SOME RESULTS ON SURJECTIVITY OF AUGMENTED DIFFERENTIAL OPERATORS

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Abstract. Considering a problem of Bonet and Domański [1, Problem 9.1], we prove that for a polynomial P on $\mathbb{R}^2$ surjectivity of the differential operator $P(D)$ on $\mathscr{D}'(X)$ implies surjectivity of the augmented operator $P^+(D)$ on $\mathscr{D}'(X \times \mathbb{R})$, where $P^+(x_1, x_2, x_3) := P(x_1, x_2)$. Moreover we give a sufficient geometrical condition on an open subset X of $\mathbb{R}^d$ such that an analogous implication is true for arbitrary dimension d in case of P being homogeneous, semi-elliptic, or of principal type.

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1. Introduction

For an open subset $X \subset \mathbb{R}^d$ and $P \in \mathbb{C}[X_1, \dots, X_d]$ a non-zero polynomial consider the corresponding differential operator $P(D)$ on $\mathscr{D}'(X)$, where as usual $D_j = -i\frac{\partial}{\partial x_j}$. For $(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ we set $P^+(x_1, \dots, x_{d+1}) := P(x_1, \dots, x_d)$ and call $P^+(D)$ the augmented operator, i.e. $P(D)$ acting "on the first d variables" on $\mathscr{D}'(X \times \mathbb{R})$.

In [1, Problem 9.1] Bonet and Domański asked if surjectivity of the constant coefficient differential operator $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ passes on to surjectivity of $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$. This question is closely connected with the parameter dependence of solutions of the differential equation

$$P(D)u_\lambda = f_\lambda,$$

see [1]. Bonet and Domański proved in [1, Proposition 8.3] that for a surjective differential operator $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ the augmented operator $P^+(D)$ is surjective if and only if the kernel of $P(D)$ has the linear topological invariant $(P\Omega)$.

By a classical result due to Hörmander [4] $P(D)$ is surjective on $\mathscr{D}'(X)$ if and only if X is $P(D)$-convex for supports as well as for singular supports. These are some kind of geometric properties of X reflecting properties of the transposed operator $P(D)^t = P(-D)$ acting on the space $\mathscr{E}'(X)$ of distributions in X with compact support. A different characterization of the surjectivity of $P(D)$ on $\mathscr{D}'(X)$ in terms of the existence of certain shifted fundamental solutions was given only recently by Wengenroth [15]. Roughly speaking, $P(D)$ is surjective on $\mathscr{D}'(X)$ if and only if for every $\xi \in X$ near the boundary of X there is $E \in \mathscr{D}'(\mathbb{R}^d)$ such that, in a large relatively compact open subset of X, $P(D)E = \delta_\xi$ and E is a C^k-function

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there with E and its derivatives up to order k being small, where k is somewhat arbitrary.

This kind of condition on the existence of shifted fundamental solutions with additional properties was also used in articles by Meise, Taylor, and Vogt [12], [13] in order to characterize the existence of continuous linear right inverses of $P(D)$ on $\mathscr{E}(X)$ and $\mathscr{D}'(X)$, respectively. In place of $E \in \mathscr{D}'(X)$ being regular in the above sense, one has to require that E vanishes in X except perhaps close to its boundary. Moreover, Langenbruch characterized in [10] (see also [11]) surjectivity of $P(D)$ on the space of real analytic functions $A(X)$ over X, where the existence of shifted fundamental solutions having additional properties plays an important rôle, too.

Because the above result of Wengenroth seems rather difficult to apply in concrete situations, we will treat the problem of Bonet and Domański by using Hörmanders classical approach. Thus we are interested in whether $X \times \mathbb{R}$ is P^+-convex for supports as well as P^+-convex for singular supports in case of X being P-convex for supports as well as P-convex for singular supports. In [3, Proposition 1] it is shown that P-convexity for supports of X is passed on to P^+-convexity for supports of $X \times \mathbb{R}$. Moreover, it is shown in [3, Example 9] that an analogous implication for P-convexity for singular supports is not true in general but in this example the set X is not P-convex for supports.

In this paper we give some positive results on the above problem under certain conditions. Namely, we prove that for every open $X \subset \mathbb{R}^2$ and every polynomial $P \in \mathbb{C}[X_1, X_2]$ surjectivity of $P(D)$ on $\mathscr{D}'(X)$ passes on to surjectivity of $P^+(D)$ on $\mathscr{D}'(X \times \mathbb{R})$. To be more precise, we show that P-convexity for supports of X is equivalent to P^+-convexity for singular supports of $X \times \mathbb{R}$. Moreover, we show that for arbitrary dimension the question posed by Bonet and Domański has a positive answer on special open subsets X if P is homogeneous, semi-elliptic, or of principal type.

However, it will be shown in a forthcoming paper that the answer to the problem of Bonet and Domański in general is in the negative [9].

The paper is organized as follows. In section 2 we give some sufficient condition for P-convexity by means of exterior cone conditions. These are formally similar to the sufficient condition for surjectivity on $A(X)$ for operators $P(D)$ with locally hyperbolic principal part P_m involving the local propagation cone for P_m in [10]. The exterior cone conditions are then used in section 3 to give an affirmative result to the above problem for special open subsets X of $\mathbb{R}^d$ in arbitrary dimensions d and the previously mentioned classes of polynomials. Finally, in section 4 we show that in two dimensions, i.e. when $d = 2$ the above question always has a positive answer.

Apart from standard notation we use the following. For an affine subspace V of $\mathbb{R}^d$ we denote by $V^\perp$ the orthogonal space to the subspace parallel to V. In particular, for a hyperplane $H = \{x; \langle x, N\rangle = \alpha\}$ in $\mathbb{R}^d$, where $N \in \mathbb{R}^d\backslash\{0\}$ and $\alpha \in \mathbb{R}$ we have that $H^\perp$ is the one-dimensional subspace spanned by N. Moreover, for $x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1}$ we set $x' = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and more generally, we write $M' = \{x'; x \in M\}$ for a subset M of $\mathbb{R}^{d+1}$. Furthermore, a cone is always assumed to be non-empty.

2. Exterior cone conditions for P-convexity

In this section we present some sufficient conditions for an open subset X of $\mathbb{R}^d$ to be P-convex for supports as well as P-convex for singular supports in terms of exterior cone conditions. A similar sufficient condition for the P^+-convexity for singular supports of $X \times \mathbb{R}$ is also given (see Theorem 11 below).

Recall that a cone C is called *proper* if it does not contain any affine subspace of dimension one. Moreover, recall that for an open convex cone $\Gamma \subset \mathbb{R}^d$ its *dual cone* is defined as

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d;\ \forall\, y \in \Gamma : \langle y, \xi\rangle \geq 0\}.$$

It is a closed proper convex cone in $\mathbb{R}^d$. On the other hand, every closed proper convex cone C in $\mathbb{R}^d$ is the dual cone of a unique open convex cone which is given by

$$\Gamma := \{y \in \mathbb{R}^d;\ \forall \xi \in C\backslash\{0\} : \langle y, \xi\rangle > 0\}.$$

The proof can be done by the Hahn-Banach Theorem (cf. [6, p. 257, vol. I]). Therefore, we use the notation Γ° also for arbitrary closed convex proper cones.

The main tool not only in this section but throughout the whole paper will be the following notion introduced by Hörmander in connection with continuation of differentiability (cf. [6, Section 11.3, vol. II]). For a subspace V of $\mathbb{R}^d$

$$\sigma_P(V) = \inf_{t>1} \liminf_{\xi\to\infty} \tilde{P}_V(\xi,t)/\tilde{P}(\xi,t)$$

with $\tilde{P}_V(\xi,t) := \sup\{|P(\xi+\eta)|;\ \eta \in V, |\eta| \leq t\}$, $\tilde{P}(\xi,t) := \tilde{P}_{\mathbb{R}^d}(\xi,t)$. This quantity is closely related with the localizations at infinity of the polynomial P which in turn are connected with bounds for the wave front set and the singular support of regular fundamental solutions of P. In order to simplify notation we will write $\sigma_P(y)$ instead of $\sigma_P(span\{y\})$. We recall some well-known facts in the following remark.

Remark 1. a) Clearly, if $V_1 \subset V_2$ are subspaces of $\mathbb{R}^d$ it follows from the definition that we have $\sigma_P(V_1) \leq \sigma_P(V_2)$.

b) If Q is a localization at infinity of P then there is a subspace $\{0\} \neq \Lambda(Q)$ of $\mathbb{R}^d$ such that

$$\forall\, \xi \in \mathbb{R}^d, \eta \in \Lambda(Q) : Q(\xi+\eta) = Q(\xi),$$

(cf. [6, Theorem 10.2.8, vol. II]). It follows directly from the definitions that for every subspace V of $\mathbb{R}^d$

$$\inf_{t>1} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)} \geq \sigma_P(V).$$

Hence, if $V = \Lambda(Q)$ and if Q is not constant then the left hand side of the above inequality is 0. In particular, if P has a non-constant localization at infinity then there is a subspace $\{0\} \neq V$ of $\mathbb{R}^d$ such that $\sigma_P(V) = 0$.

c) Recall that a polynomial P is hypoelliptic if and only if all of its localizations at infinity are constant (cf. proof of [6, Theorem 11.1.11, vol. II]). Therefore it follows that $\sigma_P(V) = 1$ for every subspace $\{0\} \neq V$ of $\mathbb{R}^d$ if P is hypoelliptic. Moreover, observe that a polynomial P is hypoelliptic if and only if the polynomial $\check{P}(\xi) = P(-\xi)$ is hypoelliptic (this follows e.g. from [6, Theorem 11.1.11, vol. II]). Together with [6, Corollary 11.3.3, vol. II], *a*), and *b*) this gives that for a polynomial P the following are equivalent.

i) Every open set $X \subset \mathbb{R}^d$ is P-convex for singular supports.
ii) P is hypoelliptic.
iii) $\sigma_P(V) \neq 0$ for every subspace $\{0\} \neq V$ of $\mathbb{R}^d$.
iv) $\sigma_P(y) \neq 0$ for every $y \in \mathbb{R}^d\backslash\{0\}$.

One way we use $\sigma_P(V)$ is given by the following result which is nothing but a reformulation of [6, Corollary 11.3.7, vol. II]. For a proof, see [3, Corollary 3].

Proposition 2. *Let $X_1 \subset X_2$ be open and convex, and let P be a non-constant polynomial. Then the following are equivalent:*

i) *Every $u \in \mathscr{D}'(X_2)$ satisfying $P(D)u \in C^\infty(X_2)$ as well as $u|_{X_1} \in C^\infty(X_1)$ already belongs to $C^\infty(X_2)$.*
ii) *Every hyperplane H with $\sigma_P(H^\perp) = 0$ which intersects X_2 already intersects X_1.*

An easy consequence of the above proposition is the next result. For a proof see [7, Proposition 7].

Proposition 3. *Let Γ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and P a non-constant polynomial. If for $X := x_0 + \Gamma$ no hyperplane $H = \{x; \langle x, N\rangle = \alpha\}$ with $\sigma_P(H^\perp) = 0$ intersects $\overline{X}$ only in x_0, the following holds.*

Each $u \in \mathscr{D}'(X)$ with $P(D)u \in C^\infty(X)$ which is C^∞ outside a bounded subset of X already belongs to $C^\infty(X)$.

Because we are interested in the P^+-convexity for singular supports of $X \times \mathbb{R}$ we need a second quantity apart from $\sigma_P(V)$ for a subspace V of $\mathbb{R}^d$.

We define

$$\sigma_P^0(V) := \inf_{t>1, \xi \in \mathbb{R}^d} \tilde{P}_V(\xi, t)/\tilde{P}(\xi, t).$$

This function has already been considered by Hörmander in [5, Section 5] to discuss "Hölder estimates" for solutions of partial differential equations. The reason for introducing this quantity here is given by the following lemma. For the proof see [3, Lemma 1]. Again we write $\sigma_P^0(y)$ instead of $\sigma_P^0(span\{y\})$.

Lemma 4. *Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ and let Π be the orthogonal projection of $\mathbb{R}^{d+1}$ onto the first d coordinates. For a subspace W of $\mathbb{R}^{d+1}$ we identify $W' := \Pi(W)$ with the corresponding subspace of R^d. Then the following hold.*

i) $\sigma_{P^+}(W' \times \{0\}) = \sigma_{P^+}(W' \times \mathbb{R}) = \sigma_P^0(W')$.
ii) $\sigma_{P^+}(W) = 0$ *if and only if* $\sigma_P^0(W') = 0$.

The next lemma exhibits a fundamental connection between σ_P and σ_P^0. Recall that a polynomial P with principal part P_m is of *principal type* if $\nabla P_m(\xi) \neq 0$ for all ξ with $P_m(\xi) = 0$. Moreover, P is called *semi-elliptic* if we have $P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq \mathbf{1}} a_\alpha \xi^\alpha$ with $\sum_{|\alpha:\mathbf{m}|=\mathbf{1}} a_\alpha \xi^\alpha \neq 0$ for any $\xi \neq 0$. Here $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ and $|\alpha : \mathbf{m}| = \sum_{j=1}^d \alpha_j/m_j$.

Lemma 5. *Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be a non-constant polynomial with principal part P_m and $V \subset \mathbb{R}^d$ a subspace.*

i) $\sigma_P^0(V) \leq \sigma_P(V)$.
ii) *If* $V \subset \{P_m = 0\}$ *then* $\sigma_P^0(V) = 0$.
iii) *Assume P is homogeneous, i.e. $P = P_m$. Then $\sigma_P^0(V) = 0$ if and only if $\sigma_P(V) = 0$ or $V \subseteq \{P = 0\}$.*
iv) *Assume that $d = 2$. Then $\sigma_P^0(V) = 0$ if and only if $V \subset \{P_m = 0\}$.*
v) *Assume P is semi-elliptic. Then $\sigma_P^0(V) = 0$ if and only if $V \subset \{P_m = 0\}$.*
vi) *Assume P is of principal type. Then $\sigma_P^0(V) = 0$ if and only if $\sigma_P(V) = 0$.*

Proof. i) is obvious from the definitions.

Obviously $\sigma_P^0(V) \leq \frac{\tilde{P}_V(0,t)}{\tilde{P}(0,t)}$ for every $t > 1$. If $P(\xi) = \sum_{0\leq|\alpha|\leq m} c_\alpha \xi^\alpha$ with $c_\alpha \neq 0$ for some α with $|\alpha| = m$, we define $P_j(\xi) := \sum_{|\alpha|=j} c_\alpha \xi^\alpha, 0 \leq j \leq m$. Thus, $P(\xi) = \sum_{j=0}^m P_j(\xi)$, each P_j is a homogeneous polynomial of degree j and P_m is the principal part of P.

If $V \subset \{P_m = 0\}$ it follows for $t > 1$

$$\frac{\tilde{P}_V(0,t)}{t^m} = \sup_{x\in V, |x|\leq t} |\sum_{j=0}^{m} \frac{1}{t^m} P_j(x)| = \sup_{x\in V, |x|\leq 1} |\sum_{j=0}^{m-1} \frac{1}{t^{m-j}} P_j(x)|.$$

Moreover, for $t > 1$ we have

$$\tilde{P}(0,t) = t^m \sup_{|x|\leq 1} |\sum_{j=0}^{m} \frac{1}{t^{m-j}} P_j(x)|,$$

so that

$$\lim_{t\to\infty} \frac{\tilde{P}_V(0,t)}{\tilde{P}(0,t)} = 0$$

proving ii).

In order to show iii) observe that by i) and ii) we only have to prove that $\sigma_P^0(V) = 0$ implies $\sigma_P(V) = 0$ or $V \subseteq \{P = 0\}$. By the homogeneity of P we have $|P(\xi + tx)| = t^m |P(\frac{\xi}{t} + x)|$ for every $t \geq 1, \xi, x \in \mathbb{R}^d$. This implies

$$\frac{\tilde{P}_V(\xi,t)}{\tilde{P}(\xi,t)} = \frac{\tilde{P}_V(\frac{\xi}{t},1)}{\tilde{P}(\frac{\xi}{t},1)}$$

for every $\xi \in \mathbb{R}^d, t \geq 1$. Thus, if $\sigma_P^0(V) = 0$ we have

$$0 = \inf_{\xi\in\mathbb{R}^d} \frac{\tilde{P}_V(\xi,1)}{\tilde{P}(\xi,1)} = \lim_{n\to\infty} \frac{\tilde{P}_V(\xi_n,1)}{\tilde{P}(\xi_n,1)}, \tag{1}$$

where $(\xi_n)_{n\in\mathbb{N}}$ is a suitably chosen sequence. If $(\xi_n)_{n\in\mathbb{N}}$ is unbounded we may pass to a subsequence if necessary and consider the corresponding localization at infinity Q. But then equality (1) implies $Q_{|V} = 0$ so that

$$0 = \inf_{t\geq 1} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}. \tag{2}$$

By [7, Lemma 2] we have $\sigma_P(V) = \inf_{t\geq 1} \inf_{Q'\in L(P)} \frac{\tilde{Q'}_V(0,t)}{\tilde{Q}'(0,t)}$ so that (2) implies $\sigma_P(V) = 0$.

If on the other hand $(\xi_n)_{n\in\mathbb{N}}$ is bounded we can assume without loss of generality that $\lim_{n\to\infty} \xi_n = \xi$. Using the continuity of $(\eta, 1) \mapsto \tilde{P}_V(\eta, 1)$ equality (1) then gives

$$0 = \lim_{n\to\infty} \frac{\tilde{P}_V(\xi_n,1)}{\tilde{P}(\xi_n,1)} = \frac{\tilde{P}_V(\xi,1)}{\tilde{P}(\xi,1)}.$$

But this implies $0 = \sup_{|\theta|\leq 1} |P(\xi + \theta x)|$ for every $x \in V, |x| = 1$, i.e. for fixed $x \in V$ the polynomial $P(\xi + tx)$ in $t \in \mathbb{R}$ vanishes for every $t \in \mathbb{R}$. But then again

$$0 = \frac{|P(\xi + tx)|}{\tilde{P}(\xi,t)} = \frac{|P(\frac{\xi}{t} + x)|}{\tilde{P}(\frac{\xi}{t},1)}$$

for all $t \neq 0$ and $x \in V$ so that $0 = P(\frac{\xi}{t} + x)$. Letting t tend to infinity gives $P(x) = 0$ for all $x \in V$.

iv) While sufficiency follows from ii) necessity is [5, Remark following Theorem 6.3].

v) is [5, Theorem 6.8] or [3, Theorem 1].

vi) is part of [5, Theorem 6.9]. □

Remark 6. A result similar to Lemma 5 iii) for arbitrary polynomials is not true in general. This will be used in the forthcoming paper [9] to give an example of a surjective $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ such that $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is not surjective, thus solving the problem of Bonet and Domański in the negative.

In order to get a sufficient condition for the P^+-convexity for singular supports of $X\times\mathbb{R}$ we aim at a result similar to Proposition 3. Before we are able to formulate and proof this, some preparations have to be made.

The following proposition (cf. [7, Proposition 8]) contains some elementary geometric results which will be usefull in the sequel.

Proposition 7. *Let $\Gamma^\circ \neq \{0\}$ be a closed proper convex cone in $\mathbb{R}^d$ and $N \in S^{d-1}$. For $c \in \mathbb{R}$ let $H_c := \{x; \langle x, N\rangle = c\}$. Then the following are equivalent.*

- i) *$H_0 \cap \Gamma^\circ = \{0\}$.*
- ii) *$N \in \Gamma$ or $-N \in \Gamma$.*
- iii) *If $x \in \mathbb{R}^d$ and $H_c \cap (x + \Gamma^\circ) \neq \emptyset$ then $H_c \cap (x + \Gamma^\circ)$ is bounded.*
- iv) *If $x \in H_c$ then $H_c \cap (x + \Gamma^\circ) = \{x\}$.*

Proposition 8. *Let $\Gamma \neq \mathbb{R}^d$ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and $N \in S^{d-1}$ such that $\pi := \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$ is a supporting hyperplane of $x_0+\overline{\Gamma}$ intersecting $x_0 + \overline{\Gamma}$ only in x_0 and $x_0 + \Gamma \subset \{x \in \mathbb{R}^d; \langle x, N\rangle > \alpha\}$. For $\beta > \alpha$ set $\tilde{X_1} := \{x \in x_0 + \Gamma; \langle x, N\rangle > \beta\}, X_1 := \tilde{X_1} \times \mathbb{R}$, and $X_2 := (x_0 + \Gamma) \times \mathbb{R}$.*

If $H = \{x \in \mathbb{R}^{d+1}; \langle x, M\rangle = c\}$ is a hyperplane with $X_2 \cap H \neq \emptyset$ as well as $X_1 \cap H = \emptyset$ then the hyperplane $H_{x_0} := \{x \in \mathbb{R}^{d+1}; \langle x, M\rangle = \langle x_0, M'\rangle\}$ is a supporting hyperplane of $\overline{X_2}$ with $H_{x_0} \cap \overline{X_2} = \{x_0\} \times \mathbb{R}$ and $M_{d+1} = 0$. Moreover, $H'_{x_0} = \{x \in \mathbb{R}^d; \langle x, M'\rangle = \langle x_0, M'\rangle\}$ is a supporting hyperplane of $x_0 + \overline{\Gamma}$ such that $H'_{x_0} \cap (x_0 + \overline{\Gamma}) = \{x_0\}$.

Proof. Without loss of generality, let $x_0 = 0$. In this case, $\alpha = 0$ and H_0 contains 0. Suppose H_0 is not a supporting hyperplane of $\overline{X_2}$. Because of $0 \in H_0\cap\overline{X_2}$ this means that there are $v, w \in \overline{X_2} = \overline{\Gamma}\times\mathbb{R}$ such that $\langle v, M\rangle < 0 < \langle w, M\rangle$, hence $\langle x, M\rangle < 0 < \langle y, M\rangle$ for some $x, y \in \Gamma \times \mathbb{R}$.

Set $P := (N, 0) \in \mathbb{R}^{d+1}$. Then $|P| = 1$ and because of $\Gamma \subset \{v \in \mathbb{R}^d; \langle v, N\rangle > 0\}$ we have $X_2 \subset \{v \in \mathbb{R}^{d+1}; \langle v, P\rangle > 0\}$. Therefore, $\lambda_1 := \langle x, P\rangle > 0$ as well as $\lambda_2 := \langle y, P\rangle > 0$. Since X_2 is a cone we have $x_1 := \frac{\beta+1}{\lambda_1}x, y_1 := \frac{\beta+1}{\lambda_2}y \in X_2$ and from $X_1 = \{v \in X_2; \langle v, P\rangle > \beta\}$ we get $x_1, y_1 \in X_1$.

From $\langle x_1, M\rangle < 0 < \langle y_1, M\rangle$ we get a $t > 1$ such that

$$\langle tx_1, M\rangle < c < \langle ty_1, M\rangle.$$

Hence there is $\lambda \in (0, 1)$ with

$$\langle \lambda tx_1 + (1 - \lambda)ty_1, M\rangle = c,$$

i.e. $\lambda tx_1 + (1-\lambda)ty_1 \in H$. Obviously, X_1 is convex and for every $x \in X_1$ and $t > 1$ we have $tx \in X_1$. Therefore we have $\lambda tx_1 + (1 - \lambda)ty_1 \in H \cap X_1$ which contradicts our hypothesis.

So, H_0 is a supporting hyperplane of $\overline{X_2} = \overline{\Gamma} \times \mathbb{R}$. This immediately implies that $M_{d+1} = 0$ and that H'_0 is a supporting hyperplane of $\overline{\Gamma}$. Moreover, $M_{d+1} = 0$ implies that $H' = \{x \in \mathbb{R}^d; \langle x, M'\rangle = c\}$ intersects Γ but not X'_1. Because Γ is a proper cone and $\Gamma\backslash X'_1 = \{x \in \Gamma; \langle x, N\rangle \leq \beta\}$ this implies that $H' \cap \overline{\Gamma}$ is bounded. Since H'_0 is a supporting hyperplane of $\overline{\Gamma}$ this yields $H'_0 \cap \overline{\Gamma} = \{0\}$ by Proposition 7 *b*), hence $H_0 \cap \overline{X_2} = (H'_0 \times \mathbb{R}) \cap (\overline{\Gamma} \times \mathbb{R}) = \{0\} \times \mathbb{R}$. □

Proposition 9. *Let $\Gamma \neq \mathbb{R}^d$ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and let X_1 and X_2 be as in Proposition 8. Moreover, let P be a non-constant polynomial. Assume that no hyperplane H in $\mathbb{R}^d$ with $\sigma^0_P(H^\perp) = 0$ intersects $x_0 + \overline{\Gamma}$ only in x_0.*

Then for every hyperplane H in $\mathbb{R}^{d+1}$ with $H \cap X_2 \neq \emptyset$ and $\sigma_{P^+}(H^\perp) = 0$ it follows that $H \cap X_1 \neq \emptyset$.

PROOF. Let $H = \{x \in \mathbb{R}^{d+1}; \langle x, M\rangle = \beta\}$ be a hyperplane with $H \cap X_2 \neq \emptyset$ but $H \cap X_1 = \emptyset$. We have to show that $\sigma_{P^+}(M) \neq 0$.

From Proposition 8 it follows that $M = (M', 0)$ and $H'_{x_0} = \{x \in \mathbb{R}^d; \langle x, M'\rangle = \langle x_0, M'\rangle\}$ is a supporting hyperplane of $x_0 + \overline{\Gamma}$ with $H'_{x_0} \cap (x_0 + \overline{\Gamma}) = \{x_0\}$. In particular, the hypothesis gives $\sigma_P^0(M') \neq 0$. With Lemma 4 we get

$$0 \neq \sigma_P^0(M') = \sigma_{P^+}(span\{M'\} \times \{0\}) = \sigma_{P^+}(M),$$

proving the proposition. □

Now, we can prove an analogue result to Proposition 3.

Proposition 10. *Let $\Gamma \neq \mathbb{R}^d$ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and $P \in \mathbb{C}[X_1, \ldots, X_d]$ a non-constant polynomial. Assume that no hyperplane H with $\sigma_P^0(H^\perp) = 0$ intersects $x_0 + \overline{\Gamma}$ only in x_0.*

Then, every $u \in \mathscr{D}'((x_0 + \Gamma) \times \mathbb{R})$ with $P^+(D)u \in C^\infty((x_0 + \Gamma) \times \mathbb{R})$ for which there is a bounded subsets B of $x_0 + \Gamma$ such that u is C^∞ outside $B \times \mathbb{R}$ already satisfies $u \in C^\infty((x_0 + \Gamma) \times \mathbb{R})$.

PROOF. Without restriction, assume $x_0 = 0$. Let $u \in \mathscr{D}'(\Gamma \times \mathbb{R})$ with $P^+(D)u \in C^\infty(\Gamma \times \mathbb{R})$ and let $B \subset \Gamma$ be bounded such that $u_{|\Gamma\backslash B\times\mathbb{R}} \in C^\infty(\Gamma\backslash B\times\mathbb{R})$. Because Γ is a proper cone in $\mathbb{R}^d$ there is a hyperplane $H_1 = \{x \in \mathbb{R}^d; \langle x, N\rangle = 0\}$ intersecting $\overline{\Gamma}$ only in 0. Let $\tilde{X}_1$ be the intersection of Γ with a halfspace whose boundary is parallel to H_1 such that $\tilde{X}_1$ is unbounded and $B \subset \Gamma\backslash\tilde{X}_1$.

Let $X_1 := \tilde{X}_1 \times \mathbb{R}$, and $X_2 := \Gamma \times \mathbb{R}$. Then $X_1 \subset X_2$ are open convex subsets of $\mathbb{R}^{d+1}$ and it follows from Proposition 9 that for every hyperplane H in $\mathbb{R}^{d+1}$ with $\sigma_{P^+}(H^\perp) = 0$ and $H \cap X_2 \neq \emptyset$ already $H \cap X_1 \neq \emptyset$. Since $u \in \mathscr{D}'(X_2), P^+(D)u \in C^\infty(X_2)$ and $u_{|X_1} \in C^\infty(X_1)$ it follows from Proposition 2 that $u \in C^\infty(X_2)$. □

We are now able to prove the main result of this section. Parts i) and ii) of the next theorem are taken from [7, Theorem 9]

Theorem 11. *Let X be an open, connected subset of $\mathbb{R}^d$ and $P \in \mathbb{C}[X_1, \ldots, X_d]$ a non-constant polynomial with principal part P_m.*

i) *X is P-convex for supports if for every $x \in \partial X$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $P_m(y) \neq 0$ for all $y \in \Gamma$.*
ii) *X is P-convex for singular supports if for every $x \in \partial X$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_P(y) \neq 0$ for all $y \in \Gamma$.*
iii) *$X \times \mathbb{R}$ is P^+-convex for singular supports if for every $x \in \partial X$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_P^0(y) \neq 0$ for all $y \in \Gamma$.*

PROOF. For the proofs of i) and ii) see [7, Theorem 9]. In order to prove iii), let $u \in \mathscr{E}'(X \times \mathbb{R})$. Recall that by extending any compactly supported distribution by zero to all of $\mathbb{R}^{d+1}$ we have $\mathscr{E}'(X \times \mathbb{R}) \subset \mathscr{D}'(\mathbb{R}^{d+1})$ and thus $\mathscr{E}'(X \times \mathbb{R}) \subset \mathscr{D}'(\mathbb{R}^{d+1}) \subset \mathscr{D}'(Y)$ for every open subset $Y \subseteq \mathbb{R}^{d+1}$.

We set $K := \operatorname{sing\,supp} P^+(-D)u$ and $\delta := dist(K, X^c \times \mathbb{R})$. By [6, Theorem 10.7.3, vol. II], we have to show that $dist(\operatorname{sing\,supp} u, X^c \times \mathbb{R}) \geq \delta$. Let $x_0 \in \partial(X \times \mathbb{R}) = \partial X \times \mathbb{R}$ and let Γ be as in the hypothesis for $x'_0 \in \partial X$. Then $(x_0 + (\Gamma^\circ \times \mathbb{R})) \cap (X \times \mathbb{R}) = \emptyset$, thus $(x_0 + y + (\Gamma^\circ \times \mathbb{R})) \cap K = \emptyset$ for all $y \in \mathbb{R}^{d+1}$ with $|y| < \delta$. Therefore, for fixed y with $|y| < \delta$, there is an open proper convex cone $\tilde{\Gamma}$ in $\mathbb{R}^d$ with $\tilde{\Gamma} \supset \Gamma^\circ\backslash\{0\}$ such that $(x_0 + y + (\tilde{\Gamma} \times \mathbb{R})) \cap K = \emptyset$. Hence,

$u\in\mathscr{E}'(X\times\mathbb{R})\subset\mathscr{D}'(x_0+y+(\tilde\Gamma\times\mathbb{R}))$ satisfies $P^+(-D)u\in C^\infty(x_0+y+(\tilde\Gamma\times\mathbb{R}))$. We show that $u\in C^\infty(x_0+y+(\tilde\Gamma\times\mathbb{R}))$ by applying Proposition 10.

Let $H=\{v\in\mathbb{R}^d;\langle v,N\rangle=\alpha\}$ be a hyperplane with $\sigma_P^0(N)=0$. As $\overline{\tilde\Gamma}$ is a closed proper convex cone with non-empty interior, it is the dual cone of some open proper convex cone Γ_1. It follows from $\Gamma_1^\circ=\overline{\tilde\Gamma}\supset\Gamma^\circ$ that $\Gamma_1\subset\Gamma$. Because $\sigma_P^0(N)=0$ it follows from the hypothesis on Γ that $\{N,-N\}\cap\Gamma=\emptyset$, hence $\{N,-N\}\cap\Gamma_1=\emptyset$, so that by Proposition 7 H does not intersect $x_0'+y'+\overline{\tilde\Gamma}$ only in $x_0'+y'$.

Since $u\in\mathscr{E}'(X\times\mathbb{R})$ we have that $\operatorname{sing\,supp} u$ is compact. Moreover $P^+(-D)u\in C^\infty(x_0+y+(\tilde\Gamma\times\mathbb{R}))$, so that $u\in C^\infty(x_0+y+(\tilde\Gamma\times\mathbb{R}))$ by Proposition 10. Since $x_0\in\partial X\times\mathbb{R}$ and y with $|y|<\delta$ were chosen arbitrarily, it follows that $dist(\operatorname{sing\,supp} u, X^c\times\mathbb{R})\geq\delta$, which proves iii). □

3. Some partial results in arbitrary dimensions

In this section we will show that for some special cases of X the sufficient conditions for P-convexity in Theorem 11 are also necessary. As a consequence, we will see that surjectivity of $P(D)$ on $\mathscr{D}'(X)$ implies surjectivity of $P^+(D)$ on $\mathscr{D}'(X\times\mathbb{R})$ for P being homogeneous, semi-elliptic, or of principal type.

Recall that a real valued function f defined on a subset M of $\mathbb{R}^d$ is said to *satisfy the minimum principle in the closed subset F of $\mathbb{R}^d$* if for every compact subset $K\subset F\cap M$ it holds that $\inf_{x\in K}f(x)=\inf_{x\in\partial_F K}f(x)$, where $\partial_F K$ denotes the boundary of K relative F.

For a subset M of $\mathbb{R}^d$ let $d_M:M\to\mathbb{R},x\mapsto dist_{\mathbb{R}^d\backslash M}(x)$ be the distance to its complement.

Proposition 12. *Let $\Gamma^\circ\neq\{0\}$ be a closed proper convex cone in $\mathbb{R}^d$ and $N\in S^{d-1}$. Assume that $d_{\mathbb{R}^d\backslash\Gamma^\circ}$ satisfies the minimum principle in every hyperplane $H_c=\{x;\langle x,N\rangle=c\},c\in\mathbb{R}$. Then $\{N,-N\}\cap\Gamma=\emptyset$.*

Proof. If $\{N,-N\}\cap\Gamma\neq\emptyset$ it follows from Proposition 7 that $H_0\cap\Gamma^\circ=\{0\}$.

Let $c\neq0$ be arbitrary. We first show that $H_c\cap\Gamma^\circ=\emptyset$ if and only if $H_{-c}\cap\Gamma^\circ\neq\emptyset$. Indeed, if $H_c\cap\Gamma^\circ=\emptyset$ the convexity of Γ° implies that either $\Gamma^\circ\subset\{x;\langle x,N\rangle<c\}$ or $\Gamma^\circ\subset\{x;\langle x,N\rangle>c\}$. Without restriction we only consider the first case. Since $0\in\Gamma^\circ$ we have $0<c$. Moreover, because Γ° is a cone, it follows for every $x\in\Gamma^\circ\backslash\{0\}$ and $t>0$ that $t\langle x,N\rangle<c$. Obviously, this implies $\langle x,N\rangle<0$ for every $x\in\Gamma^\circ\backslash\{0\}$. Therefore, $-c/\langle x,N\rangle>0$ so that $-c/\langle x,N\rangle\, x\in\Gamma^\circ$ for every $x\in\Gamma^\circ\backslash\{0\}$. In particular, there is $x\in\Gamma^\circ\cap H_{-c}$.

On the other hand, let $H_{-c}\cap\Gamma^\circ\neq\emptyset$. If $H_c\cap\Gamma^\circ\neq\emptyset$ it follows from $c\neq0$ that there are $x,y\in\Gamma^\circ\backslash\{0\}$ such that for some $\lambda\in(0,1)$ we have $\lambda x+(1-\lambda)y\in H_0$. The convexity of Γ° together with $H_0\cap\Gamma^\circ=\{0\}$ implies $\lambda x+(1-\lambda)y=0$. Therefore, $-x\in\Gamma^\circ\backslash\{0\}$ which contradicts the fact that Γ° is proper.

So, for arbitrary $c\neq0$ we can therefore assume that $H_c\cap\Gamma^\circ=\emptyset$ as well as $H_{-c}\cap\Gamma^\circ\neq\emptyset$. Because of $H_0\cap\Gamma^\circ=\{0\}$ it follows from Proposition 7 that the non-empty set $H_{-c}\cap\Gamma^\circ$ is bounded. So there is $R>|c|$ such that $\emptyset\neq H_{-c}\cap\Gamma^\circ\subset B_R(0)$. In particular, $K:=H_c\cap B_R(0)$ is a non-empty, compact subset of $H_c\cap\mathbb{R}^d\backslash\Gamma^\circ$ with

$$d_{\mathbb{R}^d\backslash\Gamma^\circ}(K)=dist_{\Gamma^\circ}(K)\leq dist_{\{0\}}(K)=|c|.$$

Obviously, $x-cN\in H_0$ for all $x\in H_c$, so that $M:=\{x-cN;\,x\in H_c\cap\partial B_R(0)\}\subset H_0$ is compact, and because $R>|c|$, M does not contain 0. Since $H_0\backslash\{0\}\cap\Gamma^\circ=\emptyset$ we obtain

$$\delta:=\inf_{v\in M}dist_{\Gamma^\circ}(v)>0.$$

We have

$$\begin{aligned}\forall\, x \in H_c, y \in \Gamma^\circ :\, |x-y|^2 &= |(x-cN)-(y-cN)|^2 \\ &= c^2 + |(x-cN)-y|^2 - 2c\langle N, y\rangle.\end{aligned}$$

Again, by the convexity of Γ° and $H_c \cap \Gamma^\circ = \emptyset$ we have either $\Gamma^\circ \subset \{x; \langle x, N\rangle < c\}$ or $\Gamma^\circ \subset \{x; \langle x, N\rangle > c\}$. From $0 \in \Gamma^\circ$ it therefore follows that $c\langle N, y\rangle \leq 0$ for all $y \in \Gamma^\circ$ so that we get

$$\forall\, x \in H_c, y \in \Gamma^\circ :\, |x-y|^2 \geq c^2 + |(x-cN)-y|^2.$$

Therefore,

$$\begin{aligned}d_{\mathbb{R}^d\backslash\Gamma^\circ}(\partial_{H_c}K) &= dist_{\Gamma^\circ}(\partial_{H_c}K) = \inf_{x\in H_c\cap\partial B_R(0)} dist_{\Gamma^\circ}|x| \\ &\geq (c^2 + \inf_{x\in H_c\cap\partial B_R(0)} dist_{\Gamma^\circ}|x-cN|^2)^{1/2} \\ &= (c^2 + \inf_{v\in M} dist_{\Gamma^\circ}(v)^2)^{1/2} \\ &= (c^2+\delta^2)^{1/2} > |c| \geq dist_{\Gamma^\circ}(K) \\ &= d_{\mathbb{R}^d\backslash\Gamma^\circ}(K),\end{aligned}$$

so that $d_{R^d\backslash\Gamma^\circ}$ does not satisfy the minimum principle in H_c contradicting the hypothesis. □

Combining the previous proposition with Theorem 11 gives the next result.

Theorem 13. *Let $\Gamma \neq \mathbb{R}^d$ be an open convex cone in $\mathbb{R}^d$ and $X := \mathbb{R}^d\backslash\Gamma^\circ$. Let P be a non-constant polynomial with principal part P_m.*

i) *X is P-convex for supports if and only if $P_m(y) \neq 0$ for all $y \in \Gamma$.*
ii) *X is P-convex for singular supports if and only if $\sigma_P(y) \neq 0$ for all $y \in \Gamma$.*
iii) *$X \times \mathbb{R}$ is P^+-convex for singular supports if and only if $\sigma_P^0(y) \neq 0$ for all $y \in \Gamma$.*

Proof. For the proof of i) recall that a necessary condition for P-convexity for supports for an arbitrary open set Y in $\mathbb{R}^d$ is that d_Y satisfies the minimum principle in every characteristic hyperplane, i.e. in every hyperplane $H = \{x; \langle x, N\rangle = c\}$ with $P_m(N) = 0$ (cf. [6, Theorem 10.8.1, vol. II]). So, if X is P-convex for supports it follows from Proposition 12 that $P_m(y) \neq 0$ for every $y \in \Gamma$.

On the other hand, for every $x \in \partial X = \partial\Gamma^\circ$ and $y \in \Gamma^\circ$ we have $x + y = 2(1/2\,x + 1/2\,y) \in \Gamma^\circ$ since Γ° is a closed convex cone, hence $(x + \Gamma^\circ) \cap X = \emptyset$ for every $x \in \partial X$. Therefore, if $P_m(y) \neq 0$ for every $y \in \Gamma$ it follows from Theorem 11 i) that X is P-convex for supports, which proves i).

For the proof of ii) recall that a necessary condition for P-convexity for singular supports for an arbitrary open set Y in $\mathbb{R}^d$ is that d_Y satisfies the minimum principle in every affine subspace V with $\sigma_P(V^\perp) = 0$ (cf. [6, Corollary 11.3.2, vol. II]). In particular, if X is P-convex for singular supports, it follows that d_X satisfies the minimum principle in every hyperplane $H = \{x; \langle x, N\rangle = c\}$ with $\sigma_P(H^\perp) = 0$. Thus, by Proposition 12 we get $\sigma_P(y) \neq 0$ for every $y \in \Gamma$.

Sufficiency of the condition stated in ii) is proved analogously to the proof of i).

Finally, to prove iii) observe that by [6, Corollary 11.3.2, vol. II] P^+-convexity for singular supports of $X \times \mathbb{R}$ in particular implies that $d_{X\times\mathbb{R}}$ satisfies the minimum principle in every affine subspace $H = \{x \in \mathbb{R}^d; \langle x, N\rangle = c\} \times \{0\}$ with $0 = \sigma_{P^+}(span\{N\} \times \mathbb{R}) = \sigma_P^0(N)$, where we used Lemma 4. Hence d_X satisfies the minimum principle in every hyperplane $H = \{x \in \mathbb{R}^d; \langle x, N\rangle = c\}$ with $\sigma_P^0(N) = 0$, so that $\sigma_P^0(y) \neq 0$ for every $y \in \Gamma$ due to Proposition 12. This proves necessity in iii). Again, sufficiency is proved as in i). □

As an immediate consequence we obtain the next result.

Corollary 14. *Let $X_0 \subset \mathbb{R}^d$ be open and convex and let $\Gamma_1, \Gamma_2, \dots$ be a sequence of open convex cones, all different from $\mathbb{R}^d$. Moreover, let $x_1, x_2 \dots$ be a sequence in X_0. Denote by X the interior of $X_0 \cap \bigcap_{n=1}^\infty (x_n + \Gamma_n^\circ)^c$ and assume that for every $n \in \mathbb{N}$ we have $\varepsilon_n > 0$ such that*

$$B_{\varepsilon_n}(x_n) \cap (x_n + \Gamma_n^\circ)^c \subset X. \tag{3}$$

Then the following holds for a non-constant polynomial P.

i) *X is P-convex for supports if and only if $P_m(y) \neq 0$ for every $y \in \cup_{n=1}^\infty \Gamma_n$, where P_m is the principal part of P.*
ii) *X is P-convex for singular supports if and only if $\sigma_P(y) \neq 0$ for every $y \in \cup_{n=1}^\infty \Gamma_n$.*
iii) *$X \times \mathbb{R}$ is P^+-convex for singular supports if and only if $\sigma_P^0(y) \neq 0$ for every $y \in \cup_{n=1}^\infty \Gamma_n$.*

Proof. Since for non-constant polynomials Q convex sets are Q-convex for (singular) supports and the interior of arbitrary intersections of Q-convex sets for (singular) supports are again Q-convex for (singular) supports (cf. [6, Theorems 10.6.4 and 10.7.4, vol. II]) the sufficiency of the conditions follows from Theorem 13.

We only prove necessity in iii) since the corresponding proofs for parts i) and ii) are the same modulo obvious changes.

Let $X \times \mathbb{R}$ be P^+-convex for singular supports. Assume that there is $j \in \mathbb{N}$ and $y \in \Gamma_j$ such that $\sigma_P^0(y) = 0$. Without restriction let $|y| = 1$. Then $H := \{x; \langle x, y\rangle = \langle x_j, y\rangle\}$ is a hyperplane through x_j with $\sigma_P^0(H^\perp) = 0$ and $H \cap (x_j + \Gamma_j^\circ) = \{x_j\}$ by Proposition 7. Without loss of generality we can assume that $x_j + \Gamma_j^\circ \subset \{x; \langle x, y\rangle \geq \langle x_j, y\rangle\}$.

For $c > 0$ set $H_c := \{x; \langle x, y\rangle = \langle x_j, y\rangle - c\}$ and $K_c := H_c \cap B_{2c}(x_j)$. Then $K_c \neq \emptyset$ is compact and due to condition (1) we have

$$\forall\, 0 < c < \varepsilon_j/4 : K_c \subset X \text{ as well as } d_X(K_c) = d_{\mathbb{R}^d \backslash (x_j + \Gamma_j^\circ)}(K_c).$$

As in the proof of Proposition 12 it follows that

$$d_{\mathbb{R}^d \backslash (x_j + \Gamma_j^\circ)}(K_c) = c < d_{\mathbb{R}^d \backslash (x_j + \Gamma_j^\circ)}(\partial_{H_c} K_c).$$

Hence by Lemma 4 for $0 < c < \varepsilon/4$ the affine subspace $H_c \times \{0\}$ of $\mathbb{R}^{d+1}$ satisfies $\sigma_{P^+}((H_c \times \{0\})^\perp) = \sigma_P^0(H_c^\perp) = \sigma_P^0(y) = 0$ but for the compact subset $K_c \times \{0\}$ of $(H_c \times \{0\}) \cap (X \times \mathbb{R})$ we have

$$\begin{aligned} d_{X\times\mathbb{R}}(K_c \times \{0\}) &= d_X(K_c) = d_{\mathbb{R}^d \backslash (x_j + \Gamma_j^\circ)}(K_c) = c \\ &< d_{\mathbb{R}^d \backslash (x_j + \Gamma_j^\circ)}(\partial_{H_c} K_c) \\ &= d_{X\times\mathbb{R}}(\partial_{H_c \times \{0\}}(K_c \times \{0\})). \end{aligned}$$

So the minimum principle for $d_{X\times\mathbb{R}}$ is not valid in $H_c \times \{0\}$ which contradicts the P^+-convexity for singular supports of $X \times \mathbb{R}$ by [6, Corollary 11.3.2, vol. II]. □

Remark 15. Observe that for sufficiency of the above conditions instead of X_0 being convex, in part i) one only needs X_0 to be P-convex for supports while in parts ii) and iii) it suffices to let X_0 be P-convex for singular supports, resp. $X_0 \times \mathbb{R}$ be P^+-convex for singular supports. For necessity of the above conditions, X_0 can be arbitrary.

Recall that P is of *real principal type* if it is of principal type and the coefficients in its principal part are real.

Corollary 16. *Let $X_0 \subset \mathbb{R}^d$ be open and convex let $\Gamma_1, \Gamma_2, \dots$ be a sequence of open convex cones, all different from $\mathbb{R}^d$. Moreover, let $x_1, x_2 \dots$ be a sequence in X_0. Denote by X the interior of $X_0 \cap \bigcap_{n=1}^{\infty}(x_n + \Gamma_n^\circ)^c$ and assume that for every $n \in \mathbb{N}$ we have $\varepsilon_n > 0$ such that*

$$B_{\varepsilon_n}(x_n) \cap (x_n + \Gamma_n^\circ)^c \subset X. \tag{4}$$

If the non-constant polynomial P is homogeneous, semi-elliptic, or of principal type the following are equivalent.

i) *$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective.*
ii) *$P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective.*

If P is homogeneous, semi-elliptic, or of real principal type then the above are also equivalent to

iii) *$X \times \mathbb{R}$ is P^+-convex for singular supports.*

Proof. Assume that i) holds. Then X is P-convex for supports as well as for singular supports. By Corollary 14 it follows that $P_m(y) \neq 0$ and $\sigma_P(y) \neq 0$ for all $y \in \cup_{i=1}^{\infty}\Gamma_i$, hence $X \times \mathbb{R}$ is P^+-convex for singular supports by Corollary 14 and Lemma 5 iii), v), or vi), respectively. Since X is P-convex for supports it follows that $X \times \mathbb{R}$ is P^+-convex for supports by [3, Proposition 1], so that ii) follows.

Now assume that ii) holds. For $v \in \mathscr{D}'(X)$ there is $w \in \mathscr{D}'(X \times \mathbb{R})$ such that $P^+(D)w = v \otimes \delta_0$. Choose $\psi \in \mathscr{D}(\mathbb{R})$ with $\psi(0) = 1$. Then $u(\varphi) = w(\varphi \otimes \psi)$ for $\varphi \in \mathscr{D}(X)$ defines a distribution with $P(D)u = v$ proving i). Note that for this implication neither the special form of X nor the special properties of P are needed.

Now, we assume that P is homogeneous, semi-elliptic, or of real principal type. Clearly, ii) implies iii). If iii) holds it follows from Corollary 14 iii) that $\sigma_P^0(y) \neq 0$ for all $y \in \cup_{n\in\mathbb{N}}\Gamma_n$. If P is homogeneous it follows from Lemma 5 iii) that $P_m(y) \neq 0$ and $\sigma_P(y) \neq 0$ for all $y \in \cup_{n\in\mathbb{N}}\Gamma_n$ so that i) follows from Corollary 14 i) and ii). If P is semi-elliptic we have $P_m(y) \neq 0$ for all $y \in \cup_{n\in\mathbb{N}}\Gamma_n$ by Lemma 5 v). Hence X is P-convex for supports by Corollary 14 i), so that iii) implies i) also for semi-elliptic P. Finally, if P is of real principal type, it follows from Lemma 5 vi) that $\sigma_P(y) \neq 0$ for all $y \in \cup_{n\in\mathbb{N}}\Gamma_n$. Therefore, X is P-convex for singular supports by Corollary 14 ii). It is shown in the proof of [6, Corollary 10.8.10] that P-convexity for singular supports implies P-convexity for supports if P is of real principal type, so that i) follows from iii) in this case, too. □

4. The two-dimensional case

Recall that for elliptic P every open subset $X \subset \mathbb{R}^d$ is P-convex for supports. The next theorem is [6, Theorem 10.8.3, vol. II].

Theorem 17. *If P is non-elliptic then the following conditions on an open connected set $X \subset \mathbb{R}^2$ are equivalent.*

i) *X is P-convex for supports.*
ii) *The intersection of X with every characteristic hyperplane is convex.*
iii) *For every $x_0 \in \partial X$ there is a closed proper convex cone $\Gamma^\circ \neq \{0\}$ with $(x_0 + \Gamma^\circ) \cap X = \emptyset$ and no characteristic hyperplane intersects $x_0 + \Gamma^\circ$ only in x_0.*

In view of Proposition 7 the above condition iii) clearly is equivalent to the following condition.

iii') *For every $x_0 \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ with $(x_0 + \Gamma^\circ) \cap X = \emptyset$ and $P_m(y) \neq 0$ for all $y \in \Gamma$, where P_m denotes the principal part of P.*

An analogous theorem to Theorem 17 for P-convexity for singular supports is the following. Recall that by Remark 1 c) a polynomial P is hypoelliptic if and only if $\sigma_P(H^\perp) \neq 0$ for every hyperplane H.

Theorem 18. *If P is non-hypoelliptic then the following conditions on an open connected set $X \subset \mathbb{R}^2$ are equivalent.*

i) *X is P-convex for singular supports.*
ii) *The intersection of X with every hyperplane H satisfying $\sigma_P(H^\perp) = 0$ is convex.*
iii) *For every $x_0 \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ with $(x_0+\Gamma^\circ)\cap X = \emptyset$ and $\sigma_P(y) \neq 0$ for all $y \in \Gamma$.*

The proof of the above theorem is very similar to the proof of [6, Theorem 10.8.3, vol. II] and can be found in [7, Theorem 11].

Theorem 19. *If P is non-constant with principal part P_m then the following conditions on an open connected set $X \subset \mathbb{R}^2$ are equivalent.*

i) *$X \times \mathbb{R}$ is P^+-convex for singular supports.*
ii) *The intersection of X with every characteristic hyperplane is convex.*
iii) *For each $x_0 \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ with $(x_0+\Gamma^\circ)\cap X = \emptyset$ and $P_m(y) \neq 0$ for all $y \in \Gamma$.*

Proof. By Lemma 5 iv) we have $\sigma_P^0(y) = 0$ if and only in $y \in \{P_m = 0\}$. Hence, that iii) implies i) is just Theorem 11.

Observe that if $X \times \mathbb{R}$ is P^+-convex for singular supports it follows as in the proof of Theorem 13 that d_X satisfies the minimum principle in every hyperplane H in $\mathbb{R}^2$ with $\sigma_P^0(H^\perp) = 0$. By Lemma 5 iv) d_X therefore satisfies the minimum principle in every characteristic hyperplane. Now that i) implies ii) follows as in the proof of [6, Theorem 10.8.3, vol. II].

That ii) implies iii) follows immediately from Theorem 17 where one has to replace iii) by iii') □

A result of Vogt (cf. [14, Proposition 2.5]) says that the kernel of an elliptic differential operator on $\mathscr{D}'(X)$ always has the linear topological invariant (Ω). Since the kernel of an elliptic differential operator is a Fréchet-Schwartz space it has property (Ω) if and only if it has property $(P\Omega)$. Therefore, it follows from [1, Proposition 8.3] that for an elliptic polynomial P the augmented operator $P^+(D)$ is surjective on $\mathscr{D}'(X\times\mathbb{R})$. This interpretation of Vogt's result is the next theorem. A proof based on the techniques used here can be found in [3, Corollary 14].

Theorem 20. *Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be elliptic. Then for every $X \subset \mathbb{R}^d$ open $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective.*

Combining the last four theorems and [7, Theorem 1] we obtain the following result.

Theorem 21. *Let X be an open subset of $\mathbb{R}^2$ and P a non-constant polynomial. Then the following are equivalent.*

i) *$P(D) : C^\infty(X) \to C^\infty(X)$ is surjective.*
ii) *$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective.*
iii) *$P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ is surjective.*
iv) *$X \times \mathbb{R}$ is P^+-convex for singular supports.*
v) *The intersection of every characteristic hyperplane with any connected component of X is convex.*

Proof. Because for elliptic polynomials P every open set is P-convex for supports and singular supports and because of Theorems 19 and 20 we can assume without loss of generality that P is non-elliptic. Moreover, by passing to different components of X we can assume without restriction that X is connected. For non-elliptic P the equivalence of i), iv) and v) follows from Theorems 17 and 19 and that i) and ii) are equivalent follows from [7, Theorem 1].

If i) (and therefore also iv)) holds then X is P-convex for supports so that $X \times \mathbb{R}$ is P^+-convex for supports by [3, Proposition 1] and we obtain iii). Finally, iii) obviously implies iv) which proves the theorem. □

Remark 22. As stated in the introduction, the results of Bonet and Domański [1, Proposition 8.3] imply that for a surjective differential operator $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ the augmented operator $P^+(D)$ is surjective if and only if ker $P(D)$ has the linear topological invariant $(P\Omega)$. Combining this with Theorem 21 and Corollary 16 gives the following, respectively.

i) Let $X \subset \mathbb{R}^2$ be open and P a non-constant polynomial. If the intersection of every characteristic hyperplane with each connected component of X is convex then the kernel of
$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$$
has the property $(P\Omega)$.

ii) Let $X \subset \mathbb{R}^d$ be as in Corollary 16 and P be homogeneous, semi-elliptic, or of principal type. If
$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$$
is surjective then its kernel has the property $(P\Omega)$.

On the other hand, it is shown in [8] that i) and ii) of the above theorem are also equivalent to the surjectivity of $P(D)$ on the space of ultradistributions of Beurling type $\mathscr{D}'_{(\omega)}(X)$ in the sense of Braun, Meise, and Taylor [2] for any/some non-quasianalytic weight function ω.

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