((\varphi_k \circ f_m^{-1}) \cdot)$ are continuous seminorms on $\mathcal{N}_P(U \cup f_m(U))$. By the P-approximability of $U \cup f_m(U)$ in Ω it follows that there is $u \in \mathcal{N}_P(\Omega)$ such that

$$q_n(u - \tilde{u}) < \varepsilon / C$$

as well as

$$\max_{1 \leq k \leq n} q_{n'}((\varphi_k \circ f_m^{-1})(u - \tilde{u})) < \varepsilon / C.$$

Since $\varphi_k \in \mathscr{D}(U)$, $1 \leq k \leq n$ we have $\varphi_k \tilde{u} = \varphi_k v$ so that

$$q_n(u - v) = \max_{1 \leq k,j \leq n} \|\varphi_k(u - v)\|_j = \max_{1 \leq k,j \leq n} \|\varphi_k(u - \tilde{u})\|_j = q_n(u - \tilde{u}) < \varepsilon,$$

i.e. $u \in V$. Moreover, because $\varphi_k \circ f_m^{-1} \in \mathscr{D}(f_m(U))$, $1 \leq k \leq n$, we obtain $(\varphi_k \circ f_m^{-1})\tilde{u} = (\varphi_k \circ f_m^{-1})(w \circ f_m^{-1}), 1 \leq k \leq n$. With Corollary 3.5 and (2) applied to $w \circ f_m^{-1} - u$ it therefore follows that

$$\begin{aligned} q_n(w - u \circ f_m) &= q_n(f_m^*(w \circ f_m^{-1} - u)) \\ &\leq \max_{1 \leq k \leq n} q_{n+1}\big(f_m^*[(\varphi_k \circ f_m^{-1})(w \circ f_m^{-1} - u)]\big) \\ &\leq C \max_{1 \leq k \leq n} q_{n'}\big((\varphi_k \circ f_m^{-1})(w \circ f_m^{-1} - u)\big) \\ &= C \max_{1 \leq k \leq n} q_{n'}\big((\varphi_k \circ f_m^{-1})(\tilde{u} - u)\big) < \varepsilon, \end{aligned}$$

i.e. $u \circ f_m \in W$ so that $f_m^*(V) \cap W \neq \emptyset$. Since V, W were chosen arbitrarily the conclusion follows from [8, Theorem 1]. □

Remark 4.5. Let $P(D)$ be either the $\bar{\partial}$-, Laplace-, Heat- or Waveoperator. Moreover, consider the tempered weight functions $k_j(\xi) = (1 + |\xi|)^j, j \in \mathbb{N}$, so that we are dealing with $\mathscr{E}_P(\Omega)$ as the kernel of $P(D)$. (Note, however, that this is no restriction for the $\bar{\partial}$-, Laplace-, or Heatoperator since these are hypoelliptic operators!) Let $f : \Omega \to \Omega_1 \subset \Omega$ be a diffeomorphism such that P is f-invariant, and in case of the Waveoperator, assume that f satisfies the additional mild conditions posed in Proposition 3.6 d).

Then by Proposition 3.6 a straight forward calculation shows that for all $\varphi \in \mathscr{E}(\Omega)$ one has

$$P(D)(\varphi \circ f) = g((P(D)\varphi) \circ f),$$

where

$$g \quad = \quad \begin{cases} 2\bar{\partial} f_1 & , \text{ if } P(D) = \bar{\partial} \\ |\nabla f_1|^2 & , \text{ if } P(D) \text{ is the Laplacian} \\ \alpha^2 \neq 0 & , \text{ if } P(D) \text{ is the Heat- or Waveoperator,} \end{cases}$$

which has no zero in Ω, because f is a diffeomorphism. Hence, the following corollary covers the cases when $P(D)$ is the $\bar{\partial}$-, Laplace-, Heat- or Waveoperator.

In case of the $\bar{\partial}$-operator, i.e. when dealing with holomorphic functions, the next result is due to Bernal and Montes (Theorem 1.2), cf. [2].

Corollary 4.6. *Let Ω be an open subset of $\mathbb{R}^N$ having convex components and let P be a non-constant polynomial. Moreover, let $f_m : \Omega \to \Omega, m \in \mathbb{N}$, be diffeomorphisms of Ω such that for every $m \in \mathbb{N}$ there is $g_m \in \mathscr{E}(\Omega)$ having no zero in Ω such that $P(D)(f_m^* u) = g_m f_m^*(P(D)u)$ for every $u \in \mathscr{E}(\Omega)$ and $m \in \mathbb{N}$. Then P is f_m-invariant for every $m \in \mathbb{N}$ and the following are equivalent.*

i) *The set*

$$\{u \in \mathscr{E}_P(\Omega);\ (u \circ f_m)_{m\in\mathbb{N}} \text{ is dense in } \mathscr{E}_P(\Omega)\}$$

is a dense G_δ-subset of $\mathscr{E}_P(\Omega)$.

ii) *There is $u \in \mathscr{E}_P(\Omega)$ such that $(u \circ f_m)_{m\in\mathbb{N}}$ is dense in $\mathscr{E}_P(\Omega)$.*

iii) *For every compact subset K of Ω there is $m \in \mathbb{N}$ such that $f_m(K) \cap K = \emptyset$.*

Proof: That P is f_m-invariant for each m follows immediately. Obviously, *i)* implies *ii)*. In order to show that *iii)* implies *i)*, observe that it follows immediately from the hypothesis on Ω, that there is a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of Ω such that for every n the components of K_n are convex. By hypothesis, for every n there is m such that $f_m(K_n) \cap K_n = \emptyset$, i.e. the closures of $f_m(K_n^\circ)$ and K_n° are disjoint. The components of K_n° being convex, it follows that every $u \in \mathscr{E}'(\bar{\Omega})$ with $\operatorname{supp} P(-D)u \subset K_n^\circ$ already satisfies $\operatorname{supp} u \subset K_n^\circ$. We also show that $\operatorname{supp} u \subset f_m(K_n^\circ)$ for every $u \in \mathscr{E}'(\bar{\Omega})$ with $\operatorname{supp} P(-D)u \subset f_m(K_n^\circ)$, so that $f_m(K_n^\circ) \cup K_n^\circ$ is P-approximable in Ω by Proposition 4.3 ii). Since $(K_n)_{n\in\mathbb{N}}$ is a compact exhaustion of Ω, this will show that the hypothesis of Theorem 4.4 is satisfied, giving $i)$. In order to simplify notation, we simply write f instead of f_m from now on.

So, let $u \in \mathscr{E}'(\bar{\Omega})$ with $\operatorname{supp} P(-D)u \subset f(K_n^\circ)$. As Ω has convex components, it follows from this, that the support of u is already contained in Ω, so we can apply f^* to u. By hypothesis we have

$$P(D)(\varphi \circ f) = g((P(D)\varphi) \circ f),$$

for all $\varphi \in \mathscr{E}(\Omega)$, where g has no zero in Ω. In particular $P(D)\varphi = g((P(D)(\varphi \circ f^{-1})) \circ f)$. For $v \in \mathscr{D}'(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$ we therefore have

$$\begin{aligned}
\langle P(-D)(v \circ f), \varphi\rangle &= \langle v, |\det Jf^{-1}|(P(D)\varphi) \circ f^{-1}\rangle \\
&= \langle v, |\det Jf^{-1}|\,(g \circ f^{-1})P(D)(\varphi \circ f^{-1})\rangle \\
&= \langle P(-D)(|\det Jf^{-1}|\,(g \circ f^{-1})\,v), \varphi \circ f^{-1}\rangle \\
&= \langle P(-D)(|\det Jf^{-1}|\,(g \circ f^{-1})\,v), |\det(Jf) \circ f^{-1}||\det Jf^{-1}|\,\varphi \circ f^{-1}\rangle \\
&= \langle f^*(P(-D)(|\det Jf^{-1}|\,(g \circ f^{-1})\,v), |\det Jf|\varphi\rangle \\
&= \langle |\det Jf|\,f^*(P(-D)(|\det Jf^{-1}|\,(g \circ f^{-1})\,v), \varphi\rangle,
\end{aligned}$$

i.e. $P(-D)(f^*v) = |\det Jf|\,f^*(P(-D)(|\det Jf^{-1}|\,(g \circ f^{-1})\,v))$ for all $v \in \mathscr{D}'(\Omega)$. Therefore,

$$P(-D)u = (f^{-1})^*\left[\frac{1}{|\det Jf|}P(-D)\left[f^*(\frac{1}{|\det Jf^{-1}|(g \circ f^{-1})}u)\right]\right],$$

which shows

$$\operatorname{supp}(f^{-1})^*\left[\frac{1}{|\det Jf|}P(-D)\left[f^*(\frac{1}{|\det Jf^{-1}|(g \circ f^{-1})}u)\right]\right] = \operatorname{supp} P(-D)u \subset f(K_n^\circ).$$

From Proposition 3.1 we get

$$\operatorname{supp}\frac{1}{|\det Jf|}P(-D)\left[f^*(\frac{1}{|\det Jf^{-1}|(g \circ f^{-1})}u)\right] \subset K_n^\circ.$$

Since $1/|\det Jf| \neq 0$ we get

$$\operatorname{supp} P(-D)\left[f^*(\frac{1}{|\det Jf^{-1}|(g \circ f^{-1})}u)\right] \subset K_n^\circ,$$

so that by the convexity of the components of K_n°

$$\operatorname{supp} f^*(\frac{1}{|\det Jf^{-1}|(g \circ f^{-1})}u) \subset K_n^\circ.$$

Using Proposition 3.1 once more gives

$$\operatorname{supp}\frac{1}{|\det Jf^{-1}|(g \circ f^{-1})}u \subset f(K_n^\circ),$$

hence, $\operatorname{supp} u \subset f(K_n^\circ)$, finally giving $i)$ of the corollary.

Observe that this last conclusion is the only one where we used that $f(\Omega) = \Omega$, for if this was not the case we would only obtain

$$f(\Omega) \cap \operatorname{supp} \frac{1}{|\det Jf^{-1}|(g \circ f^{-1})} u \subset f(K_n^\circ)$$

from Proposition 3.5.

Finally, that *ii)* implies *iii)* is shown exactly as in [2, Theorem 3.5]. Assume there is a compact subset K of Ω such that $f_m(K) \cap K \neq \emptyset$ for all $m \in \mathbb{N}$. So, there are $x_m \in K$ with $f_m(x_m) \in K$ for every m. Since in every case under consideration, P is a non-constant polynomial, the function $v(x) = 1 + \max_{y \in K} |u(y)|$ belongs to $\mathscr{E}_P(\Omega)$. For every m we have

$$\max_{x \in K} |v(x) - u(f_m(x))| \geq |v(x_m) - u(f_m(x_m))| \geq |v(x_m)| - |u(f_m(x_m))| \geq 1,$$

contradicting the denseness of $(u \circ f_m)_{m \in \mathbb{N}}$ in $\mathscr{E}_P(\Omega)$. □

Example 4.7. Let $\Omega = B_1(0) \times \mathbb{R} \subset \mathbb{R}^{N+1}$ with $B_1(0) = \{x' \in \mathbb{R}^N; |x'| < 1\}$. Moreover, let $A_m \in \mathbb{R}^{N \times N}$ be an orthogonal matrix and $b_m \in \operatorname{span}\{e_{N+1}\}$, where e_{N+1} is the $(N+1)$-th unit vector in $\mathbb{R}^{N+1}$. Clearly,

$$f_m : \Omega \to \Omega, (x', x_{N+1}) \mapsto (A_m x', x_{N+1}) + b_m$$

is a well-defined diffeomorphism.

By Proposition 3.6, both polynomials $P_H(\xi) = |\xi|^2 + i\xi_{N+1}$ and $P_W(\xi) = |\xi|^2 - \xi_{N+1}^2$ are f_m-invariant for every $m \in \mathbb{N}$. $P_H(D)$ gives the Heatoperator whereas $P_W(D)$ gives the Waveoperator.

It follows from Corollary 4.6 and Remark 4.5 that there is an $(f_m)_{m \in \mathbb{N}}$-universal zero solution of the Heatoperator, respectively the Waveoperator, if and only if $\limsup_{m \to \infty} |b_m| = \infty$. While sufficiency of this condition is obvious, to show necessity assume that $(|b_m|)_{m \in \mathbb{N}}$ is bounded by a constant C. Let $K = \{x' \in \mathbb{R}^N; |x'| \leq 1/2\} \times [-C, C]$. Then $0 \in K \cap f_m(K)$ for all $m \in \mathbb{N}$, so that necessity follows from Corollary 4.6, too.

5. Dense subspaces of universal zero solutions

Under a slight modification of the hypothesis of Theorem 4.4 one can even prove the following stronger result.

Theorem 5.1. *Let Ω be an open subset of $\mathbb{R}^N$, $f_m : \Omega \to \Omega_m, m \in \mathbb{N}$, diffeomorphisms with $\Omega_m \subset \Omega$. Moreover, let P be a non-constant polynomial which is f_m-invariant for all $m \in \mathbb{N}$.*

If for every compact subset K of Ω there are $m \in \mathbb{N}$ and $U \subset \Omega$ open and bounded with $K \subset U \subset \bar{U} \subset \Omega$ such that $f_m(U) \cap U = \emptyset$ and $f_m(U) \cup U$

is P-approximable in Ω then there is a dense subspace $\mathcal{L} \subset \mathcal{N}_P(\Omega)$ with

$$\mathcal{L}\backslash\{0\} \subset \mathcal{U} := \{u \in \mathcal{N}_P(\Omega);\ (u \circ f_m)_{m\in\mathbb{N}} \textit{ is dense in } \mathcal{N}_P(\Omega)\}.$$

Proof: Let $(K_n)_{n\in\mathbb{N}}$ be a compact exhaustion of Ω. By the hypothesis we construct inductively an increasing sequence of open, bounded subsets $(U_n)_{n\in\mathbb{N}}$ of Ω with $K_n \subset U_n \subset \overline{U_n} \subset \Omega$ and a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers such that

i) $\forall\, n \in \mathbb{N}:\ f_{m_n}(U_n) \cup U_n$ is P-approximable in Ω
ii) $\forall\, n \in \mathbb{N}:\ f_{m_n}(U_n) \cap U_n = \emptyset$

For $n = 1$ there are by hypothesis $m_1 \in \mathbb{N}$ and $U_1 \subset \Omega$ open and bounded such that $K_1 \subset U_1 \subset \overline{U_1} \subset \Omega$, $f_{m_1}(U_1) \cap U_1 = \emptyset$, and $f_{m_1}(U_1) \cup U_1$ is P-approximable in Ω.

If $U_1, \ldots U_n, m_1, \ldots, m_n$ have been constructed we get by applying the hypothesis to the compact set

$$\overline{U_n} \cup \overline{f_1(U_n)} \cup \ldots \cup \overline{f_{m_n}(U_n)} \cup K_{n+1}$$

some $m_{n+1} \in \mathbb{N}$ and $U_{n+1} \subset \Omega$ open and bounded such that

$$\overline{U_n} \cup \overline{f_1(U_n)} \cup \ldots \cup \overline{f_{m_n}(U_n)} \cup K_{n+1} \subset U_{n+1} \subset \overline{U_{n+1}} \subset \Omega$$

with $f_{m_{n+1}}(U_{n+1}) \cap U_{n+1} = \emptyset$ and $f_{m_{n+1}}(U_{n+1}) \cup U_{n+1}$ P-approximable in Ω. In particular, from $f_{m_{n+1}}(U_{n+1}) \cap U_{n+1} = \emptyset$ it follows that for all $1 \leq j \leq m_n$ we have $f_{m_{n+1}}(U_n) \cap f_j(U_n) = \emptyset$ hence $m_{n+1} > m_n$.

We will now show that for the subsequence $(f_{m_n})_{n\in\mathbb{N}}$ there is a dense linear subspace $\mathcal{L}$ of $\mathcal{N}_P(\Omega)$ such that $(u \circ f_{m_n})_{n\in\mathbb{N}}$ is dense in $\mathcal{N}_P(\Omega)$ for every $u \in \mathcal{L}\backslash\{0\}$ proving the theorem. Since we will be dealing with subsequences of $(f_{m_n})_{n\in\mathbb{N}}$ we simply write $(f_m)_{m\in\mathbb{N}}$ instead of $(f_{m_n})_{n\in\mathbb{N}}$ in order to simplify notation.

Let $(f_{m_n})_{n\in\mathbb{N}}$ be an arbitrary subsequence of $(f_m)_{m\in\mathbb{N}}$. For a given compact subset $K \subset \Omega$ there is $n \in \mathbb{N}$ such that $K \subset K_{m_n} \subset U_{m_n}$ and $f_{m_n}(U_{m_n}) \cap U_{m_n} = \emptyset$ and $f_{m_n}(U_{m_n}) \cup U_{m_n}$ is P-approximable in Ω. Therefore, the set

$$\{u \in \mathcal{N}_P(\Omega);\ (u \circ f_{m_n})_{n\in\mathbb{N}} \text{ is dense in } \mathcal{N}_P(\Omega)\}$$

is a dense subset of $\mathcal{N}_P(\Omega)$ by Theorem 4.4. Since $(f_{m_n})_{n\in\mathbb{N}}$ is an arbitrary subsequence of $(f_m)_{m\in\mathbb{N}}$ it follows from [1, Theorem 2] that there is a dense linear subspace $\mathcal{L}$ of $\mathcal{N}_P(\Omega)$ such that $(u \circ f_m)_{m\in\mathbb{N}}$ is dense in $\mathcal{N}_P(\Omega)$ for every $u \in \mathcal{L}\backslash\{0\}$. □

Referring to Theorem 5.1 rather than to Theorem 4.4 the proof of Corollary 4.6 gives the next result. By Remark 4.5, in case of the $\bar{\partial}$-, Laplace-, Heat-, and Waveoperator its hypothesis on the diffeomorphisms f_m is automatically satisfied if the corresponding polynomial is f_m-invariant.

Corollary 5.2. *Let Ω be an open subset of $\mathbb{R}^N$ having convex components and let P be a non-constant polynomial. Moreover, let $f_m : \Omega \to \Omega, m \in \mathbb{N}$, be diffeomorphisms of Ω such that for every $m \in \mathbb{N}$ there is $g_m \in \mathscr{E}(\Omega)$ having no zero in Ω such that $P(D)(f_m^* u) = g_m f_m^*(P(D)u)$ for every $u \in \mathscr{E}(\Omega)$ and $m \in \mathbb{N}$. Then P is f_m-invariant for every $m \in \mathbb{N}$ and the following are equivalent.*

i) *There is a dense subspace $\mathcal{L} \subset \mathscr{E}_P(\Omega)$ with*
$$\mathcal{L}\backslash\{0\} \subset \{u \in \mathscr{E}_P(\Omega);\ (u \circ f_m)_{m\in\mathbb{N}} \text{ is dense in } \mathscr{E}_P(\Omega)\}.$$
ii) *The set*
$$\{u \in \mathscr{E}_P(\Omega);\ (u \circ f_m)_{m\in\mathbb{N}} \text{ is dense in } \mathscr{E}_P(\Omega)\}$$
is a dense G_δ-subset of $\mathscr{E}_P(\Omega)$.

iii) *There is $u \in \mathscr{E}_P(\Omega)$ such that $(u \circ f_m)_{m\in\mathbb{N}}$ is dense in $\mathscr{E}_P(\Omega)$.*

iv) *For every compact subset K of Ω there is $m \in \mathbb{N}$ such that $f_m(K) \cap K = \emptyset$.*

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References

[1] L. Bernal-González, *Densely hereditarily hypercyclic sequences and large hypercyclic manifolds*, Proc. Am. Math. Soc. 127 (1999), 3279-3285.

[2] L. Bernal-González, A. Montes-Rodríguez, *Universal Functions for Composition Operators*, Complex Variables 27 (1995), 47-56.

[3] G. D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci. Paris 189 (1929), 473-475.

[4] M. C. Calderón-Moreno, J. Müller, *Universal holomorphic and harmonic functions with additional properties*, Acta Math. Hungar. 105 (2004), 1-15.

[5] S. M. Duyos-Ruiz, *Universal functions of the structure of the space of entire functions*, Soviet Math. Doklady 30 (1984), 713-716.

[6] O. P. Dzagnidze, *The universal harmonic function in the space E_n*, Sakharthvelos SSR Mecnierebatha Akademiis Moambe 55 (1969), 41-44.

[7] P. M. Gauthier, M. R. Pouryayevali, *Universal (pluri)subharmonic functions*, Analysis 27 (2007), 273-284.

[8] K. G. Grosse-Erdmann *Universal families and hypercyclic operators* Bull. Am. Math. Soc. New Ser. 36 (1999), 345-381

[9] K. G. Grosse-Erdmann, R. Mortini, *Universal functions for composition operators with non-automorphic symbol*, J. d'Analyse Math. 107 (2009), 355-376.

[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators I and II*, Springer-Verlag, Berlin, 1983.

[11] W. Luh, *Universalfunktionen in einfach zusammenhängenden Gebieten*, Aequationes Math. 19 (1979), 183-193.

[12] R. Meise, D. Vogt, *Introduction to Functional Analysis,* Clarendon Press, Oxford, 1997.

[13] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[14] W. P. Seidel, J. L. Walsh, *On approximation by Euclidean and non-Euclidean translates of an analytic function,* Bull. Amer. Math. Soc. 47 (1941), 916-920.

[15] F. Treves, *Topological Vector spaces, Distributions and Kernels*, Academic Press, New York, 1967.

[16] P. Zappa, *On universal holomorphic functions,* Bollettino U. M. I. (7), 2-A (1989), 345-352.

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