SURJECTIVITY OF PARTIAL DIFFERENTIAL OPERATORS ON ULTRADISTRIBUTIONS OF BEURLING TYPE IN 2 DIMENSIONS

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Abstract. We show that for a partial differential operator $P(D)$ surjectivity on the space of ultradistributions $\mathscr{D}'_{(\omega)}(\Omega)$ of Beurling type is equivalent to surjectivity of $P(D)$ on $C^\infty(\Omega)$ in case of Ω being an open subset of $\mathbb{R}^2$.

1. Introduction

It is a classical result by Malgrange [10, Chapitre 1, Théorème 4] that for a polynomial $P \in \mathbb{C}[X_1, \dots, X_d]$ and for an open set $\Omega \subset \mathbb{R}^d$ the constant coefficient differential operator $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective if and only if Ω is P-convex for supports, that is, if and only if for every compact subset K of Ω there is another compact subset L of Ω such that for each $u \in \mathscr{E}'(\Omega)$ with $\operatorname{supp} P(-D)u \subset K$ it holds $\operatorname{supp} u \subset L$.

Hörmander showed in [6] that $P(D)$ is surjective as an operator on $\mathscr{D}'(\Omega)$ if and only if Ω is P-convex for supports and P-convex for singular supports, i.e. for every compact subset K of Ω there is another compact subset L of Ω such that for each $u \in \mathscr{E}'(\Omega)$ with $\operatorname{sing\,supp} P(-D)u \subset K$ it holds $\operatorname{sing\,supp} u \subset L$.

It is well-known that surjectivity of $P(D)$ as an operator on $C^\infty(\Omega)$ does not imply surjectivity of $P(D)$ as an operator on $\mathscr{D}'(\Omega)$ in general. However, Trèves conjectured [12, p. 389, Problem 2] that in the case of $\Omega \subset \mathbb{R}^2$ this implication is true. A proof of this conjecture is given in [8].

In the present paper, we prove an adaption of Trèves conjecture to the setting of ultradistributions of Beurling type associated with a non-quasianalytic weight function ω. These generalize classical distributions by allowing more flexible growth conditions for the Fourier transforms of the

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corresponding test functions than the Paley-Wiener weights. More precisely, we prove the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]$. Then the following are equivalent.*

i) $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ *is surjective.*
ii) $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ *is surjective.*
iii) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ *is surjective for each non-quasianalytic weight function* ω.
iv) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ *is surjective for some non-quasianalytic weight function* ω.

The above theorem complements the following result proved by Zampieri which shows the peculiarity of $d = 2$, too. For an open subset Ω of $\mathbb{R}^d$ we denote as usual the space of real analytic functions on Ω with $A(\Omega)$.

Theorem 1.2 (Zampieri [13])**.** *Let $\Omega \subset \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]$. The following are equivalent.*

i) $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ *is surjective.*
ii) $P(D) : A(\Omega) \to A(\Omega)$ *is surjective.*

The article is organized as follows. In the preliminary section 2 we fix notation and recall some well known facts about ultradistributions of Beurling type. In section 3 we explain the connection of continuation of ultradifferentiability and certain localizations of P at infinity. Moreover this section contains the key result which sets apart the case $d = 2$ from $d \geq 3$. Namely, we show that in $\mathbb{R}^2$ certain hyperplanes which arise in the context of continuation of ultradifferentiability are always characteristic hyperplanes for P. Section 4 provides a sufficient condition for an open subset Ω of $\mathbb{R}^d$ to be P-convex for (ω)-singular supports by means of an exterior cone condition. This condition is applied in section 5 in order to prove Theorem 1.1.

2. Preliminaries

In this section we introduce the ultradistributions of Beurling type in the sense of Braun, Meise, and Taylor [4].

Definition 2.1. A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ is called a *(non-quasianalytic) weight function* if it satisfies the following properties

(α) there exists $K \geq 1$ with $\omega(2t) \leq K(1 + \omega(t))$ for all $t \geq 0$,
(β) $\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$,

(γ) $\lim_{t\to\infty} \frac{\log t}{\omega(t)} = 0$,

(δ) $\varphi = \omega \circ \exp$ is convex.

ω is extended to $\mathbb{C}^d$ by setting $\omega(z) := \omega(|z|)$. Since we are not dealing with quasianalytic weight functions in this article we simply speak of weight functions for brevity.

For $K \subset \mathbb{R}^d$ compact let

$$\mathscr{D}_{(\omega)}(K) = \{f \in C^\infty(\mathbb{R}^d);\, \operatorname{supp} f \subset K \text{ and } \int_{\mathbb{R}^d} |\hat{f}(x)| \exp(\lambda\omega(x))\, dx < \infty \text{ for all } \lambda \geq 1\}$$

be equipped with its natural Fréchet space topology, and $\mathscr{D}_{(\omega)}(\Omega) = \bigcup \mathscr{D}_{(\omega)}(K)$, where K runs through all compact subsets of the open subset Ω of $\mathbb{R}^d$, equipped with its natural (LF)-space topology. The elements of its dual space $\mathscr{D}'_{(\omega)}(\Omega)$ are the *ultradistributions of Beurling type.*

The associated local space in the sense of Hörmander [7, 10.1.19]

$$\mathscr{E}_{(\omega)}(\Omega) = \mathscr{D}_{(\omega)}(\Omega)^{loc} = \{u \in \mathscr{D}'_{(\omega)}(\Omega);\, \varphi u \in \mathscr{D}_{(\omega)}(\Omega) \text{ for all } \varphi \in \mathscr{D}_{(\omega)}(\Omega)\}$$

is the space of *ultradifferentiable functions of Beurling type.*

Remark 2.2. i) For each weight function ω we have $\lim_{t\to\infty} \omega(t)/t = 0$ by the remark following 1.3 of Meise, Taylor, and Vogt [11].

ii) It is shown in [4] that condition (β) guarantees that $\mathscr{D}_{(\omega)}(\Omega) \neq \{0\}$ and that there are partitions of unity consisting of elements of $\mathscr{D}_{(\omega)}(\Omega)$.

iii) By [4] we have

$$\mathscr{E}_{(\omega)}(\Omega) = \{f \in C^\infty(\Omega);\, \text{for all } k \in \mathbb{N} \text{ and } K \Subset \Omega \quad |f|_{k,K} := \sup_{\alpha\in\mathbb{N}_0^d, x\in K} |f^{(\alpha)}(x)| \exp\left(-k\varphi^*\left(\frac{|\alpha|}{k}\right)\right) < \infty\},$$

where $\varphi^*(s) = \sup\{st - \varphi(t);\, t \geq 0\}$ is the Young conjugate of φ.

iv) For $\delta > 1$ the function $\omega(t) = t^{1/\delta}$ is a weight function for which the corresponding class of ultradifferentiable functions coincides with the small Gevrey class

$$\gamma^\delta(\Omega) = \{f \in C^\infty(\Omega);\, \forall\, K \Subset \Omega\, \forall\, C \geq 1 : \sup_{x\in K, \alpha\in\mathbb{N}_0^d} \frac{|f^{(\alpha)}(x)|}{\alpha!^\delta C^{|\alpha|}} < \infty\}.$$

Definition 2.3. $\mathscr{E}_{(\omega)}(\Omega)$ equipped with the seminorms $(|\cdot|_{k,K})_{k\in\mathbb{N}, K\Subset\Omega}$ is a nuclear Fréchet space. Its dual $\mathscr{E}'_{(\omega)}(\Omega)$ is equal to the space of $u \in \mathscr{D}'_{(\omega)}(\Omega)$ for which

$$\operatorname{supp} u = \mathbb{R}^d \setminus \bigcup \{B \subset \mathbb{R}^d \text{ open};\, u(\varphi) = 0 \text{ for all } \varphi \in \mathscr{D}_{(\omega)}(B)\}$$

is a compact subset of Ω.

The next theorem is a special case of a result due to Frerick and Wengenroth (see [5]), which completes a result of Bonet, Galbis, and Meise (see [2]), characterising surjectivity of convolution operators on ultradistributions of Beurling type.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^d$ be open, ω be a weight function, and $P \in \mathbb{C}[X_1, \ldots, X_d]$. Then the following are equivalent.*

i) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ *is surjective.*
ii) Ω *is P-convex for (ω)-supports as well as P-convex for (ω)-singular supports.*

Recall, that an open subset Ω of $\mathbb{R}^d$ is called P-convex for (ω)-supports if for every compact subset K of Ω there is a compact subset L of Ω such that $\operatorname{supp} \varphi \subset L$ whenever $\operatorname{supp} P(-D)\varphi \subset K$ for every $\varphi \in \mathscr{D}_{(\omega)}(\Omega)$. Analogously, Ω is called P-convex for (ω)-singular supports if for every compact subset K of Ω there is a compact subset L of Ω such that $\operatorname{sing\,supp}_{(\omega)} u \subset L$ whenever $\operatorname{sing\,supp}_{(\omega)} P(-D)u \subset K$ for every $u \in \mathscr{E}'_{(\omega)}(\Omega)$.

Remark 2.5. i) Clearly, P-convexity for supports of Ω implies P-convexity for (ω)-supports of Ω. On the other hand, $\mathscr{D}_{(\omega)}(\Omega)$ is sequentially dense in $\mathscr{D}(\Omega)$, as shown by Braun et al. [4, Proposition 3.9], so that P-convexity for supports is implied by P-convexity for (ω)-supports. Hence, P-convexity for supports and P-convexity for (ω)-supports are in fact equivalent.

ii) If P is elliptic the same is obviously true for $\check{P}$. Hence $P(-D)$ has a fundamental solution E which is analytic in $\mathbb{R}^d \backslash \{0\}$. Since analytic functions are contained in $\mathscr{E}_{(\omega)}(\Omega)$ for each weight function ω (cf. [4, Proposition 4.10]) we have in particular

$$ch(\operatorname{sing\,supp}_{(\omega)} E) = ch(\operatorname{sing\,supp}_{(\omega)} P(-D)\delta_0),$$

where $ch(A)$ denotes the convex hull of a set $A \subset \mathbb{R}^d$. By [3, Theorem 2.1] it therefore follows that for each open set $\Omega \subset \mathbb{R}^d$ and every $u \in \mathscr{D}'_{(\omega)}(\Omega)$ we have

$$\operatorname{sing\,supp}_{(\omega)} P(-D)u = \operatorname{sing\,supp}_{(\omega)} u.$$

In particular, Ω is P-convex for (ω)-singular supports. This and the well-known fact that every open subset Ω of $\mathbb{R}^d$ is P-convex for supports for elliptic P imply by Theorem 2.4 the surjectivity of

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$$

whenever P is elliptic.

From now on, let P always be an non-constant polynomial.

3. (ω)-Localizations at Infinity and Continuation of Ultradifferentiability

Obviously, P-convexity for (ω)-singular supports is closely related to the continuation of (ω)-ultradifferentiability of $P(-D)u$ to u. Analogously to the tools introduced by Hörmander in order to deal with the classical case (see e.g. [7, Section 11.3, vol. II]) Langenbruch introduced the following notions in [9]. For a polynomial P, a subspace V of $\mathbb{R}^d$, and $t > 0, \xi \in \mathbb{R}^d$ let

$$\tilde{P}_V(\xi, t) = \sup\{|P(\xi + \eta)|; \eta \in V, |\eta| \leq t\}$$

and

$$\tilde{P}(\xi, t) = \tilde{P}_{\mathbb{R}^d}(\xi, t).$$

Moreover, let

$$\sigma_{P,(\omega)}(V) := \inf_{t \geq 1} \liminf_{\xi \to \infty} \frac{\tilde{P}_V(\xi, t\omega(\xi))}{\tilde{P}(\xi, t\omega(\xi))}.$$

If we formally set $\omega \equiv 1$, we obtain Hörmander's classical definition of $\sigma_P(V)$, [7, Section 11.3, vol. II]. In order to simplify notation we write $\sigma_{P,(\omega)}(N)$ instead of $\sigma_{P,(\omega)}(\operatorname{span}\{N\})$ for $N \in S^{d-1}$.

The next theorem is almost an immediate consequence of [9, Theorem 2.5].

Theorem 3.1. *Let $\Omega_1 \subset \Omega_2$ be open convex subsets of $\mathbb{R}^d$. Assume that every hyperplane $H = \{x \in \mathbb{R}^d; \langle x, N \rangle = \alpha\}$, $N \in S^{d-1}, \alpha \in \mathbb{R}$ with $\sigma_{P,(\omega)}(N) = 0$ which intersects Ω_2 already intersects Ω_1.*

Then for every $u \in \mathscr{D}'_{(\omega)}(\Omega_2)$ satisfying $\operatorname{sing\,supp}_{(\omega)} P(D)u = \emptyset$ as well as $\operatorname{sing\,supp}_{(\omega)} u \subset \Omega_2 \backslash \Omega_1$ we already have $\operatorname{sing\,supp}_{(\omega)} u = \emptyset$.

Proof. Let $u \in \mathscr{D}'_{(\omega)}(\Omega_2)$ satisfy $P(D)u \in \mathscr{E}_{(\omega)}(\Omega_2)$ and $u|_{\Omega_1} \in \mathscr{E}_{(\omega)}(\Omega_1)$. Since Ω_2 is convex it follows from the Theorem of supports (see e.g. [7, Theorem 4.3.3, vol. I]) and [2, Theorem A] that there is $v \in \mathscr{E}_{(\omega)}(\Omega_2)$ such that $P(D)v = P(D)u$ so that $w := u - v \in \mathscr{D}'_{(\omega)}(\Omega_2)$ satisfies $P(D)w = 0$ as well as $w|_{\Omega_1} \in \mathscr{E}_{(\omega)}(\Omega_1)$. Hence, by [9, Theorem 2.5] it follows that $w \in \mathscr{E}_{(\omega)}(\Omega_2)$ which proves the theorem. $\square$

When investigating P-convexity for (ω)-singular supports by means of the above theorem it is necessary to study the zeros of $\sigma_{P,(\omega)}$ in S^{d-1}. In order to do so, recall the definition of ω-localizations of P at infinity, as introduced by Langenbruch in [9]. For a polynomial P and $\xi \in \mathbb{R}^d$ we set $P_{\xi,\omega}(x) := P(\xi + \omega(\xi)x)$ which is again a polynomial of the same degree as P. Clearly, by $\hat{P} := \sqrt{\sum_\alpha |P^{(\alpha)}(0)|^2}$ there is a norm given on the vector space

$\mathbb{C}[X_1,\dots,X_d]$. From now on let $\mathbb{C}[X_1,\dots,X_d]$ be equipped with the vector space topology induced by this norm. The set of all limits in $\mathbb{C}[X_1,\dots,X_d]$ of the normalized polynomials

$$x\mapsto\frac{P_{\xi,\omega}(x)}{\hat{P}_{\xi,\omega}}$$

as ξ tends to infinity is denoted by $L_\omega(P)$. More precisely, if $N\in S^{d-1}$ then the set of limits where $\xi/|\xi|\to N$ (with ξ tending to infinity) is denoted by $L_{\omega,N}(P)$. Obviously, $L_\omega(P)$ as well as $L_{\omega,N}(P)$ are closed subsets of the unit sphere of all polynomials in d variables, equipped with the norm $Q\mapsto\hat{Q}$, of degree not exceeding the degree of P. The non-zero multiples of elements of $L_\omega(P)$ (resp. of $L_{\omega,N}(P)$) are called *ω-localizations of P at infinity* (resp. *ω-localizations of P at infinity in direction N*). Since $\omega(\xi)=\omega(|\xi|)$, $Q\in L_{\omega,N}(\check{P})$ if and only if $\check{Q}\in L_{\omega,-N}(P)$. Again, if we formally set $\omega\equiv 1$ we obtain the well known set $L(P)$ of localizations of P at infinity (see Hörmander [7, Definition 10.2.6]).

For the classical case, i.e. if formally $\omega\equiv 1$, the next lemma is proved in [8]. The proof here is almost the same, but we include it for the reader's convenience.

Lemma 3.2. *Let P be of degree m with principal part P_m.*

i) *For every subspace V of $\mathbb{R}^d$ and $t\geq 1$ we have*
$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t\omega(\xi))}{\tilde{P}(\xi,t\omega(\xi))}=\inf_{Q\in L_\omega(P)}\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

ii) *Let $N\in S^{d-1}$ and $Q\in L_{\omega,N}(P)$. If $P_m(N)\neq 0$ then Q is constant.*

iii) *If P is non-elliptic then for every subspace V of $\mathbb{R}^d$ and $t\geq 1$ we have*
$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t\omega(\xi))}{\tilde{P}(\xi,t\omega(\xi))}=\inf_{N\in S^{d-1},P_m(N)=0}\ \inf_{Q\in L_{\omega,N}(P)}\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

Proof. i) Since for every subspace V and each $t>0$ the maps $R\mapsto\tilde{R}_V(0,t)$ are continuous seminorms on $\mathbb{C}[X_1,\dots,X_d]$ and because $\tilde{P}_V(\xi,t\omega(\xi))=(\tilde{P_{\xi,\omega}})_V(0,t)$ it follows immediately from the definition that

$$\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}\geq\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t\omega(\xi))}{\tilde{P}(\xi,t\omega(\xi))}$$

for every $Q\in L_\omega(P)$.

Moreover, if $(\xi_n)_{n\in\mathbb{N}}$ tends to infinity such that

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t\omega(\xi))}{\tilde{P}(\xi,t\omega(\xi))}=\lim_{n\to\infty}\frac{\tilde{P}_V(\xi_n,t\omega(\xi_n))}{\tilde{P}(\xi_n,t\omega(\xi_n))}=\lim_{n\to\infty}\frac{(\tilde{P}_{\xi_n,\omega})_V(0,t)}{\tilde{P}_{\xi_n,\omega}(0,t)}$$

we can extract a subsequence of $(\xi_n)_{n\in\mathbb{N}}$ which we again denote by $(\xi_n)_{n\in\mathbb{N}}$ such that the sequence of normalized polynomials $P_{\xi_n,\omega}/\hat{P}_{\xi_n,\omega}$ converges in the compact unit sphere of all polynomials in d variables of degree at most m. This limit belongs to $L_\omega(P)$ and we get

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t\omega(\xi))}{\tilde{P}(\xi,t\omega(\xi))}\geq\inf_{Q\in L_\omega(P)}\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}$$

completing the proof of i).

The proof of ii) is an easy application of Taylor's formula. Let $P=\sum_{j=0}^m P_j$, where P_j is a homogeneous polynomial of degree j. Let $(\xi_n)_{n\in\mathbb{N}}$ tend to infinity with $\lim_{n\to\infty}\xi_n/|\xi_n|=N$ and $P_m(N)\neq 0$. Then

$$\begin{aligned}
P_{\xi_n,\omega}(\eta) &= \sum_{0\leq|\alpha|\leq j\leq m}\frac{P_j^{(\alpha)}(\xi_n)}{\alpha!}\omega(\xi_n)^{|\alpha|}\eta^\alpha\\
&= |\xi_n|^m\left(\sum_{0\leq j\leq m}\frac{|\xi_n|^j}{|\xi_n|^m}P_j(\frac{\xi_n}{|\xi_n|})+\sum_{0<|\alpha|\leq j\leq m}\frac{|\xi_n|^{j-|\alpha|}\omega(\xi_n)^{|\alpha|}}{|\xi_n|^m\alpha!}P_j^{(\alpha)}(\frac{\xi_n}{|\xi_n|})\eta^\alpha\right).
\end{aligned}$$

Moreover

$$\begin{aligned}
\hat{P}_{\xi_n,\omega} &= \sqrt{\sum_{0\leq|\alpha|\leq m}|\sum_{j=|\alpha|}^m P_j^{(\alpha)}(\xi_n)|^2\omega(\xi_n)^{2|\alpha|}}\\
&= |\xi_n|^m\sqrt{|\sum_{j=0}^m P_j(\frac{\xi_n}{|\xi_n|})\frac{|\xi_n|^j}{|\xi_n|^m}|^2+\sum_{0<|\alpha|\leq m}|\sum_{j=|\alpha|}^m P_j^{(\alpha)}(\frac{\xi_n}{|\xi_n|})\frac{|\xi_n|^{j-|\alpha|}\omega(\xi_n)^{|\alpha|}}{|\xi_n|^m}|^2},
\end{aligned}$$

which implies by $\omega(\xi_n)=o(|\xi_n|)$ as n tends to infinity

$$\lim_{n\to\infty}\frac{P_{\xi_n,\omega}(\eta)}{\hat{P}_{\xi_n,\omega}}=\frac{P_m(N)}{|P_m(N)|}$$

for every $\eta\in\mathbb{R}^d$ showing ii).

iii) is an immediate consequence of $\liminf_{\xi\to\infty}\tilde{P}_V(\xi,t\omega(\xi))/\tilde{P}(\xi,t\omega(\xi))\leq 1$, i), and ii). □

Before we continue, recall the following definition (cf. Hörmander [7, Section 10.2]). Let

$$\Lambda(P)=\{\eta\in\mathbb{R}^d;\,\forall\,\xi\in\mathbb{R}^d,t\in\mathbb{R}:\;P(\xi+t\eta)=P(\xi)\},$$

which is obviously a subspace of $\mathbb{R}^d$ which coincides with $\mathbb{R}^d$ if and only if P is constant. In case of $\omega\equiv 1$ the corresponding result of the next proposition is due to Hörmander [7, Theorem 10.2.8, vol. II] and its proof uses the Tarski-Seidenberg theorem. In our case, the proof is rather elementary.

Lemma 3.3. *If $Q\in L_{\omega,N}(P)$ then $N\in\Lambda(Q)$.*

PROOF. Since $\omega(\xi) = \omega(|\xi|)$ we can assume without loss of generality that $N = e_1 = (1, 0, \dots, 0)$. We denote the degree of P by m. In case of $P^{(e_1)} \equiv 0$ we clearly have by Taylor's theorem that $e_1 \in \Lambda(P)$ which clearly implies $e_1 \in \Lambda(Q)$ by the definition of $L_\omega(P)$.

Now, if $P^{(e_1)}$ does not vanish identically it follows that $P^{(e_1)}_{\xi,\omega}$ does not vanish identically either, for every $\xi \in \mathbb{R}^d$. Since $P \mapsto \sum_\alpha |P^{(\alpha)}(0)|$ is a norm on the space of all polynomials in d variables, it follows that for every $\xi \in \mathbb{R}^d$

$$0 \neq \sum_\alpha |P^{(e_1)}_{\xi,\omega}(0)| = \sum_\alpha |P^{(\alpha+e_1)}(\xi)|\omega(\xi)^{|\alpha|} = \sum_{0\leq|\alpha|\leq m-1} |P^{(\alpha+e_1)}(\xi)|\omega(\xi)^{|\alpha|},$$

because P has degree m. Hence, for every $\xi \in \mathbb{R}^d, t \in \mathbb{R}$ we have by Taylor's theorem

$$\begin{aligned}
0 &\leq \frac{|P^{(e_1+\alpha)}(\xi + \omega(\xi)(x + se_1))|}{\sum_\alpha |P^{(\alpha)}(\xi)|\omega(\xi)^{|\alpha|}} = \frac{|\sum_{0\leq|\alpha|\leq m-1} P^{(\alpha+e_1)}(\xi)\omega(\xi)^{|\alpha|}\frac{1}{\alpha!}(x + se_1)^\alpha|}{\sum_\alpha |P^{(\alpha)}(\xi)|\omega(\xi)^{|\alpha|}} \\
&\leq \frac{\sum_{0\leq|\alpha|\leq m-1} |P^{(\alpha+e_1)}(\xi)|\omega(\xi)^{|\alpha|}\frac{1}{\alpha!}|(x + se_1)^\alpha|}{\sum_{0\leq|\alpha|\leq m-1} |P^{(\alpha+e_1)}(\xi)|\omega(\xi)^{1+|\alpha|}} \\
&\leq \frac{\max_{0\leq|\alpha|\leq m-1} \frac{1}{\alpha!}|(x + se_1)^\alpha|}{\omega(\xi)}.
\end{aligned}$$

Since $Q \in L_\omega(P)$ there is $(\xi_n)_{n\in\mathbb{N}}$ tending to infinity such that

$$Q(x) = \lim_{n\to\infty} \frac{P(\xi_n + \omega(\xi_n)x)}{\hat{P}_{\xi_n,\omega}}$$

in the vector space topology of the polynomials in d variables of degree not exceeding m. In particular, we also have

$$Q^{(e_1)}(x) = \lim_{n\to\infty} \frac{P^{(e_1)}(\xi_n + \omega(\xi_n)x)}{\hat{P}_{\xi_n,\omega}}.$$

The space of all polynomials in d variables of degree not exceeding m being finite dimensional, all norms on it are equivalent. Therefore, by passing to a subsequence of $(\xi_n)_{n\in\mathbb{N}}$ if necessary, there is $c > 0$ such that for every $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$

$$\begin{aligned}
|Q^{(e_1)}(x + se_1)| &= \lim_{n\to\infty} \frac{|P^{(e_1)}(\xi_n + \omega(\xi_n)(x + se_1))|}{\hat{P}_{\xi_n,\omega}} \\
&\leq c \lim_{n\to\infty} \frac{|P^{(e_1)}(\xi_n + \omega(\xi_n)(x + se_1))|}{\sum_\alpha |P^{(\alpha+e_1)}(\xi_n)|\omega(\xi_n)^{|\alpha|}} \\
&\leq c \lim_{n\to\infty} \frac{\max_{0\leq|\alpha|\leq m-1} \frac{1}{\alpha!}|(x + se_1)^\alpha|}{\omega(\xi_n)} \\
&= 0.
\end{aligned}$$

Hence, for each $x \in \mathbb{R}^d$ the polynomial $q_x : \mathbb{R} \to \mathbb{C}, s \mapsto Q(x + se_1)$ satisfies $q_x'(s) = Q^{(e_1)}(x + se_1) = 0$. Thus q_x is constant which shows that $e_1 \in \Lambda(Q)$. □

Now we are able to prove the main result of this section. In the classical case, i.e. if we formally set $\omega \equiv 1$, the corresponding result was proved in [8]. Again the proofs are almost identical but we include it here for completeness' sake.

Lemma 3.4. *Let $P \in \mathbb{C}[X_1, X_2]$ be of degree m with principal part P_m. Then*

$$\{y \in S^1;\ \sigma_{P,(\omega)}(y) = 0\} \subset \{y \in S^1;\ P_m(y) = 0\}.$$

Proof. By Lemma 3.2 i) and ii) we can assume without loss of generality that P is not elliptic. Since we are in $\mathbb{R}^2$ the principal part P_m can only have a finite numbers of zeros in S^1. Let $\{N \in S^1;\ P_m(N) = 0\} = \{N_1, \ldots, N_l\}$. For each $1 \leq j \leq l$ choose $x_j \in S^1$ orthogonal to N_j. Without loss of generality, let $\{y \in S^1;\ \sigma_P(y) = 0\} \neq \emptyset$. By Lemma 3.2 there is a non-constant $Q \in L_{\omega,N_j}(P)$ for some $1 \leq j \leq l$. By Lemma 3.3 we have $Q(\xi + sN_j) = Q(\xi)$ for any $\xi \in \mathbb{R}^2, s \in \mathbb{R}$. Hence $Q(\xi) = Q(\langle \xi, x_j \rangle x_j)$ for all $\xi \in \mathbb{R}^2$. Defining

$$q : \mathbb{R} \to \mathbb{C}, s \mapsto Q(sx_j)$$

it follows that for fixed $y \in S^1$

$$\begin{aligned} \tilde{Q}_{span\{y\}}(0,t) &= \sup\{|Q(\lambda y)|;\ |\lambda| \leq t\} = \sup\{|Q(\lambda \langle y, x_j \rangle x_j)|;\ |\lambda| \leq t\} \\ &= \sup\{|q(\lambda t \langle y, x_j \rangle)|;\ |\lambda| \leq 1\}, \end{aligned}$$

and because $|x_j| = 1$ we also have

$$\begin{aligned} \tilde{Q}(0,t) &= \sup\{|Q(\xi)|;\ \xi \in \mathbb{R}^2, |\xi| \leq t\} = \sup\{|Q(\langle \xi, x_j \rangle x_j)|;\ \xi \in \mathbb{R}^2, |\xi| \leq t\} \\ &= \sup\{|Q(\lambda x_j)|;\ |\lambda| \leq t\} = \sup\{|q(\lambda t)|;\ |\lambda| \leq 1\}. \end{aligned}$$

Since $Q \in L_\omega(P)$ it follows that q is a polynomial of degree at most m. Because on the finite dimensional space of all polynomials in one variable of degree at most m the norms $\sup_{|s| \leq 1} |p(s)|$ and $\sum_{k=0}^m |p^{(k)}(0)|$ are equivalent there is $C > 0$ such that

$$C \sup_{|s| \leq 1} |p(s)| \geq \sum_{k=0}^{m} |p^{(k)}(0)| \geq 1/C \sup_{|s| \leq 1} |p(s)|$$

for all $p \in \mathbb{C}[X]$ with degree at most m. Applying this to the polynomials $s \mapsto q(st)$ and $s \mapsto q(st\langle y, x_j\rangle)$ gives

$$\begin{aligned} \frac{\tilde{Q}_{span\{y\}}(0,t)}{\tilde{Q}(0,t)} &\geq \frac{\sum_{k=0}^m |q^{(k)}(0)|t^k|\langle y, x_j\rangle|^k}{C^2\sum_{k=0}^m |q^{(k)}(0)|t^k} \\ &\geq |\langle y, x_j\rangle|^m/C^2, \end{aligned}$$

where we used $|\langle y, x_j\rangle| \leq 1$ in the last inequality. We conclude that for every $1 \leq j \leq l$

$$\inf_{Q \in L_{\omega,N_j}(P)} \frac{\tilde{Q}_{span\{y\}}(0,t)}{\tilde{Q}(0,t)} \geq \frac{|\langle y, x_j\rangle|^m}{C^2},$$

where C only depends on the degree m of P. It follows from Lemma 3.2 iii) and $\{N \in S^1;\, P_m(N) = 0\} = \{N_1, \ldots, N_l\}$ that for all $t \geq 1$

$$\liminf_{\xi \to \infty} \frac{\tilde{P}_{span\{y\}}(\xi, t\omega(\xi))}{\tilde{P}(\xi, t\omega(\xi))} = \min_{1 \leq j \leq l} \inf_{Q \in L_{\omega,N_j}(P)} \frac{\tilde{Q}_{span\{y\}}(0,t)}{\tilde{Q}(0,t)} \geq \min_{1 \leq j \leq l} \frac{|\langle y, x_j\rangle|^m}{C^2}.$$

Therefore, if for $y \in S^1$

$$0 = \sigma_{P,(\omega)}(y) = \inf_{t \geq 1} \liminf_{\xi \to \infty} \frac{\tilde{P}_{span\{y\}}(\xi, t\omega(\xi))}{\tilde{P}(\xi, t\omega(\xi))}$$

it follows that y is orthogonal to some x_j, hence $y \in \{N_j, -N_j\}$ since $|y| = 1 = |N_j|$ which shows $P_m(y) = 0$. $\square$

In particular it follows that for $P \in \mathbb{C}[X_1, X_2] \backslash \{0\}$ the set

$$\{y \in S^1; \sigma_{P,(\omega)}(y) = 0\}$$

is finite. Moreover, it follows immediately from the above lemma that in case of $d = 2$ every hyperplane $H = \{x; \langle x, N\rangle = \alpha\}, N \in S^{d-1}, \alpha \in \mathbb{R}$, with $\sigma_{P,(\omega)}(N) = 0$ is characteristic for P. That this is not the case in general for $d \geq 3$ is shown by the next example.

Example 3.5. Let $d > 2$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ be given by

$$P(x_1, \ldots, x_d) = x_1^2 - x_2^2 - \ldots - x_d^2.$$

It follows that for each weight function ω an ω-localization of P at infinity in direction $1/\sqrt{2}\,(1, 1, 0, \ldots, 0)$ is given by $Q(x_1, \ldots, x_d) = (x_1 - x_2)/\sqrt{2}$. Hence it follows for $e_d = (0, \ldots, 0, 1)$ that $\tilde{Q}_{span\{e_d\}}(0, t) = 0$ for every $t \geq 1$ so that in particular $\sigma_{P,(\omega)}(e_d) = 0$ by Lemma 3.2. On the other hand, we clearly have $P_2(e_d) = P(e_d) = -1$.

4. A sufficient Condition for P-convexity for (ω)-singular Supports

In this section we will prove a sufficient condition for an open subset Ω of $\mathbb{R}^d$ to be P-convex for (ω)-singular supports in terms of an exterior cone condition, similar to those proved in [8].

Recall that a cone C is called *proper* if it does not contain any affine subspace of dimension one. Moreover, recall that for an open convex cone $\Gamma \subset \mathbb{R}^d$ its *dual cone* is defined as

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d ; \forall\, y \in \Gamma : \langle y, \xi\rangle \geq 0\}.$$

For $\Gamma \neq \emptyset$ it is a closed proper convex cone in $\mathbb{R}^d$. On the other hand, every closed proper convex cone C in $\mathbb{R}^d$ is the dual cone of a unique non-empty, open, convex cone which is given by

$$\Gamma := \{y \in \mathbb{R}^d ; \forall \xi \in C\backslash\{0\} : \langle y, \xi\rangle > 0\}.$$

The proof can be done by the Hahn-Banach Theorem (cf. [7, p. 257, vol. I]). Therefore, we use the notation Γ° also for arbitrary closed convex proper cones. Moreover, from now on we assume all open convex cones Γ to be non-empty.

As a first result we obtain from Theorem 3.1 the next proposition which is an analogue result to [7, Corollary 8.6.11, vol. I].

Lemma 4.1. *Let Γ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$. If for $\Omega := x_0 + \Gamma$ no hyperplane $H = \{x \in \mathbb{R}^d ; \langle x, N\rangle = \alpha\}, N \in S^{d-1}, \alpha \in \mathbb{R}$, with $\sigma_{P,(\omega)}(N) = 0$ intersects $\overline{\Omega}$ only in x_0, the following holds.*

Each $u \in \mathscr{D}'_{(\omega)}(\Omega)$ with $\operatorname{sing\,supp}_{(\omega)} P(D)u = \emptyset$ and $\operatorname{sing\,supp}_{(\omega)} u$ bounded already satisfies $\operatorname{sing\,supp}_{(\omega)} u = \emptyset$.

Proof. Let $u \in \mathscr{D}'_{(\omega)}(\Omega)$ satisfy $P(D)u \in \mathscr{E}_{(\omega)}(\Omega)$ and assume that u is $\mathscr{E}_{(\omega)}$ outside a bounded subset of Ω. Since Γ is a proper cone, there is a hyperplane π intersecting Ω only in x_0. Let H_π be a halfspace with boundary parallel to π such that $\Omega_1 := \Omega \cap H_\pi \neq \emptyset$ is unbounded and $u|_{\Omega_1} \in \mathscr{E}_{(\omega)}(\Omega_1)$. Denoting $\Omega_2 := \Omega$ we have convex sets $\Omega_1 \subset \Omega_2$ and by the hypothesis, each hyperplane $H = \{x \in \mathbb{R}^d ; \langle x, N\rangle = \alpha\}, N \in S^{d-1}, \alpha \in \mathbb{R}$, with $\sigma_{P,(\omega)}(N) = 0$ and $H \cap \Omega_2 \neq \emptyset$ already intersects Ω_1. Theorem 3.1 now gives $\operatorname{sing\,supp}_{(\omega)} u = \emptyset$. $\square$

Before we come to the main result of this section, we need one more result.

Theorem 4.2. i) *If* $u \in \mathscr{E}'_{(\omega)}(\mathbb{R}^d)$ *then*

$$ch(sing\,supp_{\,(\omega)}u) = ch(sing\,supp_{\,(\omega)}P(-D)u).$$

ii) *For an open subset* Ω *of* $\mathbb{R}^d$ *the following are equivalent.*

a) Ω *is* P*-convex for* (ω)*-singular supports.*

b) *For each* $u \in \mathscr{E}'_{(\omega)}(\Omega)$ *one has*

$$dist(sing\,supp_{\,(\omega)}u, \Omega^c) = dist(sing\,supp_{\,(\omega)}P(-D)u, \Omega^c).$$

Proof. i) By a result of Bonet et al. [3, Remark 2.10] one has for a convex compact subset K of $\mathbb{R}^d$ and $u \in \mathscr{E}_{(\omega)}(\mathbb{R}^d)$ that the inclusion $\operatorname{sing\,supp}_{\,(\omega)}u \subset K$ is equivalent to the existence of $b > 0$ such that for each $m \in \mathbb{N}$ there is $C_m > 0$ such that

$$|\hat{u}(\zeta)| \leq C_m \exp(H_K(\operatorname{Im}\zeta) + b\omega(\zeta))$$

for all $\zeta \in \mathbb{C}^d$ with $|\operatorname{Im}\zeta| \leq m\omega(\zeta)$ and $|\zeta| \geq C_m$, where H_K denotes the supporting function of K. Moreover, by [3, Remark 1.2 (c)] we can assume without loss of generality that $\omega \geq 1$.

Since by Braun et al. [4, Lemma 1.2] there is some constant $K > 0$ such that $\omega(\zeta + \eta) \leq K(1 + \omega(\zeta) + \omega(\eta))$ for all $\zeta, \eta \in \mathbb{C}^d$ it follows for all $\zeta \in \mathbb{C}^d$ with $|\operatorname{Im}\zeta| \leq m\omega(\zeta)$ and all $z \in \mathbb{C}, |z| = 1$ that

$$\begin{aligned} |\operatorname{Im}(\zeta + ze_1)| &\leq m\omega(\zeta) + 1 = m\omega(\zeta + ze_1 - ze_1) + 1 \\ &\leq m\omega(|\zeta + ze_1| + 1) + 1 \leq Km(1 + \omega(\zeta + ze_1) + \omega(1)) + 1 \\ &\leq Km\omega(\zeta + ze_1) + (Km(1 + \omega(1)) + 1)\,\omega(\zeta + ze_1) \\ &= (Km(2 + \omega(1)) + 1)\,\omega(\zeta + ze_1). \end{aligned}$$

Hence, if $|\operatorname{Im}\zeta| \leq m\omega(\zeta)$ for some $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that

$$|\operatorname{Im}(\zeta + ze_1)| \leq k\omega(\zeta + ze_1) \text{ for all } z \in \mathbb{C}, |z| = 1. \tag{1}$$

Now, for $u \in \mathscr{E}'_{(\omega)}(\Omega)$ set $f := P(-D)u$ and let K be the convex hull of $\operatorname{sing\,supp}_{\,(\omega)}f$. Clearly, we have $ch(\operatorname{sing\,supp}_{\,(\omega)}u) \supset K$. In order to show that $ch(\operatorname{sing\,supp}_{\,(\omega)}u) \subset K$ observe that by [3, Remark 2.10] there is $b > 0$ such that for all $m \in \mathbb{N}$ there is $C_m > 0$ such that

$$|P(-\zeta)\hat{u}(\zeta)| = |\hat{f}(\zeta)| \leq C_m \exp(H_K(\operatorname{Im}\zeta) + b\omega(\zeta))$$

for all $\zeta \in \mathbb{C}^d$ with $|\zeta| \geq C_m$ and $|\operatorname{Im}\zeta| \leq m\omega(\zeta)$. By [7, Lemma 7.3.3, vol. I] there is $a > 0$ such that

$$a|\hat{u}(\zeta)| \leq \sup_{|z|=1} |\hat{f}(\zeta + ze_1)|$$

for all $\zeta \in \mathbb{C}^d$. Hence, for all $\zeta \in \mathbb{C}^d$ such that $|\zeta + ze_1| \geq C_m$ and $|\mathrm{Im}(\zeta + ze_1)| \leq m\omega(\zeta + ze_1)$ for every $|z| = 1$ we obtain

$$\begin{aligned} a|\hat{u}(\zeta)| &\leq \sup_{|z|=1} C_m \exp(H_K(\mathrm{Im}(\zeta + ze_1)) + b\omega(\zeta + ze_1)) \\ &\leq \sup_{|z|=1} C_m \exp(H_K(\mathrm{Im}\zeta) + H_K(\mathrm{Im} ze_1) + bK(1 + \omega(\zeta) + \omega(1))) \\ &= \sup_{|z|=1} C_m \exp(H_K(\mathrm{Im} ze_1) + bK(1 + \omega(1))) \exp(H_K(\mathrm{Im}\zeta) + bK\omega(\zeta)). \end{aligned}$$

Combining this and inequality (4.1) gives $\tilde{b} > 0$ such that for all $m \in \mathbb{N}$ there is $\tilde{C}_m > 0$ such that

$$|\hat{u}(\zeta)| \leq \tilde{C}_m \exp(H_K(\mathrm{Im}\zeta) + \tilde{b}\omega(\zeta))$$

for all $\zeta \in \mathbb{C}^d$ with $|\zeta| \geq \tilde{C}_m$ and $|\mathrm{Im}\zeta| \leq m\omega(\zeta)$, proving $ch(\operatorname{sing\,supp}_{(\omega)} u) \subset K$, hence i).

Using i), ultradifferentiable cut-off functions, and taking into account that $\mathscr{E}_{(\omega)}(\Omega)$ is an algebra with continuous multiplication (cf. [4, Proposition 4.4]), the proof of ii) follows along the same lines as the proofs of [7, Theorem 10.6.3 and/or Theorem 10.7.3, vol. II]. □

The following proposition (cf. [8]) contains some elementary geometric facts which will be used in the sequel.

Lemma 4.3. *Let $\Gamma^\circ \neq \{0\}$ be a closed proper convex cone in $\mathbb{R}^d$ and $N \in S^{d-1}$. For $c \in \mathbb{R}$ let $H_c := \{x \in \mathbb{R}^d; \langle x, N\rangle = c\}$. Then the following are equivalent.*

i) *$N \in \Gamma$ or $-N \in \Gamma$.*

ii) *If $x \in H_c$ then $H_c \cap (x + \Gamma^\circ) = \{x\}$.*

We are now able to prove the main result of this section. Compare also [8, Theorem 9].

Theorem 4.4. *Let Ω be an open connected subset of $\mathbb{R}^d$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ a non-constant polynomial with principal part P_m. Then Ω is P-convex for (ω)-singular supports if for every $x \in \partial\Omega$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap \Omega = \emptyset$ and $\sigma_{P,(\omega)}(y) \neq 0$ for all $y \in \Gamma$.*

Proof. Let $u \in \mathscr{E}'_{(\omega)}(\Omega)$. We set $K := \operatorname{sing\,supp}_{(\omega)} P(-D)u$ and $\delta := dist(K, \Omega^c)$. We will show that

$$dist(\operatorname{sing\,supp}_{(\omega)} u, \Omega^c) \geq \delta$$

which by

$$\operatorname{sing\,supp}_{(\omega)} u \supset \operatorname{sing\,supp}_{(\omega)} P(-D)u$$

will imply

$$dist(\operatorname{sing\,supp}_{(\omega)} u, \Omega^c) = \delta,$$

hence P-convexity for (ω)-singular supports of Ω by Theorem 4.2.

Let $x_0 \in \partial\Omega$ and let Γ be as in the hypothesis for $x_0 \in \partial\Omega$. Then $(x_0 + \Gamma^\circ) \cap \Omega = \emptyset$, thus $(x_0 + y + \Gamma^\circ) \cap K = \emptyset$ for all $y \in \mathbb{R}^d$ with $|y| < \delta$. Therefore, for fixed y with $|y| < \delta$, there is an open proper convex cone $\tilde{\Gamma}$ in $\mathbb{R}^d$ with $\tilde{\Gamma} \supset \Gamma^\circ \backslash \{0\}$ such that $(x_0 + y + \tilde{\Gamma}) \cap K = \emptyset$. Hence, $u \in \mathscr{E}'_{(\omega)}(\Omega) \subset \mathscr{D}'_{(\omega)}(x_0 + y + \tilde{\Gamma})$ satisfies $P(-D)u \in \mathscr{E}_{(\omega)}(x_0 + y + \tilde{\Gamma})$.

We will show that $u \in \mathscr{E}_{(\omega)}(x_0 + y + \tilde{\Gamma})$ by applying Lemma 4.1. Hence, let $H = \{v \in \mathbb{R}^d; \langle v, N \rangle = \alpha\}$ be a hyperplane with $\sigma_{P,(\omega)}(N) = 0$. As $\overline{\tilde{\Gamma}}$ is a closed proper convex cone with non-empty interior, it is the dual cone of some open proper convex cone Γ_1. It follows from $\Gamma_1^\circ = \overline{\tilde{\Gamma}} \supset \Gamma^\circ$ that $\Gamma_1 \subset \Gamma$. Because $\sigma_{P,(\omega)}(N) = 0$ it follows from the hypothesis that $\{N, -N\} \cap \Gamma = \emptyset$, hence $\{N, -N\} \cap \Gamma_1 = \emptyset$, so that by Lemma 4.3 H does not intersect $x_0 + y + \overline{\tilde{\Gamma}}$ only in $x_0 + y$. Since $u \in \mathscr{E}'_{(\omega)}(\Omega)$ we have that $\operatorname{sing\,supp} u$ is compact. Moreover $P(-D)u \in \mathscr{E}_{(\omega)}(x_0 + y + \tilde{\Gamma})$, so that $u \in \mathscr{E}_{(\omega)}(x_0 + y + \tilde{\Gamma})$ by Lemma 4.1. Since $x_0 \in \partial\Omega$ and y with $|y| < \delta$ were chosen arbitrarily, we conclude that $dist(\operatorname{sing\,supp}_{(\omega)} u, \Omega^c) \geq \delta$, which proves the theorem. □

5. Proof of the main Theorem

Recall that for elliptic P every open subset $\Omega \subset \mathbb{R}^d$ is P-convex for supports. In case of $d = 2$ a complete characterization of P-convexity for supports is known. It is due to Hörmander, see e.g. [7, Theorem 10.8.3, vol. II].

Theorem 5.1. *If P is non-elliptic then the following conditions on an open connected set $\Omega \subset \mathbb{R}^2$ are equivalent.*

i) *Ω is P-convex for supports.*
ii) *The intersection of every characteristic hyperplane with Ω is convex.*
iii) *For every $x_0 \in \partial\Omega$ there is a closed proper convex cone $\Gamma^\circ \neq \{0\}$ with $(x_0 + \Gamma^\circ) \cap \Omega = \emptyset$ and no characteristic hyperplane intersects $x_0 + \Gamma^\circ$ only in x_0.*

It is not hard to see that in the above theorem condition iii) is equivalent to the following condition (see [8]).

iii') *For every $x_0 \in \partial\Omega$ there is an open convex cone $\Gamma \neq \mathbb{R}^2$ with $(x_0 + \Gamma^\circ) \cap \Omega = \emptyset$ and $P_m(y) \neq 0$ for all $y \in \Gamma$, where P_m denotes the principal part of P.*

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^2$ be open, ω a weight function, and $P \in \mathbb{C}[X_1, X_2]$. If Ω is P-convex for supports then Ω is P-convex for (ω)-singular supports.*

Proof. Without loss of generality we can assume that P is not elliptic. Clearly, by passing to the different components of Ω if necessary, we can assume that Ω is connected. Since P is not elliptic, it follows from Theorem 5.1 with iii'), Lemma 3.4, and Theorem 4.4 that Ω is P-convex for (ω)-singular supports. □

As a corollary we now obtain Theorem 1.1.

Proof of Theorem 1.1. That i) and ii) are equivalent is shown in [8]. Clearly, iii) implies iv). By Theorem 2.4 and Remark 2.5 i), iv) implies that Ω is P-convex for supports, so that i) follows from iv). So, all that remains to be shown is that i) implies iii). But this implication follows from Theorem 5.2 and Theorem 2.4. □

Combining Theorems 1.2, 5.1, and 1.1 gives the next result.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]$. The following are equivalent.*

i) $P(D) : A(\Omega) \to A(\Omega)$ *is surjective.*
ii) $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ *is surjective.*
iii) $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ *is surjective.*
iv) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ *is surjective for some non-quasianalytic weight function ω.*
v) $P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega)$ *is surjective for each non-quasianalytic weight function ω.*
vi) *The intersection of every characteristic hyperplane with any connected component of Ω is convex.*

The next example shows that for $d \geq 3$ an analogous result to Theorem 1.1 is not true in general. See also Langenbruch [9, Example 3.13], where it is shown that surjectivity of $P(D)$ on $\mathscr{D}'_{(\omega)}(\Omega)$ for $d \geq 3$ depends explicitly on the weight function ω in general.

Example 5.4. Let $d > 2$ and $P(x_1, \ldots, x_d) = x_1^2 - x_2^2 - \ldots - x_d^2$. Moreover, let $\Gamma := \{x \in \mathbb{R}^d;\ x_d > (x_1^2 + \ldots + x_{d-1}^2)^{1/2}\}$. Then Γ is an open convex cone with $\Gamma^\circ = \overline{\Gamma}$. Set $\Omega := \mathbb{R}^d \backslash \overline{\Gamma}$. Then it is not hard to show that Ω is P-convex for supports. This follows for example by [8, Theorem 9 i)]. Hence, $P(D)$ is surjective on $C^\infty(\Omega)$ but $P(D)$ is not surjective on $\mathscr{D}'(\Omega)$ (see [8, Example 12]).

Moreover, it follows from Example 3.5 and Lemma 3.2 that

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_{span\{e_d\}}(\xi,\omega(\xi))}{\tilde{P}(\xi,\omega(\xi))}=0,$$

where $e_d=(0,\ldots,0,1)$. Setting $H=\{x\in\mathbb{R}^d;\,\langle x,e_d\rangle=-1\}$ and

$$K:=H\cap\{x\in\mathbb{R}^d;|x|\leq 2\}$$

it is easily seen that the distance of $\partial\Omega=\partial\Gamma$ to K is 1 while the distance of $\partial\Gamma$ to $\partial_H K$, i.e. to the boundary of K relative H, strictly increases 1. Hence, it follows from [9, Corollary 2.7] that $P(D)$ cannot be surjective on $\mathscr{D}'_{(\omega)}(\Omega)$.

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