HYPERCYCLICITY AND MIXING FOR COSINE OPERATOR FUNCTIONS GENERATED BY SECOND ORDER PARTIAL DIFFERENTIAL OPERATORS

T. KALMES

Abstract. For $\Omega \subset \mathbb{R}^d$ open, we characterize when cosine operator functions generated by second order partial differential operators on $L^p(\Omega, \mu)$ and $C_{0,\rho}(\Omega)$, respectively, are hypercyclic and prove that this happens if and only if they are weakly mixing. In the case of $d = 1$ we give an easy to check characterization of when this happens. Moreover, mixing of these cosine operator functions is also characterized.

Keywords: hypercyclicity; transitivity; mixing; cosine operator functions

MSC 2000: primary 47D09, secondary: 47A16, 47F05.

1. Introduction

A continuous linear operator T on a separable Banach space X is called *hypercyclic* if there is a hypercyclic vector $x \in X$ for T which means that $\{T^n x;\, n \in \mathbb{N}\}$ is dense in X. There are a number of articles dealing with hypercyclic operators, for a survey see e.g. [11], [12].

Analogously, a family $(T_\iota)_{\iota\in I}$ of continuous linear operators on X, is called *hypercyclic* if there exists an element $x \in X$ such that $\{T_\iota x;\, \iota \in I\}$ is dense in X. In this case x is again called *hypercyclic vector* for the family $(T_\iota)_{\iota\in I}$. Apart from single operators, there are various results on hypercyclic C_0-semigroups, see e.g. [8], [2], [5], [4], [6], [13], [14] [1], [7].

A notion closely related to hypercyclicity is that of *transitivity*. A family of continuous linear operators $(T_\iota)_{\iota\in I}$ on a Banach space X is called *transitive* if for each pair of non-empty, open subsets U, V of X there is $\iota \in I$ such that $T_\iota^{-1}(U) \cap V \neq \emptyset$. It was shown by Grosse-Erdmann that $(T_\iota)_{\iota\in I}$ is transitive if and only if $(T_\iota)_{\iota\in I}$ is hypercyclic and the set of hypercyclic vectors is dense [10, Satz 1.2.2 and its proof]. Moreover, Peris proved that a commuting family of continuous linear operators $(T_\iota)_{\iota\in I}$ for which each T_ι has dense range is hypercyclic if and only if the set of hypercyclic vectors is dense [11, Proposition 1]. In particular, an arbitrary commuting family of continuous linear operators $(T_\iota)_{\iota\in I}$ for which each T_ι has dense range is hypercyclic if and only if it is transitive.

A family of continuous linear operators $(T_\iota)_{\iota\in I}$ on a Banach space X is called *weakly mixing* if $(T_\iota \oplus T_\iota)_{\iota\in I}$ is transitive on $X \oplus X$. And finally, a family of continuous linear operators $(T_t)_{t\in\mathbb{R}}$ is called *mixing* if for each pair of non-empty, open subsets U, V of X there is $t_0 \in \mathbb{R}$ such that $T_t^{-1}(U) \cap V \neq \emptyset$ for every $t \geq t_0$.

A cosine operator function on a Banach space X is a strongly continuous mapping C from the real line into the space of continuous linear operators on X satisfying $C(0) = id$ and the d'Alembert functional equation $2C(t)C(s) = C(t+s) + C(t-s)$ for all $s, t \in \mathbb{R}$. If T is a C_0-group it is easily seen that $C(t) := \frac{1}{2}(T(t) + T(-t))$ defines a cosine operator function. The generator of a cosine operator function is defined as $Af := \lim_{t\to 0} \frac{2}{t^2}(C(t)f - f)$ for $f \in D(A)$,

Published in: Journal of Mathematical Analysis and Applications, Volume 365, Issue 1, Pages 363–375
DOI: 10.1016/j.jmaa.2009.10.063
https://doi.org/10.1016/j.jmaa.2009.10.063.

i.e. for those f for which the limit exists. If T is a C_0-group with generator $(A, D(A))$ then the cosine operator function defined by $C(t) = \frac{1}{2}(T(t) + T(-t))$ has generator $(A^2, D(A^2))$.

Transitive cosine operator functions on Banach spaces were first considered by Bonilla and Miana in [3]. Among other things they gave a sufficient condition for the translation cosine function on $L^p_\rho(\mathbb{R})$ and $C_{0,\rho}(\mathbb{R})$, respectively, to be transitive and characterized when it is mixing. Moreover, they showed that there is a topologically mixing cosine operator function on any separable infinite dimensional Banach space.

The paper is organized as follows. In section 2 we show that at least for cosine operator functions stemming from strongly continuous groups hypercyclicity and transitivity are equivalent. In section 3 we give sufficient conditions for hypercyclicity of cosine operator functions generated by second order partial differential operators on space of integrable functions and continuous functions, respectively. Moreover, we show that under some mild additional conditions these sufficient conditions are also necessary and that then hypercyclicity is equivalent to weak mixing. Furthermore, mixing of the same cosine operator functions is characterized as well. Since the given conditions might be difficult to check for concrete examples we concentrate on the one-dimensional case in section 4 and considerably simplify the conditions characterizing hypercyclicity and mixing. Several examples are given to illustrate the given results.

2. A general observation

In this short section we show that for cosine operator functions defined via a C_0-group hypercyclicity is indeed equivalent to transitivity. We begin with a general proposition.

Proposition 1. *Let T be a C_0-group on the Banach space X and define $C(t) := \frac{1}{2}(T(t) + T(-t))$, $t \in \mathbb{R}$. If the cosine operator function $C = (C(t))_{t\in\mathbb{R}}$ is hypercyclic then $\sigma_p(T(t)^*) = \emptyset$ for all $t > 0$, where $\sigma_p(T(t)^*)$ denotes the point spectrum of the transpose of $T(t)$.*

Proof: Assume there is $t_0 > 0$ such that $\sigma_p(T(t_0)^*) \neq \emptyset$. Let $(A, D(A))$ be the generator of T. Since $T(t_0)$ is one-to-one and onto it follows from the spectral mapping theorem for the residual spectrum (cf. [9, Theorems IV.3.7 and 3.8]), that there are $\lambda \in \sigma_p(A^*)$ and $x' \in X'\backslash\{0\}$ such that $e^{t\lambda} \in \sigma_p(T(t)^*)$ and $T(t)^*x' = e^{t\lambda}x'$ for all $t > 0$. From this we get $C(t)^*x' = \cosh(t\lambda)x'$ for $t > 0$.

Let x be a hypercyclic vector for C. Then, since $x' \neq 0$ we get

$$\mathbb{K} = \overline{\{x'(C(t)x);\, t \geq 0\}} = \overline{\{\cosh(t\lambda)x'(x);\, t \geq 0\}} = \overline{\{\cosh(t\lambda);\, t \geq 0\}}x'(x)$$

giving a contradiction. □

Corollary 2. *Let T be a C_0-group on the Banach space X and let $C(t) := \frac{1}{2}(T(t) + T(-t))$, $t \in \mathbb{R}$. If the cosine operator function $C = (C(t))_{t\in\mathbb{R}}$ is hypercyclic then the set of hypercyclic vectors for C is a dense G_δ-set in X. In particular, C is hypercyclic if and only if C is transitive.*

Proof: We have $C(t) = \frac{1}{2}(T(2t) + id)T(-t)$. Because $T(-t)$ is one-to-one and onto, $C(t)$ has dense range if $T(2t) + id$ has dense range, i.e. if $-1 \notin \sigma_p(T(2t)^*)$ which is true by the above proposition. Since $C(s)C(r) = C(r)C(s)$ for all $r, s \in \mathbb{R}$ it follows from [11, Proposition 1] that the set of hypercyclic vectors for C is dense in X. From [3, Theorem 1.1] we obtain that C is hypercyclic if and only if C is transitive. □

Remark 3. It seems to be still unknown whether for general cosine operator functions hypercyclicity and transitivity are equivalent properties.

3. Characterizations of hypercyclicity and mixing in arbitrary dimensions

In this section we characterize when cosine operator functions generated by second order differential operators are hypercyclic or mixing, respectively. Observe that by taking $t = 0$ in the d'Alembert equation we get $C(s) = C(-s)$ for all $s \in \mathbb{R}$ so that C is hypercyclic (mixing) if and only if $(C(s))_{s\geq 0}$ is hypercyclic (mixing).

We consider an open subset Ω of $\mathbb{R}^d$ and a locally Lipschitz continuous vector field F on Ω such that for every $x_0 \in \Omega$ the unique solution $\varphi(\cdot, x_0)$ of the initial value problem

$$\dot{x} = F(x), x(0) = x_0$$

is defined on $\mathbb{R}$. Moreover, let $h : \Omega \to \mathbb{R}$ be a continuous function.

We call a locally finite Borel measure μ on Ω *p-admissible for F and h*, if $T(t)f(x) := \exp(\int_0^t h(\varphi(r, x))\, dr)f(\varphi(t, x)), t \in \mathbb{R}$, defines a C_0-group on $L^p(\mu)$, where $p \in [1,\infty)$.

For $t \in \mathbb{R}$ we define the Borel measures $\nu_{p,t}(B) := \int_{\varphi(-t,B)} h_t^p\, d\mu$, where $h_t(x) := \exp(\int_0^t h(\varphi(r, x))\, dr)$ for $t \in \mathbb{R}$. Note that these are well defined since $\varphi(t, \cdot)$ is a homeomorphism of Ω with $\varphi(t, \cdot)^{-1}(B) = \varphi(-t, B)$ for each $t \in \mathbb{R}$ and $B \subset \Omega$ Borel measurable.

Moreover, a function $\rho : \Omega \to (0,\infty)$ is called *C_0-admissible* for F and h, if $T(t)$ defined as above gives a C_0-group on $C_{0,\rho}(\Omega)$, where

$$C_{0,\rho}(\Omega) := \{f \in C(\Omega); \forall \varepsilon > 0 : \{x \in \Omega; |f(x)|\rho(x) \geq \varepsilon\} \text{ is compact}\}$$

is equipped with the norm $\|f\| := \sup_{x\in\Omega} |f(x)|\rho(x)$. Since Ω is locally compact and μ is locally finite the subspace $C_c(\Omega)$ of compactly supported continuous functions is dense in $L^p(\mu)$. The same obviously holds for $C_{0,\rho}(\Omega)$.

As [13, Theorem 4.7, Proposition 4.12, Remark 3.10 and the remark following Theorem 4.11] one proofs the following theorem which we give only for completeness' sake. Observe that by our hypotheses we have $\varphi(t, \Omega) = \Omega$ for all $t \in \mathbb{R}$ and that $\{h_t(x)\rho(x) \geq \delta\} \cap \varphi(-t, K), \delta > 0$, is always compact if ρ is upper semicontinuous. Recall that $x \mapsto \varphi(t, x)$ is continuously differentiable if F is continuously differentiable. In case of existence we denote the Jacobian of $x \mapsto \varphi(t, x)$ by $D\varphi(t, x)$.

Theorem 4. *Let μ be a locally finite Borel measure on Ω and let F and h be as above.*

a) *The following are equivalent.*
   i) *μ is p-admissible for F and h.*
   ii) *$\nu_{p,t}$ has a μ-density $g_{p,t} \in L^\infty(\mu)$ and there are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|g_{p,t}\|_\infty \leq Me^{\omega|t|}$ for all $t \in \mathbb{R}$.*

b) *Assume that μ has a positive Lebesgue density ρ. If F is continuously differentiable the following are equivalent.*
   i) *μ is p-admissible for F and h.*
   ii) *There are $M \geq 1, \omega \in \mathbb{R}$ such that for $t \in \mathbb{R}$ and λ^d-almost all $x \in \Omega$*

$$h_t^p(x)\rho(x) \leq Me^{\omega|t|}\rho(\varphi(t, x))|\det D\varphi(t, x)|,$$

   where λ^d denotes d-dimensional Lebesgue measure.

c) *Let μ be p-admissible for F and h and assume that μ has a positive Lebesgue density ρ. If F is differentiable a μ-density of $\nu_{p,t}$, resp. $\nu_{p,-t}$, is given by*

$$\frac{\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|}{\rho\, h_{-t}^p},$$

   resp.

$$\frac{\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|}{\rho\, h_t^p}.$$

d) *Let $\rho : \Omega \to (0,\infty)$. Then i) implies ii).*
   i) *ρ is C_0-admissible for F and h.*

ii) *There are constants* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that for all* $t \in \mathbb{R}$ *and* $x \in \Omega$

$$h_t(x)\rho(x) \leq Me^{\omega|t|}\rho(\varphi(t,x)).$$

Moreover, if ρ *is upper semicontinuous the above are equivalent.*

e) *Let* F *and* h *be twice continuously differentiable,* μ *be* p*-admissible, and* ρ C_0*-admissible for* F *and* h*. Let* X *be either* $L^p(\mu)$ *or* $C_{0,\rho}(\Omega)$*. The generator of the cosine operator function on* X *defined via*

$$(C(t)f)(x) = \frac{1}{2}(h_t(x)f(\varphi(t,x)) + h_{-t}(x)f(\varphi(-t,x)))$$

is given by the closure of the operator

$$C_c^2(\Omega) \to X,$$

$$f \mapsto \sum_{j,k=1}^d F_jF_k\partial_j\partial_k f + \sum_{j=1}^d(2hF_j + \sum_{k=1}^d F_k\partial_k F_j)\partial_j f + (h^2 + \sum_{j=1}^d F_j\partial_j h)f.$$

In particular, if $F \equiv a \in \mathbb{R}^d$ and $h \equiv \alpha \in \mathbb{R}$ it follows that for a p-admissible measure μ, respectively a C_0-admissible ρ, the generator of the cosine operator function under consideration is the closure of the operator

$$C_c^2(\Omega) \to X, f \mapsto \langle a, \nabla^2 f\ a\rangle + 2\alpha\langle a, \nabla f\rangle + \alpha^2 f,$$

where ∇^2 denotes the Hessian of f.

Theorem 5. *Let* μ *be* p*-admissible for* F *and* h*. For the cosine operator function* $C(t) := \frac{1}{2}(T(t)+T(-t))$ *with* $T(t)f(x) = h_t(x)f(\varphi(t,x))$*, among the following, i) implies ii) and ii) implies iii).*

i) *For each compact subset* K *of* Ω *there are sequences* $(L_n^+)_{n\in\mathbb{N}}$ *and* $(L_n^-)_{n\in\mathbb{N}}$ *of Borel subsets of* K *and a sequence* $(t_n)_{n\in\mathbb{N}}$ *of positive numbers such that for* $L_n := L_n^+ \cup L_n^-$ *one has*

$$\lim_{n\to\infty}\mu(K\backslash L_n) = \lim_{n\to\infty}\nu_{p,t_n}(L_n) = \lim_{n\to\infty}\nu_{p,-t_n}(L_n) = 0$$

and

$$\lim_{n\to\infty}\nu_{p,2t_n}(L_n^+) = \lim_{n\to\infty}\nu_{p,-2t_n}(L_n^-) = 0.$$

ii) C *is weakly mixing on* $L^p(\mu)$*.*

iii) C *is hypercyclic on* $L^p(\mu)$*.*

Moreover, if for every compact subset K *of* Ω *one has* $\lim_{|t|\to\infty}\mu(K\cap\varphi(t,K)) = 0$ *the above are equivalent.*

Proof: In order to show that i) implies ii) let $U_j, V_j, j = 1,2$, be open, non-empty subsets of $L^p(\mu)$ and $f_j \in U_j \cap C_c(\Omega)$, $g_j \in V_j \cap C_c(\Omega), j = 1,2$. Then $K := \operatorname{supp} f_1 \cup \operatorname{supp} f_2 \cup \operatorname{supp} g_1 \cup \operatorname{supp} g_2$ is compact. Choose $(L_n^+)_n$, $(L_n^-)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ as in i) for K. We can assume without loss of generality that $L_n^+ \cap L_n^- = \emptyset$.

Setting for $n \in \mathbb{N}, j = 1,2$

$$v_{j,n} := h_{t_n}(\cdot)g_j(\varphi(t_n,\cdot))\chi_{L_n^+}(\varphi(t_n,\cdot)) + h_{-t_n}(\cdot)g_j(\varphi(-t_n,\cdot))\chi_{L_n^-}(\varphi(-t_n,\cdot))$$

it follows from

$$\begin{aligned}\|v_{j,n}\| &\leq \left(\int h_{t_n}^p|g_j(\varphi(t_n,\cdot))|^p\chi_{\varphi(-t_n,L_n^+)}\,d\mu\right)^{1/p}\\ &\quad+\left(\int h_{-t_n}^p|g_j(\varphi(-t_n,\cdot))|^p\chi_{\varphi(t_n,L_n^-)}\,d\mu\right)^{1/p}\\ &\leq \|g_j\|_\infty\left(\nu_{p,t_n}(L_n^+)^{1/p} + \nu_{p,-t_n}(L_n^-)^{1/p}\right),\end{aligned}$$

(where by $\|\cdot\|_\infty$ we denote the sup-norm) that $(f_j\chi_{L_n} + v_{j,n})_{n\in\mathbb{N}}$ converges to f_j in $L^p(\mu)$.

Moreover,

$$\begin{aligned} C(t_n)(f_j\chi_{L_n}+v_{j,n}) = g_j\chi_{L_n} &+ \frac{1}{2}\Big(h_{t_n}(\cdot)f_j(\varphi(t_n,\cdot))\chi_{\varphi(-t_n,L_n)} \\ &+ h_{-t_n}(\cdot)f_j(\varphi(-t_n,\cdot))\chi_{\varphi(t_n,L_n)} \\ &+ h_{2t_n}(\cdot)g_j(\varphi(2t_n,\cdot))\chi_{\varphi(-2t_n,L_n^+)} \\ &+ h_{-2t_n}(\cdot)g_j(\varphi(-2t_n,\cdot))\chi_{\varphi(2t_n,L_n^-)}\Big), \end{aligned}$$

so that

$$\begin{aligned} \|C(t_n)(f_j\chi_{L_n}+v_{j,n})-g_j\| &\le \|g_j\|_\infty \mu(K\backslash L_n)^{1/p} \\ &+\frac{\|f_j\|_\infty}{2}\left(\nu_{p,t_n}(L_n^+)^{1/p}+\nu_{p,-t_n}(L_n^-)^{1/p}\right) \\ &+\frac{\|g_j\|_\infty}{2}\left(\nu_{p,2t_n}(L_n^+)^{1/p}+\nu_{p,-2t_n}(L_n^-)^{1/p}\right). \end{aligned}$$

Hence, $(C(t_n)(f_j\chi_{L_n}+v_{j,n})-g_j)_{n\in\mathbb{N}}$ converges to g_j in $L^p(\mu)$ which shows that $C(t_n)(U_j)\cap V_j\neq\emptyset$ for $j=1,2$ and sufficiently large n, i.e. C is weakly mixing.

Obviously, ii) implies iii).

Now, assume that $\lim_{|t|\to\infty}\mu(K\cap\varphi(t,K))=0$ for every compact subset K of Ω. In order to show that iii) implies i) let K be a compact subset of Ω and $\varepsilon\in(0,1)$. By Corollary 2 there are $v\in L^p(\mu)$ and $t>0$ such that $\|v-\chi_K\|^p<\varepsilon^2$ and $\|C(t)v+\chi_K\|^p<\varepsilon^2$ and without loss of generality we can assume that $\mu(K\cap\varphi(2t,K))<\varepsilon^2$ as well as $\mu(K\cap\varphi(-2t,K))<\varepsilon^2$.

By the continuity of the mapping $L^p(\mu,\mathbb{C})\to L^p(\mu,\mathbb{R}), f\mapsto Ref$ and the fact that C commutes with it, we can assume without loss of generality that v is real-valued.

Furthermore, for measurable subsets $B\subseteq\Omega$ we have $\|C(t)(f\chi_B)\|\le\|C(t)f\|$ for arbitrary $t\in\mathbb{R}$ and all $f\in L^p(\mu)$. Obviously the mapping $L^p(\mu,\mathbb{R})\to L^p(\mu,\mathbb{R}), f\mapsto f^+$, where $f^+:=\max\{0,f\}$, satisfies $\|(f+g)^+\|\le\|f^++g^+\|$ and commutes with C so that for measurable $A\subset\Omega$

$$\begin{aligned} \|(C(t)(v^+\chi_B))\chi_A\| &\le \|(C(t)v)^+\| = \|(C(t)v-(-\chi_K)+(-\chi_K))^+\| \\ &\le \|(C(t)v-(-\chi_K))^+\|+\|(-\chi_K)^+\| \\ &= \|(C(t)v-(-\chi_K))^+\| \le \|C(t)v+\chi_K\| < \varepsilon^{2/p} \end{aligned}$$

and $\|v-\chi_K\|^p<\varepsilon^2$ implies

$$\begin{aligned} \|v^-\chi_B\| &\le \|v^-\| = \|(-v)^+\| = \|(\chi_K-v-\chi_K)^+\| \\ &\le \|\chi_K-v\|+\|(-\chi_K)^+\| = \|\chi_K-v\| < \varepsilon^{2/p}, \end{aligned}$$

where $v^-:=\max\{0,-v\}$.

Setting $L:=K\cap\{|1-v|^p\le\varepsilon\}\cap\{|1+C(t)v|^p\le\varepsilon\}$ it follows that $\mu(K\backslash L)<2\varepsilon$ as well as $v_{|L}\ge 1-\varepsilon^{1/p}>0$ and $(C(t)v)_{|L}\le\varepsilon^{1/p}-1<0$.

Now, define $L^-:=\{x\in L;\ (T(t)v)(x)\le\varepsilon^{1/p}-1\}$ and $L^+:=L\backslash L^-$.

Using the fact that $\int f\,d\nu_{p,t}=\int h_t^p(\cdot)f(\varphi(t,\cdot))\,d\mu$ for positive, measurable f we obtain

$$\begin{aligned} \varepsilon^2 &> \|C(t)(v^+\chi_L)\|^p \\ &\ge \int h_t^p v^+(\varphi(t,\cdot))^p\chi_L(\varphi(t,\cdot))\,d\mu+\int h_{-t}^p(\cdot)v^+(\varphi(-t,\cdot))^p\chi_L(\varphi(-t,\cdot))\,d\mu \\ &= \int_L (v^+)^p\,d\nu_{p,t}+\int_L (v^+)^p\,d\nu_{p,-t} \ge (1-\varepsilon^{1/p})^p(\nu_{p,t}(L)+\nu_{p,-t}(L)), \end{aligned}$$

so that the first part of condition $i)$ follows, since ε was arbitrary.

By definition of L^- we have $(T(t)v)(x)\le\varepsilon^{1/p}-1$ for $x\in L^-$ and it follows from $(C(t)v)_{|L}\le\varepsilon^{1/p}-1$ that $(T(-t)v)(x)\le\varepsilon^{1/p}-1$ for $x\in L^+$. These inequalities give $1-\varepsilon^{1/p}\le(T(t)v^-)_{|L^-}$ which implies by bijectivity of $\varphi(-t,\cdot)$ and $h_t(\varphi(-t,\cdot))=1/h_{-t}$ that

$$1-\varepsilon^{1/p}\le(T(t)v^-)(\varphi(-t,x))=h_{-t}(\varphi(-t,x))v^-(x)=v^-(x)/h_{-t}(x)$$

for $x \in \varphi(t, L^-)$. Analogously it follows that $v^-(x)/h_t(x) \geq 1 - \varepsilon^{1/p}$ for $x \in \varphi(-t, L^+)$.

Using this, $h_r(x)h_s(\varphi(r,x)) = h_{r+s}(x)$ for all $r, s \in \mathbb{R}$, and the positivity of the operator $T(-t)$ we have

$$\begin{aligned}
(1-\varepsilon^{1/p})^p \nu_{p,2t}(L^+) &= \int (1-\varepsilon^{1/p})^p h_{2t}^p(x)\, \chi_{L^+}(\varphi(2t,x))\, d\mu(x) \\
&= \int (1-\varepsilon^{1/p})^p h_t^p(x) h_t(\varphi(t,x))^p \chi_{\varphi(-t,L^+)}(\varphi(t,x))\, d\mu(x) \\
&= \int (1-\varepsilon^{1/p})^p h_t^p(x) \chi_{\varphi(-t,L^+)}(x)\, d\nu_{p,t}(x) \\
&\leq \int (v^-)^p(x)/h_t^p(x)\ h_t^p(x) \chi_{\varphi(-t,L^+)}(x)\, d\nu_{p,t}(x) \\
&= \int h_t^p(x)(v^-(\varphi(t,x))^p \chi_{\varphi(-t,L^+)}(\varphi(t,x))\, d\mu(x) \\
&= \int_{\varphi(-2t,L^+)} (T(t)v^-)^p(x)\, d\mu(x) \\
&\leq 2^{p+1} \int_{\varphi(-2t,L^+)} (C(t)v^-)^p(x)\, d\mu(x) \\
&= 2^{p+1} \|(C(t)v^-)\chi_{\varphi(-2t,L^+)}\|^p \\
&= 2^{p+1} \|(C(t)(v^+ - v))\chi_{\varphi(-2t,L^+)}\|^p \\
&= 2^{p+1} \|(C(t)v^+)\chi_{\varphi(-2t,L^+)} - (C(t)v + \chi_K)\chi_{\varphi(-2t,L^+)} \\
&\qquad + \chi_{K\cap\varphi(-2t,L^+)}\|^p \\
&\leq 2^{p+1}(2^p\|C(t)v^+\|^p + 2^p\|C(t)v + \chi_K\|^p \\
&\qquad + 2^p\|\chi_{K\cap\varphi(-2t,L^+)}\|^p) \\
&\leq 2^{3p+1}(2\varepsilon^2 + \mu(K \cap \varphi(-2t, K))) \\
&< 2^{3(p+1)}\varepsilon^2.
\end{aligned}$$

In the same way one shows

$$(1-\varepsilon^{1/p})^p \nu_{p,-2t}(L^-) < 2^{3(p+1)}\varepsilon^2,$$

so that the second part of condition i) follows as well. □

Remark 6. Note that in the above proof we did not need neither the strong continuity of $t \mapsto T(t)$ nor the group law $T(t)T(s) = T(t+s)$.

In fact, we only need μ to be a locally finite Borel measure on a locally compact, σ-compact Hausdorff space Ω such that $T(t)f = h_t(\cdot)\, f(\varphi_t(\cdot))$ is a continuous operator for every t from some index set, where h_t is a positive continuous function on Ω and φ_t a homeomorphism of Ω. For example, one could equip $\Omega = \mathbb{Z}$ with the discrete topology and a measure μ with a positive counting density $(\beta_n)_{n\in\mathbb{Z}}$ on $\mathbb{Z}$, define $\varphi_t(n) = n+t, n \in \mathbb{Z}$, for all $t \in \mathbb{Z}$. Then $T(t)(x_n)_{n\in\mathbb{Z}} = (x_{n+t})_{n\in\mathbb{Z}}$. Obviously, $T := T(1)$ is a well-defined operator on $\ell^p(\beta)$ if and only if $\sup_{n\in\mathbb{Z}} \beta_n/\beta_{n+1} < \infty$. An analogue of the above theorem then reads that under the assumption of $\sup_{n\in\mathbb{Z}} \beta_n/\beta_{n+1} < \infty$ the sequence of operators $(T^n + T^{-n})_{n\in\mathbb{N}}$ is hypercyclic on $\ell_p(\mathbb{Z}, \beta)$ if and only if for each finite subset K of $\mathbb{Z}$ there are a strictly increasing sequence $(n_l)_{l\in\mathbb{N}}$ of natural numbers and a sequence $(\sigma_l)_{l\in\mathbb{N}}$ in $\{-1, 1\}$ such that

$$\lim_{l\to\infty} \sum_{k\in K} \beta_{k+n_l} = \lim_{l\to\infty} \sum_{k\in K} \beta_{k-n_l} = \lim_{l\to\infty} \sum_{k\in K} \beta_{k+2\sigma_l n_l} = 0.$$

From $\sup_{n\in\mathbb{Z}} \beta_n/\beta_{n+1} < \infty$ it is easily deduced that the last condition is equivalent to that for every $k \in \mathbb{Z}$ there are a strictly increasing sequence $(n_l)_{l\in\mathbb{N}}$ of natural numbers and a sequence $(\sigma_l)_{l\in\mathbb{N}}$ in $\{-1,1\}$ such that

$$\lim_{l\to\infty}\beta_{k+n_l}=\lim_{l\to\infty}\beta_{k-n_l}=\lim_{l\to\infty}\beta_{k+2\sigma_l n_l}=0,$$

(compare [13, Example 2.7]).

An obvious modification of the proof of Theorem 5 gives the following result.

Theorem 7. *Let μ be p-admissible for F and h. For the cosine function C defined by $C(t):=\frac{1}{2}(T(t)+T(-t))$ with $T(t)f(x)=h_t(x)f(\varphi(t,x))$, the following condition i) implies ii).*

i) *For each compact subset K of Ω there are families $(L_t^+)_{t\geq 0}$ and $(L_t^-)_{t\geq 0}$ of Borel subsets of K such that with $L_t:=L_t^+\cup L_t^-$*

$$\lim_{t\to\infty}\mu(K\backslash L_t)=\lim_{t\to\infty}\nu_{p,t}(L_t)=\lim_{t\to\infty}\nu_{p,-t}(L_t)=0$$

and

$$\lim_{t\to\infty}\nu_{p,2t}(L_t^+)=\lim_{t\to\infty}\nu_{p,-2t}(L_t^-)=0$$

ii) *C is mixing on $L^p(\mu)$.*

If additionally $\lim_{|t|\to\infty}\mu(K\cap\varphi(t,K))=0$ for all compact subsets K of Ω the above are equivalent.

Corollary 8. *Let μ be p-admissible for F and h such that $\lim_{|t|\to\infty}\mu(K\cap\varphi(t,K))=0$ for every compact subset K of Ω.*

a) *If the cosine operator function C defined by $C(t):=\frac{1}{2}(T(t)+T(-t))$ with $T(t)f(x)=h_t(x)f(\varphi(t,x))$ is hypercyclic on $L^p(\mu)$ then the C_0-semigroup $(T(t))_{t\geq 0}$ is hypercyclic, too.*

b) *If the cosine operator function C defined by $C(t):=\frac{1}{2}(T(t)+T(-t))$ with $T(t)f(x)=h_t(x)f(\varphi(t,x))$ is mixing on $L^p(\mu)$ then the C_0-semigroup $(T(t))_{t\geq 0}$ is mixing, too.*

PROOF. From hypercyclicity, resp. mixing, of C it follows from Theorem 5, resp. Theorem 7 that

$$\lim_{n\to\infty}\mu(K\backslash L_n)=\lim_{n\to\infty}\nu_{p,t_n}(L_n)=\lim_{t\to\infty}\nu_{p,-t_n}(L_n)=0$$

for suitable $(t_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$, resp.

$$\lim_{t\to\infty}\mu(K\backslash L_t)=\lim_{t\to\infty}\nu_{p,t}(L_t)=\lim_{t\to\infty}\nu_{p,-t}(L_t)=0$$

for suitable $(L_t)_{t\geq 0}$. Applying [13, Theorem 4.10], resp. [13, Theorem 5.1 a)], now gives the corollary. □

For the case of continuous functions one has the following result.

Theorem 9. *Let ρ be a C_0-admissible function for F and h on Ω. For the cosine operator function C defined by $C(t):=\frac{1}{2}(T(t)+T(-t))$ with $T(t)f(x)=h_t(x)f(\varphi(t,x))$, among the following i) implies ii) and ii) implies iii).*

i) *For every compact subset K of Ω there are sequences of positive numbers $(t_n)_{n\in\mathbb{N}}$ and open subsets $(U_n^+)_{n\in\mathbb{N}},(U_n^-)_{n\in\mathbb{N}}$ of Ω with $K\subset U_n^+\cup U_n^-$ for every $n\in\mathbb{N}$ such that*

$$\lim_{n\to\infty}\sup_{x\in K}\frac{\rho(\varphi(-t_n,x))}{h_{-t_n}(x)}=\lim_{n\to\infty}\sup_{x\in K}\frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)}=0$$

as well as

$$\lim_{n\to\infty}\sup_{x\in K\cap U_n^-}\frac{\rho(\varphi(-2t_n,x))}{h_{-2t_n}(x)}=\lim_{n\to\infty}\sup_{x\in K\cap U_n^+}\frac{\rho(\varphi(2t_n,x))}{h_{2t_n}(x)}=0.$$

ii) *C is weakly mixing on $C_{0,\rho}(\Omega)$.*

iii) *C is hypercyclic on $C_{0,\rho}(\Omega)$.*

Moreover, if for every compact subset K of Ω $\lim_{|t|\to\infty}\sup_{x\in K\cap\varphi(t,K)}\rho(x)=0$ and $\inf_{x\in K}\rho(x)>0$ hold, the above are equivalent.

Proof: In order to show that i) implies ii) let $W_j,V_j\subset C_{0,\rho}(\Omega)$ be open and non-empty, $j=1,2$. Let $f_j\in W_j\cap C_c(\Omega), g_j\in V_j\cap C_c(\Omega), j=1,2$ and define $K:=\operatorname{supp} f_1\cup\operatorname{supp} f_2\cup\operatorname{supp} g_1\cup\operatorname{supp} g_2$. Choose $(U_n^+)_{n\in\mathbb{N}}, (U_n^-)_{n\in\mathbb{N}}$, and $(t_n)_{n\in\mathbb{N}}$ as in i) for K. Since $K\subset U_n^+\cup U_n^-$ there are C^∞-functions $\psi_n^+\geq 0$ and $\psi_n^-\geq 0$ such that $\operatorname{supp}\psi_n^+\subset U_n^+$, $\operatorname{supp}\psi_n^-\subset U_n^-$ and $\psi_n^++\psi_n^-\equiv 2$ in a neighbourhood of K.

We define for $n\in\mathbb{N}$ and $j=1,2$

$$v_{j,n}:=h_{t_n}(\cdot)g_j(\varphi(t_n,\cdot))\psi_n^-(\varphi(t_n,\cdot))+h_{-t_n}(\cdot)g_j(\varphi(-t_n,\cdot))\psi_n^+(\varphi(-t_n,\cdot)).$$

Then, $v_{j,n}\in C_{0,\rho}(\Omega)$ and taking into account that $\psi_n^++\psi_n^-\equiv 2$ in a neighbourhood of K, a straightforward calculation gives

$$\begin{aligned} C(t_n)v_{j,n} &= \frac{1}{2}(h_{2t_n}(\cdot)g_j(\varphi(2t_n,\cdot))\psi_n^-(\varphi(2t_n,\cdot))\\ &\quad +h_{-2t_n}(\cdot)g_j(\varphi(-2t_n,\cdot))\psi_n^+(\varphi(-2t_n,\cdot)))+g_j.\end{aligned}$$

Since $h_{2t_n}(\varphi(2t_n,x))=1/h_{-2t_n}(x)$ it follows that

$$\begin{aligned} &\sup_{x\in\Omega}h_{2t_n}(x)|g_j(\varphi(2t_n,x))|\psi_n^-(\varphi(2t_n,x))\rho(x)\\ =&\sup_{x\in\varphi(-2t_n,K)}h_{2t_n}(x)|g_j(\varphi(2t_n,x))|\psi_n^-(\varphi(2t_n,x))\rho(x)\\ =&\sup_{x\in K}h_{2t_n}(\varphi(-2t_n,x))|g_j(x)|\psi_n^-(x)\rho(\varphi(-2t_n,x))\\ \leq& 2\|g_j\|_\infty\sup_{x\in K\cap U_n^-}\frac{\rho(\varphi(-2t_n,x))}{h_{-2t_n}(x)}\end{aligned}$$

and analogously

$$\sup_{x\in\Omega}h_{-2t_n}(x)|g_j(\varphi(-2t_n,x))|\psi_n^+(\varphi(-2t_n,x))\rho(x)\leq 2\|g_j\|_\infty\sup_{x\in K\cap U_n^+}\frac{\rho(\varphi(2t_n,x))}{h_{2t_n}(x)},$$

which implies $\lim_{n\to\infty}C(t_n)v_{j,n}=g_j$ in $C_{0,\rho}(\Omega)$.

In the same way one shows that $\lim_{n\to\infty}v_{j,n}=0$ in $C_{0,\rho}(\Omega)$.

Because

$$\begin{aligned} &\sup_{x\in\Omega}|h_{t_n}(x)f_j(\varphi(t_n,x))+h_{-t_n}(x)f_j(\varphi(-t_n,x))|\rho(x)\\ \leq&\sup_{x\in\varphi(-t_n,K)}h_{t_n}(x)|f_j(\varphi(t_n,x))|\rho(x)+\sup_{x\in\varphi(t_n,K)}h_{-t_n}(x)|f_j(\varphi(-t_n,x))|\rho(x)\\ \leq&\|f_j\|_\infty(\sup_{x\in K}\frac{\rho(\varphi(-t_n,x))}{h_{t_n}(x)}+\sup_{x\in K}\frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)})\end{aligned}$$

we have $\lim_{n\to\infty}C(t_n)f_j=0$ in $C_{0,\rho}(\Omega)$.

Altogether this gives

$$\lim_{n\to\infty}(f_j+v_{j,n})=f_j$$

and

$$\lim_{n\to\infty}C(t_n)(f_j+v_{j,n})=g_j,\ j=1,2$$

so that $C(t_n)(W_j)\cap V_j\neq\emptyset$ for $j=1,2$ and sufficiently large n so that ii) follows.

Trivially, ii) implies iii).

Now we assume that $\lim_{|t|\to\infty}\sup_{x\in K\cap\varphi(t,K)}\rho(x)=0$ and $\inf_{x\in K}\rho(x)>0$ hold for every compact subset K of Ω. In order to prove that iii) implies i) let K be a compact subset of Ω and $\varepsilon\in(0,\inf_{x\in K}\rho(x))$. Let $f\in C_c(\Omega)$ be such that $0\leq f\leq 1$ and $f\equiv 1$ in a neighbourhood of K.

By Corollary 2 there are $t > 0, v \in C_{0,\rho}(\Omega)$ with $\|v - f\| < \varepsilon$ and $\|C(t)v + f\| < \varepsilon$. Without loss of generality we can assume that

$$\sup_{x \in M \cap \varphi(2t,M)} \rho(x) + \sup_{x \in M \cap \varphi(-2t,M)} \rho(x) < \varepsilon$$

where $M := \operatorname{supp} f$.

As in the proof of Theorem 5 we can assume v to be real-valued and we obtain $\|C(t)v^+\| < \varepsilon$ and $\|v^-\| < \varepsilon$.

Because of

$$\varepsilon > \|C(t)v + f\| \geq \sup_{x \in K} |C(t)v + 1|\rho(x)$$

and the choice of ε we get

$$\forall\, x \in K : \, C(t)v(x) < \frac{\varepsilon}{\rho(x)} - 1 < -\frac{1}{2}.$$

In the same way one derives from $\varepsilon > \|v - f\|$ that

$$\forall\, x \in K : \, v(x) > 1 - \frac{\varepsilon}{\rho(x)} > \frac{1}{2},$$

i.e. $v^+ > 1/2$ on K.

From this we obtain

$$\begin{aligned}
\varepsilon \quad &> \quad \|C(t)v^+\| = \frac{1}{2} \sup_{x \in \Omega} (h_t(x)v^+(\varphi(t,x)) + h_{-t}(x)v^+(\varphi(-t,x)))\rho(x) \\
&\geq \quad \frac{1}{4}(\sup_{x \in \varphi(-t,K)} h_t(x)v^+(\varphi(t,x))\rho(x) + \sup_{x \in \varphi(t,K)} h_{-t}(x)v^+(\varphi(-t,x))\rho(x)) \\
&\geq \quad \frac{1}{8}(\sup_{x \in K} \frac{\rho(\varphi(-t,x))}{h_{-t}(x)} + \sup_{x \in K} \frac{\rho(\varphi(t,x))}{h_t(x)}).
\end{aligned}$$

Since $T(t)v$ and $T(-t)v$ are continuous functions it follows that the sets $U^+ := \{x \in \Omega;\, (T(t)v)(x) < -1/4\}$ and $U^- := \{x \in \Omega;\, (T(-t)v)(x) < -1/4\}$ are open and because of $C(t)v < -1/2$ on K we have $K \subset U^+ \cup U^-$.

Because of $\varphi(t,\cdot)$ and $\varphi(-t,\cdot)$ are one-to-one and onto we obtain

$$\forall\, x \in \varphi(t, U^+) : \, \frac{1}{2} \leq (T(t)v^-)(\varphi(-t,x)) = \frac{v^-(x)}{h_{-t}(x)}$$

and

$$\forall\, x \in \varphi(-t, U^-) : \, \frac{1}{2} \leq (T(-t)v^-)(\varphi(t,x)) = \frac{v^-(x)}{h_t(x)}.$$

Having in mind that $h_t(x)h_{-t}(\varphi(t,x)) = 1$ for every $x \in \Omega$ we get

$$\begin{aligned}
\frac{1}{2}\sup_{x\in K\cap U^-}\frac{\rho(\varphi(-2t,x))}{h_{-2t}(x)} &= \frac{1}{2}\sup_{x\in\varphi(-t,K\cap U^-)}\frac{\rho(\varphi(-t,x))}{h_{-2t}(\varphi(t,x))}\\
&\leq \sup_{x\in\varphi(-t,K\cap U^-)}\frac{v^-(x)\rho(\varphi(-t,x))}{h_t(x)h_{-2t}(\varphi(t,x))}\\
&= \sup_{x\in\varphi(-t,K\cap U^-)}\frac{h_t(\varphi(-t,x))v^-(\varphi(t,\varphi(-t,x)))}{h_t(\varphi(-t,x))h_{-t}(\varphi(t,\varphi(-t,x)))}\rho(\varphi(-t,x))\\
&= \sup_{x\in\varphi(-t,K\cap U^-)}(T(t)v^-)(\varphi(-t,x))\rho(\varphi(-t,x))\\
&= \sup_{x\in\varphi(-2t,K\cap U^-)}(T(t)v^-)(x)\rho(x)\\
&\leq 2\sup_{x\in\varphi(-2t,K\cap U^-)}(C(t)v^-)(x)\rho(x)\\
&= 2\sup_{x\in\varphi(-2t,K\cap U^-)}(C(t)(v^+-v))(x)\rho(x)\\
&\leq 2\Big(\sup_{x\in\varphi(-2t,K\cap U^-)}(C(t)v^+)(x)\rho(x)\\
&\quad+\sup_{x\in\varphi(-2t,K\cap U^-)}|(C(t)v)(x)+f(x)|\rho(x)\\
&\quad+\sup_{x\in\varphi(-2t,K\cap U^-)}|f(x)|\rho(x)\Big)\\
&\leq 2\Big(\|C(t)v^+\|+\|C(t)v+f\|+\sup_{x\in\varphi(-2t,M)\cap M}\rho(x)\Big)\\
&\leq 6\varepsilon.
\end{aligned}$$

In the same way one verifies

$$\frac{1}{2}\sup_{x\in K\cap U^+}\frac{\rho(\varphi(2t,x))}{h_{2t}(x)} < 6\varepsilon.$$

Since ε was chosen arbitrarily small, i) finally follows. □

Obvious modifications of the above proof yield the next result.

Theorem 10. *Let ρ be a C_0-admissible function for F and h on Ω. For the cosine operator function $C(t) := \frac{1}{2}(T(t)+T(-t))$ with $T(t)f(x) = h_t(x)f(\varphi(t,x))$, the following condition i) implies ii).*

i) *For every compact subset K of Ω there are open subsets $(U_t^+)_{t\geq 0}, (U_t^-)_{t\geq 0}$ of Ω with $K \subset U_t^+ \cup U_t^-$ for every $t \geq 0$ such that*

$$\lim_{t\to\infty}\sup_{x\in K}\frac{\rho(\varphi(-t,x))}{h_{-t}(x)} = \lim_{t\to\infty}\sup_{x\in K}\frac{\rho(\varphi(t,x))}{h_t(x)} = 0$$

as well as

$$\lim_{t\to\infty}\sup_{x\in K\cap U_t^-}\frac{\rho(\varphi(-2t,x))}{h_{-2t}(x)} = \lim_{t\to\infty}\sup_{x\in K\cap U_t^+}\frac{\rho(\varphi(2t,x))}{h_{2t}(x)} = 0.$$

ii) *C is mixing on $C_{0,\rho}(\Omega)$.*

Moreover, if for every compact subset K of Ω $\lim_{|t|\to\infty}\sup_{x\in K\cap\varphi(t,K)}\rho(x) = 0$ and $\inf_{x\in K}\rho(x) > 0$ hold, the above are equivalent.

4. The one-dimensional case

In case of $d = 1$, that is $\Omega \subset \mathbb{R}$, we can considerably simplify the conditions characterizing hypercyclicity, resp. mixing, derived in the previous section. One tool for this will be the next lemma. For a proof see [14, Lemma 7]. In this section we simply write $\partial_2\varphi(t,x)$ for the Jacobian of $x \mapsto \varphi(t,x)$.

Lemma 11. *Let $\Omega \subset \mathbb{R}$ be open and $[a,b] \subset \{F \neq 0\}$. Assume that $\rho : \Omega \to (0,\infty)$ is measurable and satisfies $h_t^p(x)\rho(x) \leq Me^{\omega t}\rho(\varphi(t,x))|\partial_2\varphi(t,x)|$ for some constants $M \geq 1, \omega \geq 0$ and for every $t \geq 0, x \in [a,b]$.*

Then there is $C > 0$ such that $1/C < \rho(y) < C$ for all $y \in [a,b]$ and

$$\begin{aligned} & h_t^p(\varphi(-t,c))\rho(\varphi(-t,c))|\partial_2\varphi(-t,c)|\chi_{\varphi(t,\Omega)}(c) \\ \leq\ & Ch_t^p(\varphi(-t,y))\rho(\varphi(-t,y))|\partial_2\varphi(-t,y)|\chi_{\varphi(t,\Omega)}(y) \\ \leq\ & C^2h_t^p(\varphi(-t,d))\rho(\varphi(-t,d))|\partial_2\varphi(-t,d)|\chi_{\varphi(t,\Omega)}(d) \end{aligned}$$

as well as

$$\begin{aligned} h_t^{-p}(c)\rho(\varphi(t,c))|\partial_2\varphi(t,c)| &\leq Ch_t^{-p}(y)\rho(\varphi(t,y))|\partial_2\varphi(t,y)| \\ &\leq C^2h_t^{-p}(d)\rho(\varphi(t,d))|\partial_2\varphi(t,d)|. \end{aligned}$$

for all $t \geq 0$, where $c := a, d := b$ if $F_{|[a,b]} > 0$, respectively $c := b, d := a$ if $F_{|[a,b]} < 0$.

Now we come to a characterization of hypercyclicity on $L^p(\mu)$ which is more applicable in concrete situations than the one given by Theorem 5. We denote by λ^m m-dimensional Lebesgue measure and simply write λ instead of λ^1.

Theorem 12. *Let $\Omega \subset \mathbb{R}$ be open and F continuously differentiable. Assume the locally finite p-admissible measure μ has a positive Lebesgue density ρ. Then the following are equivalent.*

i) *The cosine operator function C defined via*

$$(C(t)f)(x) = \frac{1}{2}(h_t(x)f(\varphi(t,x)) + h_{-t}(x)f(\varphi(-t,x)))$$

is weakly mixing on $L^p(\mu)$.

ii) *The cosine operator function C is hypercyclic on $L^p(\mu)$.*

iii) *$\lambda(\{F = 0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are m different components $\Omega_1, \dots, \Omega_m$ of $\Omega\backslash\{F = 0\}$, for λ^m-almost all choices of $x_j \in \Omega_j, j = 1, \dots, m$, there are a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity and a sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{1,-1\}^{\mathbb{N}}$ such that*

$$\lim_{n\to\infty} h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))\partial_2\varphi(t_n,x_j) = 0,$$

$$\lim_{n\to\infty} h_{-t_n}^{-p}(x_j)\rho(\varphi(-t_n,x_j))\partial_2\varphi(-t_n,x_j) = 0,$$

and

$$\lim_{n\to\infty} h_{2\sigma_n t_n}^{-p}(x_j)\rho(\varphi(2\sigma_n t_n,x_j))\partial_2\varphi(2\sigma_n t_n,x_j) = 0$$

for $j = 1, \dots, m$.

Proof: That i) implies ii) is again trivial. In order to show that ii) implies iii) observe that $\varphi(t,x) = x$ if $F(x) = 0$ so that $h_t(x)f(\varphi(t,x)) = \exp(th(x))f(x)$ for every $f \in L^p(\mu)$ on $\{F = 0\}$. From this it follows easily that C cannot be hypercyclic if $\lambda(\{F = 0\}) > 0$. Hence, $L^p(\mu) = L^p(\Omega\backslash\{F = 0\},\mu)$. Because of $\varphi(t,\Omega\backslash\{F = 0\}) \subset \Omega\backslash\{F = 0\}$ we can therefore consider C on $L^p(\Omega\backslash\{F = 0\},\mu)$ rather than on $L^p(\mu)$. Obviously, C is hypercyclic on $L^p(\Omega\backslash\{F = 0\},\mu)$ by ii). But for a compact subset of $\Omega\backslash\{F = 0\}$ we obviously have $K \cap \varphi(t,K) = \emptyset$ for $|t|$ large enough, in particular $\lim_{|t|\to\infty}\mu(K \cap \varphi(t,K)) = 0$.

Let $x_1, \ldots, x_m$ be from different components of $\Omega\backslash\{F=0\}$ which, by Theorem 4 b), we assume without loss of generality to satisfy

$$h_t(x_j)\rho(x_j) \leq Me^{\omega t}\rho(\varphi(t,x_j))|\partial_2\varphi(t,x_j)|$$

for all $t \geq 0, j = 1, \ldots, m$. Since Ω is open there is $r < 0$ such that $\varphi(t, x_j)$ is well defined for all $t \in [r,\infty), j = 1, \ldots, m$ and the aforementioned inequality is valid for $\varphi(r, x_j)$ in place of x_j, too. For $j = 1, \ldots, m$ we define $K_j := \{\varphi(t, x_j); 0 \leq t \leq 1\}$ if $F(x_j) > 0$, respectively $K_j := \{\varphi(t, x_j); r \leq t \leq 0\}$ if $F(x_j) < 0$. Then the K_j's are compact intervals contained in $\Omega\backslash\{F = 0\}$ satisfying $\lambda(K_j) > 0$, since $F(x_j) \neq 0$, and $K_j = [x_j, \varphi(1, x_j)]$ if $F(x_j) > 0$, respectively $K_j = [x_j, \varphi(r, x_j)]$ if $F(x_j) < 0$. In particular $\mu(K_j) > 0$.

For the compact set $K := \cup_{1\leq j\leq m}K_j$ choose measurable subsets $(L_n^+)_{n\in\mathbb{N}}, (L_n^-)_{n\in\mathbb{N}}$ and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ according to i) of Theorem 5. Without loss of generality we can assume that $L_n^+ \cap L_n^- = \emptyset$ for all $n \in \mathbb{N}$. Set $L_n := L_n^+ \cup L_n^-$.

Since C is weakly mixing, it follows from Theorem 4 b) that $\omega > 0$, because otherwise $\{\|T(t)\|; t \in \mathbb{R}\}$ was bounded, implying the boundedness of each orbit under C. Defining $L_n := L_n^+ \cup L_n^-$ and $L_{n,j} := L_n \cap K_j, n \in \mathbb{N}, 1 \leq j \leq m$ we obtain from Theorem 4 c) and Lemma 11 that for some constant $C_j > 0$

$$\begin{aligned}\nu_{p,-t_n}(L_{n,j}) &= \int_{L_{n,j}} \frac{h_{t_n}^{-p}(y)\rho(\varphi(t_n,y))|\partial_2\varphi(t_n,y)|}{\rho(y)}\,d\mu(y)\\ &\geq C_j h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))|\partial_2\varphi(t_n,x_j)|\mu(L_{n,j}).\end{aligned}$$

Because $\lim_{n\to\infty}\mu(L_{n,j}) = \mu(K_j) > 0$ it follows from $\lim_{n\to\infty}\nu_{p,t_n}(L_{n,j}) = 0$ that

$$\lim_{n\to\infty} h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))|\partial_2\varphi(t_n,x_j)| = 0$$

for all $j = 1, \ldots, m$ and the continuity of $(s, y) \mapsto h_s(y), \varphi$, and $\partial_2\varphi$ together with Lemma 11 imply that $(t_n)_{n\in\mathbb{N}}$ has to converge to infinity.

Furthermore, we get from Theorem 4 c) and Lemma 11

$$\begin{aligned}\nu_{p,t_n}(L_{n,j}) &= \int_{L_{n,j}} \frac{h_{t_n}^{p}(\varphi(-t_n,y))\rho(\varphi(-t_n,y))|\partial_2\varphi(-t_n,y)|}{\rho(y)}\,d\mu(y)\\ &\geq C_j h_{t_n}^{p}(\varphi(-t_n,x_j))\rho(\varphi(-t_n,x_j))|\partial_2\varphi(-t_n,x_j)|\mu(L_{n,j}).\end{aligned}$$

Observing that $h_{t_n}(\varphi(-t_n,\cdot)) = 1/h_{-t_n}$ this shows by the same arguments as above that

$$\lim_{n\to\infty} h_{-t_n}^{-p}(x_j)\rho(\varphi(-t_n,x_j))|\partial_2\varphi(-t_n,x_j)| = 0.$$

Moreover, by the same reasoning we obtain for some other $C_j > 0$

$$\nu_{p,2t_n}(L_n^+) \geq C_j h_{2t_n}^{-p}(x_j)\rho(\varphi(2t_n,x_j))|\partial_2\varphi(2t_n,x_j)|\mu(L_n^+)$$

and

$$\nu_{p,-2t_n}(L_n^-) \geq C_j h_{-2t_n}^{-p}(x_j)\rho(\varphi(-2t_n,x_j))|\partial_2\varphi(-2t_n,x_j)|\mu(L_n^-).$$

Since $\mu(L_n) = \mu(L_n^+) + \mu(L_n^-)$ tends to $\mu(K) > 0$ for n to infinity, iii) follows.

In order to show that iii) implies i) let K be a compact subset of Ω. Since obviously $L^p(\Omega,\mu) = L^p(\Omega\backslash\{F = 0\},\mu)$ and $\varphi(t,\Omega\backslash\{F = 0\}) \subset \Omega\backslash\{F = 0\}$ for all $t \geq 0$ we can assume without loss of generality that $K \subset \Omega\backslash\{F = 0\}$.

Therefore, there are finitely many intervals $[a_j, b_j] \subset \Omega\backslash\{F = 0\}$ such that each $[a_j, b_j]$ is contained in a different component of $\Omega\backslash\{F = 0\}$ and $K \subset \cup_{1\leq j\leq m}[a_j, b_j]$. We define $x_j := a_j$ if $F_{|[a_j,b_j]} > 0$, respectively $x_j := b_j$ if $F_{|[a_j,b_j]} < 0$, where without loss of generality we assume iii) to be true for $x_1, \ldots, x_m$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers according to iii) for $x_1, \dots, x_m$. From Lemma 11 it follows that for some $C_j > 0$

$$\begin{aligned}\nu_{p,-t_n}(K) &\leq \sum_{j=1}^m \nu_{p,t_n}([a_j,b_j]) = \sum_{j=1}^m \int_{[a_j,b_j]} \frac{h_t^{-p}(y)\rho(\varphi(t_n,y))|\partial_2\varphi(t_n,y)|}{\rho(y)}\,d\mu(y)\\ &\leq \sum_{j=1}^m C_j\mu([a_j,b_j])h_{t_n}^{-p}(x_j)\rho(\varphi(t_n,x_j))|\partial_2\varphi(t_n,x_j)|\end{aligned}$$

so that $\lim_{n\to\infty}\nu_{p,t_n}(K) = 0$.

Analogously, one shows that $\lim_{n\to\infty}\nu_{p,t_n}(K) = \lim_{n\to\infty}\nu_{p,2\sigma_n t_n}(K) = 0$ as well. Setting $L_n^+ := K, L_n^- := \emptyset$ in case of $\sigma_n = 1$ and $L_n^+ := \emptyset, L_n^- := K$ in case of $\sigma_n = -1$ now shows that condition i) of Theorem 5 is satisfied so that i) follows. □

Using the same arguments one gets the following result.

Theorem 13. *Let $\Omega \subset \mathbb{R}$ be open and F continuously differentiable. Assume the locally finite p-admissible measure μ has a positive Lebesgue density ρ. Then the following are equivalent.*

i) *The cosine operator function C defined via*
$$(C(t)f)(x) = \frac{1}{2}(h_t(x)f(\varphi(t,x)) + h_{-t}(x)f(\varphi(-t,x)))$$
is mixing on $L^p(\mu)$.

ii) *$\lambda(\{F=0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are m different components $\Omega_1, \dots, \Omega_m$ of $\Omega\backslash\{F=0\}$, for λ^m-almost all choices of $x_j \in \Omega_j, j = 1, \dots, m$, there is a family $(\sigma_t)_{t\in\mathbb{R}} \in \{1,-1\}^{\mathbb{R}}$ such that*
$$\lim_{t\to\infty} h_t^{-p}(x_j)\rho(\varphi(t,x_j))\partial_2\varphi(t,x_j) = 0,$$
$$\lim_{t\to\infty} h_{-t}^{-p}(x_j)\rho(\varphi(-t,x_j))\partial_2\varphi(-t,x_j) = 0,$$
and
$$\lim_{t\to\infty} h_{2\sigma_t t}^{-p}(x_j)\rho(\varphi(2\sigma_t t,x_j))\partial_2\varphi(2\sigma_t t,x_j) = 0$$
for $j = 1, \dots, m$.

Using the next lemma instead of Lemma 11 one can derive analogously to Theorem 12 a result for the case of continuous functions. A proof of the next lemma can be found in [13, Lemma 10].

Lemma 14. *Let $\Omega \subset \mathbb{R}$ be open and $[a,b] \subset \{F \neq 0\}$. Assume that $\rho : \Omega \to (0,\infty)$ satisfies $h_t(x)\rho(x) \leq Me^{\omega t}\rho(\varphi(t,x))$ for some $M \geq 1, \omega \in \mathbb{R}$ and all $x \in [a,b], t \geq 0$.*

Then there is $C > 0$ such that $1/C < \rho(y) < C$ for all $y \in [a,b]$ and
$$\begin{aligned}h_t^p(\varphi(-t,c))\rho(\varphi(-t,c))\chi_{\varphi(t,\Omega)}(c) &\leq Ch_t^p(\varphi(-t,y))\rho(\varphi(-t,y))\chi_{\varphi(t,\Omega)}(y)\\ &\leq C^2h_t^p(\varphi(-t,d))\rho(\varphi(-t,d))\chi_{\varphi(t,\Omega)}(d)\end{aligned}$$
as well as
$$\begin{aligned}h_t^{-p}(c)\rho(\varphi(t,c)) &\leq Ch_t^{-p}(y)\rho(\varphi(t,y))\\ &\leq C^2h_t^{-p}(d)\rho(\varphi(t,d)).\end{aligned}$$
for all $t \geq 0$ and all $y \in [a,b]$, where $c := a, d := b$ if $F_{|[a,b]} > 0$, respectively $c := b, d := a$ if $F_{|[a,b]} < 0$.

Having at hand the above lemma the proofs of the next results are so similar to the one of Theorem 12 that we omit them.

Theorem 15. *Let $\Omega \subset \mathbb{R}$ be open, F continuously differentiable and ρ be a positive function on Ω C_0-admissible for F and h. Then the following are equivalent.*

i) *The cosine operator function C defined via*

$$(C(t)f)(x) = \frac{1}{2}(h_t(x)f(\varphi(t,x)) + h_{-t}(x)f(\varphi(-t,x)))$$

is weakly mixing on $C_{0,\rho}(\Omega)$.

ii) *The cosine operator function C is hypercyclic on $C_{0,\rho}(\Omega)$.*

iii) *$\{F = 0\} = \emptyset$ and for all $x \in \Omega$, there are a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity and a sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{1,-1\}^{\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)} = \lim_{n\to\infty} \frac{\rho(\varphi(-t_n,x))}{h_{-t_n}(x)} = 0$$

and

$$\lim_{n\to\infty} \frac{\rho(\varphi(2\sigma_n t_n,x))}{h_{2\sigma_n t_n}(x)} = 0.$$

is

Theorem 16. *Let $\Omega \subset \mathbb{R}$ be open, F continuously differentiable and ρ be a positive function on Ω C_0-admissible for F and h. Then the following are equivalent.*

i) *The cosine operator function C defined via*

$$(C(t)f)(x) = \frac{1}{2}(h_t(x)f(\varphi(t,x)) + h_{-t}(x)f(\varphi(-t,x)))$$

is mixing on $C_{0,\rho}(\Omega)$.

ii) *$\{F = 0\} = \emptyset$ and for all $x \in \Omega$, there is a family $(\sigma_t)_{t\in\mathbb{R}} \in \{1,-1\}^{\mathbb{R}}$ such that*

$$\lim_{t\to\infty} \frac{\rho(\varphi(t,x))}{h_t(x)} = \lim_{t\to\infty} \frac{\rho(\varphi(-t,x))}{h_{-t}(x)} = 0$$

and

$$\lim_{t\to\infty} \frac{\rho(\varphi(2\sigma_t t,x))}{h_{2\sigma_t t}(x)} = 0.$$

For the special case of $F \equiv 1$ and $h \equiv 0$ we obtain the so-called *left translation group* $(T(t)f)(x) = f(x+t)$. Since the generator of the corresponding cosine operator function is given by the closure of the operator

$$C_c^2(\mathbb{R}) \to L^p(\mu), f \mapsto \frac{d^2}{dx^2}f$$

it is closely related to the wave equation. For this special case we have the following corollary which should be compared with [3, Theorem 2.2]

Corollary 17. *Let μ be p-admissible for $F \equiv 1$ and $h \equiv 0$ on $\mathbb{R}$, admitting a positive Lebesgue density ρ. Then for the cosine operator function C defined by $(C(t)f)(x) = \frac{1}{2}(f(x+t) + f(x-t))$ the following are equivalent.*

i) *C is hypercyclic on $L^p(\mu)$.*

ii) *For almost all $x \in \mathbb{R}$ there are a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity and a sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{1,-1\}^{\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \rho(x+t_n) = \lim_{n\to\infty} \rho(x-t_n) = \lim_{n\to\infty} \rho(x+2\sigma_n t_n) = 0.$$

Clearly, if in the above corollary for ρ there are $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $t \in \mathbb{R}$

$$\rho(x) \leq Me^{\omega|t|}\rho(x+t)$$

not only for λ-almost all x but for all x (as is the case in [3, Theorem 2.2]) then ii) is equivalent to

ii') There are a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity and a sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{1,-1\}^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} \rho(t_n) = \lim_{n\to\infty} \rho(-t_n) = \lim_{n\to\infty} \rho(2\sigma_n t_n) = 0.$$

Example (perturbed wave equation). Let $F \equiv 1, h \equiv \alpha \in \mathbb{R}$ and $\mu = \lambda$ on $\mathbb{R}$. It follows that $\varphi(t,x) = x+t$ and $h_t(x) = \exp(\alpha|t|)$ so that by Theorem 4 b) λ is p-admissible for F and h for arbitrary $p \in [1,\infty)$. By Theorem 4 e) the generator $(A, D(A))$ of the corresponding cosine operator function is given by the closure of the operator

$$C_c^2(\mathbb{R}) \to L^p(\lambda), f \mapsto f'' + 2\alpha f' + \alpha^2 f,$$

i.e. for $f \in D(A)$ we have

$$\frac{\partial^2}{\partial t^2} C(t)f(x) = \frac{\partial^2}{\partial x^2} C(t)f(x) + 2\alpha \frac{\partial}{\partial x} C(t)f(x) + \alpha^2 C(t)f(x)$$

in a generalized sense.

Since $h_t^{-p}(x) = \exp(-p\alpha|t|)$ it follows immediately from Theorem 12 that the cosine operator function is mixing, in particular hypercyclic on $L^p(\mu)$ for every $p \in [1,\infty)$ if and only if this is true for some $p \in [1,\infty)$ if and only if $\alpha > 0$.

In the same way one shows that $\rho \equiv 1$ is C_0-admissible for F and h and that C is hypercyclic on $C_{0,\rho}(\mathbb{R})$ if and only if it is mixing if and only if $\alpha > 0$.

Example (exponential translation). Let $\Omega = (0,\infty)$ and $F(x) = x, h \equiv 0$, so that $\varphi(t,x) = xe^t$. Let μ be the measure on $(0,\infty)$ with Lebesgue density $\rho(x) = \chi_{(0,1)}(x) + \frac{1}{x^2}\chi_{[1,\infty)}(x)$. Using Theorem 4 b) it is not hard to see that the locally finite measure μ is p-admissible for F and h. By Theorem 4 e) the generator $(A, D(A))$ of the corresponding cosine operator function is given by the closure of the operator

$$C_c^2(0,\infty) \to L^p(\lambda), f \mapsto (x \mapsto x^2 f''(x) + xf'(x)),$$

i.e. for $f \in D(A)$ we have

$$\frac{\partial^2}{\partial t^2} C(t)f(x) = x^2 \frac{\partial^2}{\partial x^2} C(t)f(x) + x\frac{\partial}{\partial x} C(t)f(x)$$

in a generalized sense.

Since $\lim_{t\to\infty} \rho(xe^t)e^t = \lim_{t\to\infty} \rho(xe^{-t})e^{-t} = \lim_{t\to\infty} \rho(xe^{2t})e^{2t} = 0$ for every $x \in (0,\infty)$ it follows immediately from Theorem 12 that the cosine operator function is mixing, in particular hypercyclic on $L^p(\mu)$ for every $p \in [1,\infty)$.

Moreover, ρ is C_0-admissible for F and h but it follows from Theorem 15 that C is not hypercyclic on $C_{0,\rho}(0,\infty)$.

Acknowledgement. I want to thank P. Butzer for turning my attention to cosine operator functions.

References

[1] T. Bermúdez, A. Bonilla, A. Martinón, *On the existence of chaotic and hypercyclic semigroups on Banach spaces*, Proc. Amer. Math. Soc. 131 (2002), 2435-2441

[2] T. Bermúdez, A. Bonilla, J. A. Conejero, A. Peris, *Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces*, Studia Math. 170 (2005), 57-75

[3] A. Bonilla, P. Miana, *Hypercyclic and topologically mixing cosine functions on Banach spaces*, Proc. Amer. Math. Soc. 136 (2008), 519-528

[4] J. A. Conejero, A. Peris, *Linear transitivity criteria*, Topology Appl. 153 (2005), 767-773

[5] J. A. Conejero, V. Müller, A. Peris, *Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup*, J. Funct. Anal. 244 (2007), 342-348

[6] J. A. Conejero, E. M. Mangino, *Hypercyclic semigroups generated by Ornstein-Uhlenbeck operators*, to appear in Mediterr. J. Math.

[7] G. Costakis, A. Peris, *Hypercyclic semigroups and somewhere dense orbits*, C. R. Acad. Sci. Paris, Ser. I 335 (2002), 895-898

[8] W. Desch, W. Schappacher, G. F. Webb, *Hypercyclic and chaotic semigroups of linear operators*, Ergodic Theory Dynam. Systems 17 (1997), 793-819

[9] K. J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, Berlin, Heidelberg, New York, 2000

[10] K. G. Grosse-Erdmann, *Holomorphe Monster und universelle Funktionen*, Mitt. Math. Sem. Giessen 176 (1987)

[11] K. G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. 36 (1999), no. 3, 345-381

[12] K. G. Grosse-Erdmann, *Recent developments in hypercyclicity*, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 97 (2003), no. 2, 273-286

[13] T. Kalmes, *Hypercyclic, mixing, and chaotic C_0-semigroups induced by semiflows*, Ergodic Theory Dynam. Systems, 27 (2007), no. 5, 1599-1631

[14] T. Kalmes, *Hypercyclic C_0-semigroups and evolution families generated by first order differential operators*, Proc. Amer. Math. Soc. 137 (2009), 3833-3848

E-mail address: thomas.kalmes@math.tu-chemnitz.de

FACULTY OF MATHEMATICS, CHEMNITZ TECHNICAL UNIVERSITY, 09107 CHEMNITZ, GERMANY