Examples of quantitative universal approximation

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Abstract

Let $L := (L_j)$ be a sequence of continuous maps from a complete metric space $(X, d_X)$ to a separable metric space $(Y, d_Y)$. An element $x \in X$ is called $L$-universal for a subset $M$ of $Y$ if $F(x, M, \varepsilon) < \infty$ for all $\varepsilon > 0$, where

$$F(x, M, \varepsilon) := \sup_{y \in M} \inf_{j \in \mathbb{N}} \{ d_Y(y, L_j x) < \varepsilon \}.$$ 

In this article we obtain quantitative estimates for $F(x, M, \varepsilon)$ in a variety of examples arising in the theory of universal approximation.

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1 Introduction

Let $(X, d_X)$ be a complete metric space, let $(Y, d_Y)$ a separable metric space, and let $L := (L_j)_{j \in \mathbb{N}}$ be a sequence of continuous mappings $L_j : X \to Y$. An element $x \in X$ is called $L$-universal if

$$\forall n \in \mathbb{N} \ \forall y \in Y \ \exists N \in \mathbb{N}: \ d_Y(y, L_N x) < \frac{1}{n}.$$ 

We denote the set of all $L$-universal elements by $U(L)$. It is a $G_\delta$-set, due to the separability of $Y$. The sequence $L$ is called universal if $U(L) \neq \emptyset$.

Given a subset $M \subset Y$, one might ask how fast the elements of $M$ can be approximated by some $L$-universal element $x$, that is, how many elements of the sequence $(L_j x)_{j \in \mathbb{N}}$ are needed to cover $M$ by $B(L_j x, \varepsilon)$, $j = 1, \ldots, N$, the
\(\varepsilon\)-balls around \(L_j x\)? Evidently, the answer will be expressed in terms of the numbers:

\[
F(x, \mathcal{M}, \varepsilon) = F(x, \varepsilon) := \sup_{y \in \mathcal{M}} \inf \left\{ j \in \mathbb{N} : d_y(y, L_j x) < \varepsilon \right\}.
\]

Note that \(F(x, \mathcal{M}, \varepsilon)\) also depends on the metric \(d_y\). Obviously, if \(F(x, \mathcal{M}, \varepsilon)\) is finite for every \(\varepsilon > 0\), then \(\mathcal{M}\) must be \textit{totally bounded} (that is, \(\mathcal{M}\) can be covered by a finite number of \(\varepsilon\)-balls for every \(\varepsilon > 0\)).

When \(\mathcal{Y}\) is a Fréchet space, a natural metric to consider is

\[
d_y(y, z) := \sup_{n \in \mathbb{N}} \left( \min \left\{ p_n(y - z), \frac{1}{n} \right\} \right),
\]

where \((p_n)_{n \in \mathbb{N}}\) is an increasing sequence of seminorms defining the topology on \(\mathcal{Y}\). In this case \(d_y(y, z) < 1/n\) if and only if \(p_n(y - z) < 1/n\). If \(\mathcal{Y}\) is a Fréchet space, then the totally bounded subsets \(\mathcal{M}\) of \(\mathcal{Y}\) are precisely the relatively compact ones.

The above question was first studied in [10] for sequences of \textit{composition} and \textit{differentiation operators} on spaces \(H(\Omega)\) of holomorphic functions on a simply connected domain \(\Omega\) equipped with the compact-open topology. This is the Fréchet-space topology defined by the seminorms

\[
p_n(f) := \|f - g\|_{K_n} := \max_{z \in K_n} |f(z) - g(z)|,
\]

where \((K_n)_{n \in \mathbb{N}}\) is a compact exhaustion of \(\Omega\), i.e., \(K_n \subseteq \Omega\) compact, \(K_n\) is contained in the interior of \(K_{n+1}\) for each \(n \in \mathbb{N}\), and \(\bigcup_{n \in \mathbb{N}} K_n = \Omega\). Recall that, in this situation, the totally bounded subsets of \(H(\Omega)\) are exactly the \textit{normal families}.

Consider the sequence \(\mathcal{C} := (C_n)_{n \in \mathbb{N}}\) of composition operators, defined by

\[
C_n : H(\Omega_2) \to H(\Omega_1), \; f \mapsto f \circ \varphi_n,
\]

where \((\varphi_n)_{n \in \mathbb{N}}\) is a sequence of injective holomorphic mappings \(\varphi_n : \Omega_1 \to \Omega_2\) between open subsets \(\Omega_1, \Omega_2\) of \(\mathbb{C}\). Recall that \((\varphi_n)\) is called \textit{runaway} if, for every pair of compact sets \(K \subseteq \Omega_1, L \subseteq \Omega_2\), there exists an \(N \in \mathbb{N}\) with \(\varphi_N(K) \cap L = \emptyset\). This property characterizes the existence of \(\mathcal{C}\)-universal elements when \(\Omega_1 = \Omega_2\) and \(\Omega_1\) is not conformally equivalent to \(\mathbb{C}\setminus\{0\}\), cf. [3].

Now consider the sequence of differentiation operators \(\mathcal{D} := (D^n)_{n \in \mathbb{N}}\), where

\[
D : H(\Omega) \to H(\Omega), \; f \mapsto f'.
\]

In this case, the existence of \(\mathcal{D}\)-universal elements is equivalent to \(\Omega\) being simply connected, cf. [15].

In order to summarize the main results from [10] we introduce the following notation which will be used throughout this article. For a totally bounded subset \(\mathcal{M}\) of an arbitrary metric space \(\mathcal{Y}\) we define the \textit{n-th covering number}

\[
\lambda_n := \lambda_n(\mathcal{M}) := \min \left\{ l \in \mathbb{N} : \exists y_1, \ldots, y_l \in \mathcal{Y} : \mathcal{M} \subseteq \bigcup_{j=1}^{l} B(y_j, 1/n) \right\}.
\]

Obviously, the sequence \((\lambda_n(\mathcal{M}))_{n \in \mathbb{N}}\) measures the size of \(\mathcal{M}\) in a metrical sense.
For totally bounded subsets $M$ of $Y = H(\Omega)$, i.e. for normal families over $\Omega$, we need two more sequences. The first one, $(\gamma_n)_{n \in \mathbb{N}} = (\gamma_n(M))_{n \in \mathbb{N}}$ measures the approximative behavior of the Taylor/Faber expansions and is defined as the smallest integers with

$$
\|T_{\gamma_n} f - f\|_{K_n} < \frac{1}{n} \quad \forall f \in M,
$$

where $T_k f$ denotes the $k$-th Taylor/Faber polynomial on the compact set $K_n$.

The second sequence $(\sigma_n)_{n \in \mathbb{N}} = (\sigma_n(M))_{n \in \mathbb{N}}$ measures the speed of convergence of the anti-derivatives to 0 and is defined as the smallest integers with

$$
\|(T_m f)^{(-j)}\|_{K_n} < \frac{1}{n^2} \quad \forall f \in M, m \in \mathbb{N} \cup \{0\}, j \geq \sigma_n.
$$

Using this notation, the main results in [10] are summarized in the following theorem.

**Theorem 1.** (i) In case of $C$ (composition operators): For any normal family $M$, there exists a $C$-universal function $f$ with

$$
F(f, M, 2/n) \leq n(\lambda_n + 1) \quad (n \in \mathbb{N}).
$$

The set of all $C$-universal functions satisfying the above estimate contains a $G_\delta$-set, but is never dense. The set of $C$-universal functions $f$ satisfying

$$
F(f, M, 2/n) = O(n\lambda_n) \quad (n \to \infty)
$$

is dense.

(ii) In case of $D$ (differentiation operators): Let $\Omega$ be bounded. For any normal family $M$, there exists a $D$-universal function $f$ with

$$
F(f, M, 3/n) \leq n(\lambda_n + 1)(\gamma_n + \sigma_n(\lambda_n + 1)) \quad (n \in \mathbb{N}).
$$

For $\Omega = \mathbb{D}$, the unit disk, $\gamma_n = O(n \log(nM_{2n+1}))$ and $\sigma_n = O(\log(n^2M_{2n+1}))$ as $n \to \infty$, where $M_n := \sup f \in M \|f\|_{K_n}$. Hence, in this case,

$$
F(f, M, 1/n) = O(n^2\lambda_n \log(n\lambda_n \max\{1, M_{12n\lambda_n + 1}\})) \quad (n \to \infty).
$$

We introduce a special kind of fast approximating universal behavior.

**Definition 2.** A family of operators $L$ is called $m$-polynomial universal for $M$ if there is a $L$-universal element $x$ such that

$$
F(x, M, 1/n) = O(n^m) \quad (n \to \infty).
$$

For a totally bounded set $M \subseteq Y$ with covering numbers $\lambda_n$, i.e., $\lambda_n$ functions $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in Y$ cover $M$ with their $\frac{1}{n}$-neighborhoods, the set of all $m$-polynomial universal functions is given by

$$
\bigcup_{c \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{j=1}^{\lambda_n} H^{-1}(B(f_j^{(n)}, 1/n)) \cap U(L).
$$

This is a $G_\delta$-set. It is unknown if it is also a $G_\delta$-set.
In Section 2 we consider the above question for sequences of composition operators on kernels of differential operators and obtain exactly the same estimates as in the holomorphic case (compare Theorem 1 and Theorem 5). Section 3 contains an investigation of similar questions for universal Taylor series and comparisons of the results with those from [10] for differentiation operators. Finally, in Section 4, we consider some classic examples of normal families, like the set of normalized univalent functions $S$, and their covering numbers.

2 Composition operators on kernels of differential operators

In this section, let $\Omega \subset \mathbb{R}^d$ be open and let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be a non-zero polynomial. As usual, we equip $C^\infty(\Omega)$ with the Fréchet-space topology induced by the family of semi-norms $q_{K_n,0}(f) := \max_{x \in K_n,|\alpha| \leq n} |\partial^\alpha f(x)|$, where $(K_n)_{n \in \mathbb{N}}$ is a compact exhaustion of $\Omega$. We denote this Fréchet space by $E(\Omega)$ and the metric defined in (1.1) by $d$. As the differential operator $P(D)$ is continuous on $E(\Omega)$, it follows that the kernel of $P(D)$ in $E(\Omega)$, namely $\mathcal{N}_P(\Omega) := \{ f \in E(\Omega) : P(D)f = 0 \}$, is a closed subspace of $E(\Omega)$, and hence is itself a Fréchet space in a natural way. As is well known, $E(\Omega)$ is separable, so the same is true for $\mathcal{N}_P(\Omega)$.

In case of $P$ being hypoelliptic, the above mentioned Fréchet-space topology of $\mathcal{N}_P(\Omega)$ is induced by the family of semi-norms $(q_{K_n,0})_{n \in \mathbb{N}}$, see for example [8, Theorem 4.4.2]. We denote the corresponding metric defined in (1.1) by $d_0$. In particular, when dealing with the Cauchy–Riemann operator or the Laplace operator, we consider the spaces of holomorphic functions and harmonic functions respectively, equipped with the compact-open topology. As is well known, $\mathcal{N}_P(\Omega)$ is a Montel space if $P$ is hypoelliptic (this follows for example from [8, Theorem 4.4.2]), so in this case $\mathcal{M} \subset \mathcal{N}_P(\Omega)$ is relatively compact if and only if $\mathcal{M}$ is bounded, i.e., if and only if for every compact $K \subset \Omega$ we have

$$\sup_{f \in \mathcal{M}} q_{K,0}(f) < \infty.$$  

**Definition 3.** (i) Let $\varphi : \Omega \to \Omega$ be a $C^\infty$-diffeomorphism. Then $P$ is called $\varphi$-invariant if, for any $f \in C^\infty(\Omega)$, we have $f \circ \varphi \in \mathcal{N}_P(\Omega)$ whenever $f \in \mathcal{N}_P(\Omega)$. If $P$ is $\varphi$-invariant and $\varphi^{-1}$-invariant, then we call $P$ completely $\varphi$-invariant.

(ii) An open subset $U \subset \Omega$ is called $P$-approximable in $\Omega$ if $\{ f|_U : f \in \mathcal{N}_P(\Omega) \}$ is dense in $\mathcal{N}_P(U)$.

**Remark 4.** (i) If $P$ is $\varphi$-invariant, then the mapping

$$C_\varphi : \mathcal{N}_P(\Omega) \to \mathcal{N}_P(\Omega), \; f \mapsto f \circ \varphi$$

is well-defined and linear. Moreover, for compact $K \subset \mathbb{R}^d$ and $n \in \mathbb{N}_0$, we obviously have $q_{K,n}(C_\varphi f) \leq Mq_{\varphi(K),n}(f)$ for all $f \in E(\Omega)$, where $M > 0$ is
a suitable constant depending on $K$ and $n$. Thus $C_\varphi$ is a continuous, linear operator on $\mathcal{M}_P(\Omega)$.

(ii) If, for the $C^\infty$-diffeomorphism $\varphi : \Omega \to \Omega$, there is $g \in L^1(\Omega)$ such that the set $\{x \in \Omega : g(x) = 0\}$ is nowhere dense in $\Omega$ and $P(D)(C_\varphi(f)) = gC_\varphi(P(D)f)$ for every $f \in L^1(\Omega)$, then it follows immediately that $P$ is completely $\varphi$-invariant. In case of $P(D)$ being the Cauchy–Riemann, Laplace or heat operator, it is shown in [9, Proposition 3.6] that this condition on $\varphi$ is already necessary for $P$ to be $\varphi$-invariant. Moreover, the same is true in case of $P(D)$ being the wave operator, under the mild additional assumption that $\varphi$ does not mingle the time variable with the space variables and vice versa. It should be noted that in [9] the term “$\varphi$-invariance” is used for what we call complete $\varphi$-invariance here. Nevertheless, the proof of [9, Proposition 3.6] uses only that $f \circ \varphi \in \mathcal{M}_P(\Omega)$ for every $f \in \mathcal{M}_P(\Omega)$.

Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of $C^\infty$-diffeomorphisms of $\Omega$ such that $P$ is completely $\varphi_n$-invariant for every $n \in \mathbb{N}$. There are several articles dealing with the existence of universal functions for $(C_{\varphi_n})_{n \in \mathbb{N}}$ for special $P(D)$, in particular for the Cauchy–Riemann or the Laplace operator, see e.g. [3], [4], [6]. For arbitrary $P$, a characterization is given in [9] for the case that $\Omega$ has convex components.

Our first result in this section is the following theorem.

**Theorem 5.** Let $(\varphi_m)_{m \in \mathbb{N}}$ be a sequence of $C^\infty$-diffeomorphisms on $\Omega$ such that $P$ is completely $\varphi_m$-invariant for every $m \in \mathbb{N}$. Assume that, for every compact subset $K$ of $\Omega$, there are a bounded open neighborhood $U \subset \Omega$ of $K$ with $\overline{U} \subset \Omega$ and $m \in \mathbb{N}$ such that $\varphi_m(U) \cup U$ is $P$-approximable and $\varphi_m(U) \cap U = \emptyset$. Then there is a strictly increasing sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$ such that, for any $\mathcal{M} \subset \mathcal{M}_P(\Omega)$ relatively compact, there is a universal function $u$ for $(C_{\varphi_{m_n}})_{n \in \mathbb{N}}$ such that

$$F(u, \mathcal{M}, 2/n) \leq n(\lambda_n + 1) \quad \forall n \in \mathbb{N}.$$ 

In order to make the proof of the above theorem more transparent, we first prove the following lemma.

**Lemma 6.** Under the hypotheses of Theorem 5, for any compact exhaustion $(K_n)_{n \in \mathbb{N}}$ of $\Omega$, there is a strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ of natural numbers such that, for any sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_P(\Omega)$, there is $v \in \mathcal{M}_P(\Omega)$ with

$$q_{K_n, m_n}(f_n - C_{\varphi_{m_n}}(v)) < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$ 

**Proof.** Fix a compact exhaustion $(K_n)_{n \in \mathbb{N}}$ of $\Omega$ and a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_P(\Omega)$. We simply write $C_n$ in place of $C_{\varphi_n}$.

We start by constructing a sequence of bounded, open subsets $(U_n)_{n \in \mathbb{N}}$ of $\Omega$, sequences of natural numbers $(m_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$, and a sequence $(M_n)_{n \in \mathbb{N}}$ in $(1, \infty)$, such that:

(i) $\forall n \in \mathbb{N} : K_n \subset U_n \subset \overline{U_n} \subset \Omega$,

(ii) $\forall n \in \mathbb{N} : \varphi_{m_n}(U_n) \cap U_n = \emptyset$ and $\varphi_{m_n}(U_n) \cup U_n$ is $P$-approximable in $\Omega$,

(iii) $(m_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$ are strictly increasing, with $r_n \geq n + 1$ for each $n \in \mathbb{N}$,
(iv) \((M_n)_{n \in \mathbb{N}}\) is non-decreasing,

(v) \(\forall n \in \mathbb{N}, f \in \mathcal{M}_P(\Omega) : q_{K,n}(C_m(f)) \leq M_nq_{K,n}(f),\)

(vi) \(\forall n \in \mathbb{N} : K_{r_n} \subset U_{n+1}.\)

By hypothesis, there exists a bounded open neighborhood \(U_1 \subset \Omega\) of \(K_1\) with \(\overline{U_1} \subset \Omega\), and there exists \(m_1 \in \mathbb{N}\) with \(\varphi_{m_1}(U_1) \cap U_1 = \emptyset\) and \(\varphi_{m_1}(U_1) \cup U_1\) being \(P\)-approximable in \(\Omega\). Moreover, by the continuity of \(C_{m_1}\), there are \(r_1 \in \mathbb{N}, r_1 \geq 2\) and \(M_1 > 1\) with \(q_{K,1}(C_{m_1}(f)) \leq M_1q_{K,1}(f)\).

Assume that \(U_1, \ldots, U_n, m_1, \ldots, m_n, r_1, \ldots, r_n\) and \(M_1, \ldots, M_n\) have already been constructed. For the compact set

\[
K := \overline{U_n} \cup K_{r_{n+1}} \cup \bigcup_{j=1}^{m_n} \varphi_{j}(\overline{U_n}),
\]

there exist, by hypothesis, a bounded open neighborhood \(U_{n+1} \subset \overline{U_{n+1}} \subset \Omega\) and \(m_{n+1} \in \mathbb{N}\) with \(U_{n+1} \cap \varphi_{m_{n+1}}(U_{n+1}) = \emptyset\) and \(U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1})\) being \(P\)-approximable in \(\Omega\). From \(U_{n+1} \cap \varphi_{m_{n+1}}(U_{n+1}) = \emptyset\) and the definition of \(K\), it follows that \(m_{n+1} > m_n\). By the continuity of \(C_{m_{n+1}}\), there are \(M_{n+1}\) and \(r_{n+1}\) with

\[
q_{K_{n+1},n+1}(C_{m_{n+1}}(f)) \leq M_{n+1}q_{K_{n+1},n+1}(f)
\]

for any \(f \in \mathcal{M}_P(\Omega)\), where, without loss of generality, we may assume that \(M_{n+1} \geq M_n\) and \(r_{n+1} > \max\{r_n, n + 2\}\).

We observe that, by (iii) and (vi), we have \(K_{n+1} \subseteq K_r \subseteq U_{n+1}\) for every \(n \in \mathbb{N}\).

Next, we recursively construct a sequence \((v_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}_P(\Omega)\) such that:

(a) \(\forall n \in \mathbb{N} : q_{K,n}(f_n - C_m(v_n)) < \frac{1}{2^n},\)

(b) \(\forall n \in \mathbb{N} : q_{K,n}(v_{n+1} - v_n) < \frac{1}{2^{n+1}M_{n+1}}.\)

Indeed, for \(n = 1\), consider

\[
w_1 : U_1 \cup \varphi_{m_1}(U_1) \to \mathbb{C}, \quad w_1(x) := \begin{cases} 0, & \text{if } x \in U_1, \\ f_1(\varphi_{m_1}^{-1}(x)), & \text{if } x \in \varphi_{m_1}(U_1). \end{cases}
\]

Since \(U_1 \cap \varphi_{m_1}(U_1) = \emptyset\), the map \(w_1\) is well-defined, and \(w_1 \in \mathcal{M}_P(U_1 \cup \varphi_{m_1}(U_1))\) follows from the complete \(\varphi_{m_1}\)-invariance of \(P\). Fix \(\psi_1 \in \mathcal{D}(U_1)\) such that \(\psi_1 = 1\) in a neighborhood of \(K_1\). Obviously, \(\psi_1 \circ \varphi_{m_1}^{-1} \in \mathcal{D}(\varphi_{m_1}(U_1))\), so that, for any \(f \in \mathcal{M}_P(U_1 \cup \varphi_{m_1}(U_1))\), we have \((\psi_1 \circ \varphi_{m_1}^{-1})f \in C^\infty(\Omega)\) in a natural way. Therefore,

\[
p_1(f) := q_{K,1}(\psi_1 \circ \varphi_{m_1}^{-1})f
\]

defines a continuous semi-norm on \(\mathcal{M}_P(U_1 \cup \varphi_{m_1}(U_1))\). The \(P\)-approximability of \(U_1 \cup \varphi_{m_1}(U_1)\) in \(\Omega\) and the continuity of the seminorm \(p_1\) imply the existence of \(v_1 \in \mathcal{M}_P(\Omega)\) with

\[
p_1(v_1 - w_1) < \frac{1}{4M_1}.
\]
Since, from the definition of $w_1$, we have $(\psi_1 \circ \varphi_{m_1}^{-1}) w_1 = (\psi_1 \circ \varphi_{m_1}^{-1})(f_1 \circ \varphi_{m_1}^{-1})$, and because $\psi_1 = 1$ in a neighborhood of $K_1$, this implies

$$q_{K_1,1}(f_1 - C_{m_1}(v_1)) = q_{K_1,1} \left( C_{m_1}(\psi_1 \circ \varphi_{m_1}^{-1})(f_1 \circ \varphi_{m_1}^{-1} - v_1) \right) \leq M_1 q_{K_1,1} \left( (\psi_1 \circ \varphi_{m_1}^{-1})(f_1 \circ \varphi_{m_1}^{-1} - v_1) \right) = M_1 p_1 (w_1 - v_1) < \frac{1}{4},$$

where we used (v) in the second step.

Assuming that $v_1, \ldots, v_n$ have already been constructed, we consider

$$w_{n+1} : U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1}) \to C,$$

$$w_{n+1}(x) := \begin{cases} v_n(x), & \text{if } x \in U_{n+1}, \\ f_{n+1}(\varphi_{m_{n+1}}^{-1}(x)) & \text{if } x \in \varphi_{m_{n+1}}(U_{n+1}). \end{cases}$$

Then, as for $w_1$, we have $w_{n+1} \in \mathcal{P}(U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1}))$. Fix $v_{n+1} \in \mathcal{P}(U_{n+1})$ such that $v_{n+1} = 1$ in a neighborhood of $K_n \supseteq K_{n+1}$. As above,

$$p_{n+1}(f) := q_{K_{n+1},1}(v_{n+1} \circ \varphi_{m_{n+1}}^{-1}(f)) + q_{K_n,n}(f)$$

defines a continuous semi-norm on $\mathcal{P}(U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1}))$, so that the $P$-approximability of $U_{n+1} \cup \varphi_{m_{n+1}}(U_{n+1})$ in $\Omega$ yields $v_{n+1} \in \mathcal{P}(\Omega)$ with

$$p_{n+1}(v_{n+1} - w_{n+1}) < \frac{1}{2^{n+1} M_{n+1}}.$$

Again, since $(\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1}) w_{n+1} = (\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})(f_{n+1} \circ \varphi_{m_{n+1}}^{-1})$, and as $\psi_{m_{n+1}} = 1$ in a neighborhood of $K_n \supseteq K_{n+1}$, this implies

$$q_{K_{n+1},1}(f_{n+1} - C_{m_{n+1}}(v_{n+1})) = q_{K_{n+1},1,n+1} \left( C_{m_{n+1}}(\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})(f_{n+1} \circ \varphi_{m_{n+1}}^{-1} - v_{n+1}) \right) \leq M_{n+1} q_{K_{n+1},1,n+1} \left( (\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})(f_{n+1} \circ \varphi_{m_{n+1}}^{-1} - v_{n+1}) \right) = M_{n+1} q_{K_{n+1},1,n+1} \left( (\psi_{n+1} \circ \varphi_{m_{n+1}}^{-1})(w_{n+1} - v_{n+1}) \right) \leq M_{n+1} p_{n+1}(v_{n+1} - w_{n+1}) < \frac{1}{2^{n+1} M_{n+1}} \cdot \frac{1}{2^{n+1}},$$

where we used (v) in the second step. Moreover, since $K_n \subset U_{n+1}$, and since, by definition, $v_n|U_{n+1} = w_{n+1}|U_{n+1}$, we obtain

$$q_{K_n,n}(v_{n+1} - v_n) = q_{K_n,n}(v_{n+1} - w_{n+1}) \leq p_{n+1}(v_{n+1} - w_{n+1}) < \frac{1}{2^{n+1} M_{n+1}}.$$

thereby finishing the construction of $(v_n)_{n \in \mathbb{N}}$. Because of the inclusion $K_n \supseteq K_n$, the fact that $M_n \geq 1$ and (b), we have

$$\forall n \in \mathbb{N} : q_{K_n,n}(v_{n+1} - v_n) < \frac{1}{2^{n+1}}.$$

so that $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{P}(\Omega)$, and hence convergent. We set $v := \lim_{n \to \infty} v_n$, and observe that $v = v_n + \sum_{j=n}^{\infty} (v_{j+1} - v_j)$ for every $n \in \mathbb{N}$. 7
From the continuity of $C_{m_n}$, and using (a), (v), (b), and (iv), we finally get that, for $n \in \mathbb{N}$,

$$q_{K_n,n}(f_n - C_{m_n}(v)) = q_{K_n,n}(f_n - C_{m_n}(v_n)) - \sum_{j=n}^{\infty} C_{m_n}(v_{j+1} - v_j)$$

$$\leq \frac{1}{2n} + \sum_{j=n}^{\infty} q_{K_n,n}(C_{m_n}(v_{j+1} - v_j))$$

$$\leq \frac{1}{2n} + \sum_{j=n}^{\infty} M_n q_{K_n,n}(v_{j+1} - v_j)$$

$$\leq \frac{1}{2n} + \sum_{j=n}^{\infty} M_n q_{K_n,n}(v_{j+1} - v_j)$$

$$\leq \frac{1}{2n} + \sum_{j=n}^{\infty} M_n \frac{M_n}{2^{j+1}M_{j+1}} < \frac{1}{n}.$$ 

This completes the proof of the lemma.

Proof of Theorem 5. Let $(K_n)_{n \in \mathbb{N}}$ be the compact exhaustion of $\Omega$ defining the metric $d$ on $\mathcal{M}(\Omega)$. For $n \in \mathbb{N}$, let $f_{(n)}^{(1)}, \ldots, f_{(n)}^{(\lambda_n)} \in \mathcal{M}(\Omega)$ be such that

$$M \subseteq \bigcup_{j=1}^{\lambda_n} B(f_{j}^{(n)}, 1/n),$$

and let $(g_n)_{n \in \mathbb{N}}$ be a dense sequence in $\mathcal{M}(\Omega)$. We define $(f_n)_{n \in \mathbb{N}}$ to be the sequence

$$f_{1}^{(1)}, \ldots, f_{\lambda_n}^{(1)}, g_1, f_{1}^{(2)}, \ldots, f_{\lambda_n}^{(2)}, g_2, f_{1}^{(3)}, \ldots, f_{\lambda_n}^{(3)}, g_3, \ldots$$

Applying Lemma 6 gives an increasing sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$ and $u \in \mathcal{M}(\Omega)$ such that

$$q_{K_n,n}(f_n - C_{\varphi_{m_n}}(u)) < \frac{1}{n}.$$ 

Since $\{g_n : n \in \mathbb{N}\}$ is dense in $\mathcal{M}(\Omega)$, it follows that $u$ is universal for $(C_{\varphi_{m_n}})_{n \in \mathbb{N}}$. Now fix $f \in \mathcal{M}$ and $n \in \mathbb{N}$. Then $d(f, f_{j}^{(n)}) < 1/n$ for some $1 \leq j \leq \lambda_n$. Because $f_{j}^{(n)} = f_N$ for some $n \leq N \leq \sum_{j=1}^{\lambda_n} (\lambda_j + 1) \leq n(\lambda_n + 1)$, and because

$$q_{K_N,n}(f_N - C_{\varphi_{M_N}}(u)) < \frac{1}{N},$$

that is

$$d(f_{j}^{(n)}, C_{\varphi_{m_n}}(u)) < \frac{1}{N},$$

the result follows.

In order to verify the hypothesis of Theorem 5 in some concrete situations we recall the following results about approximation of zero solutions of differential equations. Part (i) of the next theorem is the Malgrange–Lax Theorem, cf. [8, Theorem 4.4.5], while part (ii) is due to Hörmander, see e.g. [8, Theorem 10.5.2].
Theorem 7. Let $U \subseteq \Omega$ be open.

(i) Assume that $P$ is elliptic. If $\Omega \setminus U$ is not the disjoint union $F \cup K$, where $K$ is compact and non-empty and $F$ is closed in $\Omega$, then $U$ is $P$-approximable in $\Omega$.

(ii) Suppose that every $\mu \in \mathcal{E}'(\Omega)$ with $\text{supp} P(-D)\mu \subset U$ already belongs to $\mathcal{E}'(U)$. Then $U$ is $P$-approximable in $\Omega$.

Remark 8. (i) Let $\hat{\Omega}$ denote the one-point compactification of $\Omega$. It is easily seen that the condition in (i) of Theorem 7 is equivalent to $\hat{\Omega} \setminus U$ being connected while part (ii) immediately implies the $P$-approximability in $\hat{\Omega}$ of every $U \subset \Omega$ with convex components.

(ii) It is shown in [9, Proof of Corollary 4.6] that, if $\varphi$ satisfies the condition under (ii) of Remark 4, and if $K \subset \Omega$ is compact, has only convex components and satisfies $\varphi(K) \cap K = \emptyset$, then $\varphi(K^\circ) \cup K^\circ$ is $P$-approximable in $\Omega$, where $K^\circ$ denotes the interior of $K$.

(iii) Assume that $\Omega$ has only convex components and that every element of the sequence $(\varphi_m)_{m \in \mathbb{N}}$ of $C^\infty$-diffeomorphisms satisfies the condition (ii) of Remark 4. Then it follows from (ii) above that the assumption of Theorem 5 is fulfilled if and only if, for every compact subset $K$ of $\Omega$, there is $m \in \mathbb{N}$ with $\varphi_m(K) \cap K = \emptyset$.

Corollary 9. Let $(\varphi_m)_{m \in \mathbb{N}}$ be a sequence of $C^\infty$-diffeomorphisms of $\Omega$ such that $P$ is completely $\varphi_m$-invariant for every $m \in \mathbb{N}$. Assume, further, that for any compact subset $K \subset \Omega$, there is $m \in \mathbb{N}$ with $\varphi_m(K) \cap K = \emptyset$. Then there is an increasing sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$ for which the following hold:

(i) Assume that $\Omega$ is contractible and that $\Omega$ has the complementation property, i.e., given any compact subset $K \subset \Omega$, there is at most one component of $\Omega \setminus K$ whose closure in $\Omega$ is not compact. If $P$ is elliptic, then, for any relatively compact subset $M$ of $\mathcal{A}_P(\Omega)$, there is a universal function $u \in \mathcal{A}_P(\Omega)$ for $(C\varphi_m)_{m \in \mathbb{N}}$ such that $F(u, M, 2/n) \leq n(\lambda_n + 1)$ for each $n \in \mathbb{N}$.

(ii) If $P$ is arbitrary, each $\varphi_m$ satisfies the condition (ii) from Remark 4, and $\Omega$ has only convex components, then, for any relatively compact subset $M$ of $\mathcal{A}_P(\Omega)$, there is a universal function $u \in \mathcal{A}_P(\Omega)$ for $(C\varphi_m)_{m \in \mathbb{N}}$ such that $F(u, M, 2/n) \leq n(\lambda_n + 1)$ for each $n \in \mathbb{N}$.

Proof. Part (ii) follows from the hypothesis, Remark 8, and Theorem 5.

In order to show (i), it is straightforward to verify that, with

$$U_n := \{x \in \Omega : |x| < n \text{ and dist}(x, \Omega^c) > 1/n\},$$

the set $\Omega \setminus U_n$ is not the disjoint union of a non-empty, compact set $K$ and a set $F$ closed in $\Omega$. As $\varphi_m$ is a homeomorphism, the same holds for $\varphi_m(U_n)$ for arbitrary $m$. By hypothesis, there is $m_0$ such that $U_n \cap \varphi_{m_0}(U_n) = \emptyset$. The contractibility of $\Omega$ easily gives that every continuous mapping $g : \Omega \rightarrow S^1$ is homotopic to a constant. Together with the complementation property of $\Omega$, this implies the unicoherence of $\Omega$ (see e.g. [5, Theorem 4.12]), so that for the two connected and closed sets $\Omega \setminus U_n$ and $\Omega \setminus \varphi_{m_0}(U_n)$ covering $\Omega$, their intersection $\Omega \setminus (U_n \cup \varphi_{m_0}(U_n))$ is also connected. Therefore, $U_n \cup \varphi_{m_0}(U_n)$ is $P$-approximable in $\Omega$, by Theorem 7 (i). Part (i) now follows from this and from Theorem 5. \qed
3 Universal Taylor series

Let \( \Omega \subseteq \mathbb{C} \) be a simply connected domain. For \( L \subset \mathbb{C} \setminus \Omega \) compact with connected complement and \( \zeta \in \Omega \), we consider the sequence \( T_L^\zeta = (T_{L,n}^\zeta)_{n \in \mathbb{N}} \) of linear operators

\[
T_{L,n}^\zeta : H(\Omega) \to A(L), \quad f \mapsto T_{L,n}^\zeta f(z) := \sum_{\nu=0}^{n} a_{\nu}^{(\zeta)}(z - \zeta)^\nu,
\]

where \( a_{\nu}^{(\zeta)} \) denotes the \( \nu \)-th Taylor coefficient of \( f \) expanded about \( \zeta \), and \( A(L) \) denotes the space of all continuous functions on \( L \) that are holomorphic in the interior of \( L \). Endowing \( A(L) \) with the sup-norm \( \|f\|_L \), it follows from Mergelyan’s theorem that \( \{f|_L : f \in A(L)\} \) is dense in \( A(L) \) for any compact superset \( \tilde{L} \) of \( L \).

As shown in [13, Lemma 2.1], there exists a sequence \( (L_k)_{k \in \mathbb{N}} \) of compact sets \( L_k \subset \mathbb{C} \setminus \Omega \) with connected complement such that, for every compact subset \( L \subset \mathbb{C} \setminus \Omega \) with connected complement, there is \( k_0 \in \mathbb{N} \) with \( L \subset L_{k_0} \). The set of all universal Taylor series in the sense of [13] is then given by

\[
U(\zeta) := \bigcap_{k \in \mathbb{N}} U(T_{L_k}^\zeta),
\]

and it is shown in [12, Theorem 2] that

\[
U(\zeta_1) = U(\zeta_2)
\]

for any \( \zeta_1, \zeta_2 \in \Omega \). Abusing our former notation we simply write \( U(\mathcal{T}) \) for these equal sets, that is, \( f \in U(\mathcal{T}) \) if and only if the set of the Taylor polynomials of \( f \) expanded about an arbitrary \( \zeta \in \Omega \) is dense in any \( A(L) \), where \( L \subset \mathbb{C} \setminus \Omega \) is compact and has connected complement.

Our first aim is to compare how fast a normal family \( \mathcal{M} \) may be approximated by the partial sums of a universal Taylor series \( f \in U(\mathcal{T}) \) with the speed of approximation by the derivatives of a function \( g \in U(\mathcal{D}) \). In a second step we then estimate the possible speed of approximation for \( f \in U(\mathcal{T}) \). To help us in pursuit of these goals, we introduce the following notion:

**Definition 10.** Let \( \Omega \subseteq \mathbb{C} \) be open, let \( L_k \subset \mathbb{C} \setminus \Omega \) be compact, and let \( f_k \in A(L_k) \). We say \( f \in H(\Omega) \) has a **uniformly universal power series in \( \zeta_1 \in \Omega \)** for \( (f_k, L_k)_{k \in \mathbb{N}} \) if there is a sequence of natural numbers \( (N_k)_{k \in \mathbb{N}} \) such that

\[
\forall 1 \leq j \leq k \ \exists 1 \leq n \leq N_k : \|f_j - T_n(\zeta_1) f\|_{L_j} < \frac{1}{2^j}.
\]

Let \( \mathcal{P}_Q \) be the set of all polynomials with coefficients in \( \mathbb{Q} + i\mathbb{Q} \), and let \( K_{\mathbb{C}\setminus\Omega} := (L_k) \) be a sequence of compact sets in \( \mathbb{C} \setminus \Omega \) as above. If \( \Omega \) is simply connected and if \( (f_k, L_k) \) contains each element \((p, L) \in \mathcal{P}_Q \times K_{\mathbb{C}\setminus\Omega} \) infinitely often, then a uniformly universal power series \( f \) for \( (f_k, L_k) \) is a universal Taylor series, i.e. \( f \in U(\mathcal{T}) \).

**Remark 11.** (i) Let \( \Omega = \mathbb{D} \), let \( f \) be a uniformly universal power series in \( \zeta_1 = 0 \) for \( (f_k, L_k), k > 2 \), with

\[
f_k \equiv 0, \quad L_k := \left\{ z : \left| z - \frac{3}{2} k \right| \leq k \right\},
\]

and let \( f \in U(\mathcal{T}) \). Then for \( \zeta \in \Omega \), the sequence \( (T_{L,n} f(\zeta))_{n \in \mathbb{N}} \) is a Cauchy sequence in \( A(L) \). Therefore, \( f \) is a uniformly universal power series in \( \zeta = 0 \), hence \( f \in U(\mathcal{T}) \).
and let \((N_k)\) be a sequence of numbers as in Definition 10. Assume only that 
\[ \|T_{N_k}^{(0)} f\|_{L_k} \leq 1 \] for each \(k > 2\). Then the Taylor coefficients satisfy

\[ |a_\nu|^{1/\nu} \leq k \log \frac{2}{k-1} \quad \text{for all } \nu \text{ with } \tilde{N}_k := \left\lfloor \frac{N_k}{\log k} \right\rfloor + 1 \leq \nu \leq N_k, \]

cf. [7, p.84]. Thus approximation by partial sums occurs with rather large blocks of small coefficients. Assume, further, that \(f \in U(T)\), so in particular the radius of convergence of \(f\) is 1. Since

\[ \limsup_{\nu \to \infty} |a_\nu|^{1/\nu} = 0, \quad I := \mathbb{N} \cap \bigcup_{k \in \mathbb{N}} [\tilde{N}_k, N_k], \]

for every \(\varepsilon > 0\) the power series of \(f\) must also have infinitely many Taylor coefficients \(a_\nu\) with \(|a_\nu|^{1/\nu} \geq 1 - \varepsilon\), \(\nu \in \mathbb{N}\setminus I\). More precisely, the set of indices \(\kappa := \{k \in \mathbb{N} : \exists \nu \in (N_{k-1}, \tilde{N}_k) \text{ with } |a_\nu|^{1/\nu} > k \log \frac{2}{k-1} \} \subset \{k \in \mathbb{N} : N_{k-1} < \tilde{N}_k\}\) is infinite. Thus, on the infinite set \(\kappa\) we have

\[ \frac{N_k}{N_{k-1}} \geq \log k, \quad k \in \kappa. \]

The same holds if \(f\) has finite radius of convergence, without necessarily belonging to \(U(T)\).

(ii) We compare the above quotient \(N_k/N_{k-1}\) with a similar one for a function \(g \in U(D)\). In [10, Theorem 8], a function \(g \in U(D) \cap H(\Omega)\) (where \(H(\Omega)\) is endowed with the natural metric as in (1.1) and seminorms as in (1.2)) is constructed, which is fast approximating for a normal family \(M\). For appropriate functions \(f_j, j = 1, \ldots, k\), define \((N_k)\) to be a sequence of natural numbers with

\[ \forall 1 \leq j \leq k \quad \exists 1 \leq n \leq N_k : \|f_j - g^{(n)}\|_{K_j} < \frac{1}{j}. \]

For the constructed function \(g \in U(D)\), we obtain from [10, Proof of Theorem 8] that \(N_k \leq N_{k-1} + \sigma_k + \gamma_k\), where \(\sigma_k\) and \(\gamma_k\) are defined as in the paragraph preceding Theorem 1. Considering \(M = \{0\}\), i.e. \(f_j \equiv 0\), as in (i), \(\sigma_k = \gamma_k := k\) is a possible choice, and so is \(N_k := k(k + 1)\). Hence

\[ \frac{N_k}{N_{k-1}} = \frac{k + 1}{k - 1}, \]

which is bounded, and not strictly increasing to \(\infty\) on a subsequence \(\kappa\), as is the case for \(f \in U(T)\).

This simple example already illustrates the tremendous difference between the speeds of approximation by \(f \in U(T)\) and \(g \in U(D)\). To elucidate this difference, we remark that successive derivatives of a function may change rather quickly, while in universal approximation successive partial sums change rather slowly, which is expressed by large blocks of rather small coefficients, namely so-called Ostrowski gaps, cf. [7]. Even the boundedness of the partial sums on a non-polar set \(\overline{E} \subset \mathbb{C} \setminus \overline{B}\) causes small coefficients, in this case so-called Hadamard–Ostrowski gaps, as recently shown in [2].
Nevertheless, we also want to give results in the other direction by showing which speeds of approximation are possible, this time by estimating possible upper bounds, not for \( F(f, \mathcal{M}, 1/n) \), but for the numbers \( N_k \) as defined in Definition 10. In order to construct a universal function \( f \) with small \( F(f, \mathcal{M}, 1/n) \), we first find a sequence \( \{f_j\} \) containing appropriately chosen centers, whose balls \( B(f_j, 1/n) \) cover \( \mathcal{M} \). Their number is \( \lambda_n(\mathcal{M}) \), the \( n \)-covering number of \( \mathcal{M} \). Then these centers \( f_j \) will be approximated by the first \( N_k \) Taylor polynomials of \( f \), i.e., by \( T_{f_j}^{(n)} f, j \in \{1, \ldots, N_k\} \). Finally, \( F(f, \mathcal{M}, 1/n) \) and \( N_k \) are connected, since \( F(f, \mathcal{M}, 1/n) \leq N_k \) for some \( k \) which may depend on \( \lambda_n(\mathcal{M}) \).

With regard to estimate \( N_k \), we start by recalling some results on best polynomial approximation, cf. [1]. For a continuous complex-valued function \( f \) on a compact set \( K \) in the plane, let

\[
d_n := d_n(f, K) := \inf \{ \| f - p \|_K : p \in \mathcal{P}_n \},
\]

where \( \mathcal{P}_n \) is the vector space of complex polynomials of degree at most \( n \). Recall that a Green’s function \( g_K \) for \( \mathbb{C} \setminus K \) is a continuous function \( g_K : \mathbb{C} \to [0, +\infty) \) which is identically equal to zero on \( K \), harmonic on \( \mathbb{C} \setminus K \), and has a logarithmic singularity at infinity, in the sense that \( g_K(z) - \log |z| \) is harmonic at infinity.

**Theorem 12** (Walsh). Let \( K \) be a compact subset of the plane such that \( \mathbb{C} \setminus K \) is connected and has a Green’s function \( g_K \). For \( R > 1 \), let \( D_R := \{ z \in \mathbb{C} : g_K < \log R \} \). Let \( f \) be continuous on \( K \). Then \( \limsup_{n \to \infty} d_n(f, K)^{1/n} \leq 1/R \) if and only if \( f \) is the restriction to \( K \) of a function holomorphic in \( D_R \).

The proof of the “if” part of this theorem for the case \( K = [-1, 1] \), given in [1, Section 2] by the use of duality theory, in fact provides the following result which will be crucial for our considerations. We include its proof here for the reader’s convenience.

**Lemma 13.** Let \( \Omega \) be an open subset of \( \mathbb{C} \), and let \( K \) be a compact subset of \( \Omega \) such that \( \mathbb{C} \setminus K \) is connected and has a Green’s function \( g_K \). Let \( R > 1 \) be such that \( \overline{D_R} \subset \Omega \). Then, for every \( f \in H(\Omega) \), we have

\[
\forall 1 < r < \rho < R : \quad d_n(f, K) \leq \| f \|_{\partial D_n} \left( \frac{r}{\rho} \right)^n \frac{8 \lambda(D_R \setminus D_\rho)}{\pi \dist(\partial D_R, D_\rho) \dist(\partial D_r, K)},
\]

where \( \lambda \) denotes Lebesgue measure on \( \mathbb{C} \).

**Proof.** Let \( 1 < r < \rho < R \). Choose \( \phi \in \mathcal{D}(\Omega) \) with \( \text{supp} \phi \subset D_\rho \) and \( \phi = 1 \) in a neighborhood of \( D_\rho \), and set \( F := \phi f \in \mathcal{D}(\Omega) \subset \mathcal{D}(\mathbb{R}^2) \). Then it follows, as in [1, Section 2], that

\[
d_n = d_n(f, K) = \int_{D_{R} \setminus D_\rho} \tilde{\mu}(z) \frac{\partial}{\partial z} F(z) \, d\lambda(z),
\]

where \( \lambda \) denotes Lebesgue measure on \( \mathbb{C} \), and \( \tilde{\mu} \in H(\mathbb{C} \setminus K) \) satisfies

\[
\forall z \in \mathbb{C} \setminus D_r : \quad |\tilde{\mu}(z)| \leq \frac{1}{\pi \dist(\partial D_r, K)} \left( \exp(\log r - g_K(z)) \right)^n.
\]

In particular, for all \( z \in \mathbb{C} \setminus D_\rho \), we have

\[
|\tilde{\mu}(z)| \leq \frac{1}{\pi \dist(\partial D_r, K)} \left( \exp(\log r - \log \rho) \right)^n = \frac{1}{\pi \dist(\partial D_r, K)} \left( \frac{r}{\rho} \right)^n,
\]

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so that, by (3.1), by the identity \( \frac{\partial}{\partial z} F(z) = f(z) \frac{\partial}{\partial z} \phi(z) \) and by the maximum principle applied to \( f \), we have

\[
d_n \leq \frac{1}{\pi} \text{dist}(\partial D_r, K) \left( \frac{r}{\rho} \right)^n \int_{D_r \setminus D_\rho} \left| f(z) \frac{\partial}{\partial z} \phi(z) \right| d\lambda(z)
\]

\[
\leq \|f\|_{\partial D_n} \left( \frac{r}{\rho} \right)^n \frac{1}{\pi} \text{dist}(\partial D_r, K) \sup_{z \in D_n} \left| \frac{\partial}{\partial z} \phi(z) \right| \lambda(D_R \setminus D_\rho).
\]

Let \( \delta := \text{dist}(\partial D_R, D_\rho) \) be the distance from \( D_\rho \) to the complement of \( D_R \). According to [8, Proof of Theorem 1.4.2], we can choose \( \phi \) with

\[
\forall g \in \mathbb{N}^2, |g| = k, x \in \mathbb{R}^2 : |\partial^g \phi(x)| \leq 8^k/(\delta_1 \ldots \delta_k),
\]

where \( (\delta_j)_{j \in \mathbb{N}} \) is any decreasing sequence of positive numbers with \( \sum_{j=1}^{\infty} \delta_j < \delta \).

In particular, we can choose \( \phi \) such that

\[
\forall z \in \mathbb{C} : \left| \frac{\partial}{\partial z} \phi(z) \right| \leq \frac{8}{\delta_1},
\]

with \( 0 < \delta_1 < \delta \) arbitrary. Combining this with (3.2) gives

\[
\forall 0 < \delta_1 < \delta : \quad d_n \leq \|f\|_{\partial D_n} \left( \frac{r}{\rho} \right)^n \frac{8}{\pi} \text{dist}(\partial D_r, K) \frac{1}{\delta_1} \lambda(D_R \setminus D_\rho),
\]

and, letting \( \delta_1 \) tend to \( \delta \), we have

\[
\forall 1 < r < \rho < R : \quad d_n \leq \|f\|_{\partial D_n} \left( \frac{r}{\rho} \right)^n \frac{8\lambda(D_R \setminus D_\rho)}{\pi} \text{dist}(\partial D_r, D_\rho) \text{dist}(\partial D_r, K).
\]

This completes the proof. \( \square \)

To formulate our next result conveniently, we introduce the following notion. Let \( K, L \) be two non-empty, disjoint, compact subsets of \( \mathbb{C} \) such that \( \mathbb{C} \setminus (K \cup L) \) has a Green’s function \( g \). We call \( R > 1 \) separating for \( K \) and \( L \) if no component of \( D_R := \{ z \in \mathbb{C} : g(z) < \log R \} \) contains elements of both \( K \) and \( L \). That is, if \( U_R \) is the union of the components of \( D_R \) intersecting \( K \), and if \( V_R := D_R \setminus U_R \), then \( U_R, V_R \) are open, disjoint neighborhoods of \( K, L \), respectively with \( U_R \cup V_R = D_R \).

**Proposition 14.** Let \( \Omega \subseteq \mathbb{C} \) be open and simply connected, let \( \zeta \in \Omega \), and let \( (K_k)_{k \in \mathbb{N}} \) be a compact exhaustion of \( \Omega \) such that \( \zeta \in K_1 \) and \( \mathbb{C} \setminus K_k \) is connected for every \( k \in \mathbb{N} \). Also, for \( k \in \mathbb{N} \), let \( \Omega_k \subseteq \mathbb{C} \) be open, let \( L_k \subseteq \Omega_k \) be compact, and let \( f_k \in H(\Omega_k) \). Assume that \( \mathbb{C} \setminus L_k \) is connected, that \( K_k \cap L_k = \emptyset \), and that \( \mathbb{C} \setminus (K_k \cup L_k) \) has a Green’s function \( g_k \) for every \( k \in \mathbb{N} \). Let \( R_k > 1 \) be separating for \( K_k, L_k \), and suppose further that \( D_k := D_{R_k} = \{ z \in \mathbb{C} : g_k(z) < \log R_k \} \subseteq \Omega \cup \Omega_k \).

Then, for every choice of \( 1 < r_k < \rho_k < R_k \) \( (k \in \mathbb{N}) \), there exists \( f \in H(\Omega) \) with uniformly universal power series in \( \zeta \) for \((f_k, L_k)_{k \in \mathbb{N}} \) such that

\[
\forall k \geq 2 : \quad N_k < N_{k-1} + \frac{\log^+ \left( k^2 \| f - T_{N_{k-1}}(\zeta) f \|_{\mathbb{C}} \| g_k^{N_{k-1}-1} C_k \right)}{\log \left( \frac{\rho_k}{r_k} \right)} + 1,
\]
where

\[ V_k := V_{R_k}, \quad q_k := \frac{\text{diam}(K_k \cup L_k)}{\text{dist}(K_k, V_k)}, \]

and

\[ C_k := C(r_k, \rho_k, R_k) := \frac{8\lambda(D_{R_k} \setminus D_{\rho_k})}{\pi \text{dist}(\partial D_{R_k}, D_{\rho_k}) \text{dist}(\partial D_{r_k}, K_k \cup L_k)}. \]

Proof. Like in [11, Proof of Theorem 2] we begin by constructing a sequence of polynomials \((P_k)_{k \in \mathbb{N}_0}\) and a strictly increasing sequence of integers \((N_k)_{k \in \mathbb{N}_0}\) with the following properties: the degree of \(P_k\) satisfies \(\text{deg} P_k = N_k\), the point \(\zeta\) is a zero of \(P_k\) of multiplicity at least \(N_k - 1\), and

\[ \forall k \geq 1 : \|P_k\|_{K_k} < \frac{1}{k^2}, \tag{3.3} \]

as well as

\[ \forall k \geq 1 : \left\| \sum_{\nu=0}^{k-1} P_{\nu} - f_k \right\|_{L_k} < \frac{1}{k^2}. \tag{3.4} \]

We set \(P_0(z) \equiv 1\) and \(N_0 = 0\). Suppose that, for some \(k \in \mathbb{N}\), the polynomials \(P_0, \ldots, P_{k-1}\) and the integers \(N_0, \ldots, N_{k-1}\) have already been determined. Because \(R_k\) is separating for \(K_k\) and \(L_k\), we have, with \(U_k := U_{R_k}\) and \(V_k := V_{R_k}\), disjoint open neighborhoods of \(K_k\) and \(L_k\) with \(U_k \cup V_k = D_k\). Consider the function

\[ h_k : U_k \cup V_k \to \mathbb{C}, \quad z \mapsto \begin{cases} 0, & \text{if } z \in U_k, \\ f_k(z) - \sum_{\nu=0}^{k-1} P_{\nu}(z), & \text{if } z \in V_k, \end{cases} \]

which is well-defined and holomorphic. From Lemma 13, we obtain that

\[ d_n(h_k, K_k \cup L_k) \leq \|h_k\|_{\partial D_k} \left( \frac{r_k}{\rho_k} \right)^n C_k \leq \|h_k\|_{\partial V_k} \left( \frac{r_k}{\rho_k} \right)^n C_k, \]

where, in the last step, we used the maximum principle and the fact that \(h_k|U_k = 0\). Hence, in order to have

\[ d_n(h_k, K_k \cup L_k) < \frac{1}{k^2 \max_{K_k \cup L_k} |z - \zeta|^{N_{k-1}}}, \tag{3.5} \]

it suffices that

\[ k^2 \max_{K_k \cup L_k} |z - \zeta|^{N_{k-1}} \|h_k\|_{\partial V_k} C_k < \left( \frac{\rho_k}{r_k} \right)^n. \]

The latter is obviously the case if

\[ k^2 \left\| f_k - \sum_{\nu=0}^{k-1} P_{\nu} \right\|_{V_k} \left( \max_{K_k \cup L_k} \frac{|z - \zeta|}{\min_{V_k} |z - \zeta|} \right)^{N_{k-1}} C_k < \left( \frac{\rho_k}{r_k} \right)^n. \]
Moreover, \( \min_{K_k} |z - \zeta| \geq \text{dist}(K_k, V_k) \) and \( \max_{K_k \cup L_k} |z - \zeta| \leq \text{diam}(K_k \cup L_k) \), the diameter of \( K_k \cup L_k \), so that (3.5) is satisfied if

\[
\log^+ \left( k^2 \| f_k - \sum_{v=0}^{k-1} P_v \|_{T^n_k q_k}^{N_k-1} C_k \right) \geq \frac{\log (\rho_k)}{r_k} =: c(k).
\]  

(3.6)

By the above, if we fix \( n \in \mathbb{N} \cap [c(k), c(k) + 1] \), then there is \( \Pi_n \in \mathcal{P}_n \) satisfying

\[
\| \Pi_n \|_{K_k} < \frac{1}{k^2 \cdot \max_{K_k \cup L_k} |z - \zeta|^{N_k-1}}
\]

and

\[
\| \Pi_n - f_k - \sum_{v=0}^{k-1} P_v \|_{L_k} < \frac{1}{k^2 \cdot \max_{K_k \cup L_k} |z - \zeta|^{N_k-1}}.
\]

By adding a sufficiently small multiple of the identity to \( \Pi_n \), we can assume without loss of generality that \( \deg \Pi_n \geq 1 \). Setting \( P_k(z) := (z - \zeta)^{N_k-1} \Pi_n(z) \), we thus obtain that \( \zeta \) is a zero of \( P_k \) of multiplicity at least \( N_k-1 \), that \( N_k := \deg P_k \leq N_k-1 + n \) and \( \deg P_k > N_k-1 \), and that \( P_k \) fulfills (3.3) and (3.4).

With the \( P_k \) constructed, we now define \( f : \Omega \to \mathbb{C}, \ z \mapsto \sum_{k=0}^{\infty} P_k(z) \). Because of (3.3), the function \( f \) is well-defined and holomorphic in \( \Omega \). Since \( \deg P_k = N_k \) and \( P_k(z) = (z - \zeta)^{N_k-1} \Pi_k(z) \) for some polynomial \( \Pi_k \) of strictly positive degree, it follows that \( T_{N_k}^{(k)} f = \sum_{v=0}^{k} P_v \) for every \( k \in \mathbb{N} \). On the one hand, by (3.4), this implies that

\[
\forall k \geq 1 : \ \| f_k - T_{N_k}^{(k)} f \|_{L_k} < \frac{1}{k^2},
\]

(3.7)

and on the other hand, by (3.6) and the maximum principle, we have

\[
N_k = \deg P_k \leq N_k-1 + n \leq N_k-1 + \frac{\log^+ \left( k^2 \| f_k - T_{N_k-1}^{(k)} f \|_{T^n_k q_k}^{N_k-1} C_k \right)}{\log \left( \frac{\rho_k}{r_k} \right)} + 1.
\]

Thus \( f \) has all the required properties. \( \Box \)

Obviously, the result stated in Proposition 14 contains too many unknown quantities in order to allow an explicit (non-recursive) estimate for the growth of \( N_k \). But nevertheless, in the general context, we already see that the \( N_k \) grow slower if \( L_k \) is farther away from \( \Omega \) (respectively \( K_k \)), since \( q_k \) is smaller then.

Let us say that \( f \in H(\mathbb{D}) \) has a universal Taylor series in 0 for \( H(\Omega) \) if the Taylor polynomials of \( f \) about 0 are dense in \( H(\Omega) \), where \( \Omega \subset \mathbb{C} \setminus \mathbb{D} \) is open. Instead of constructing a holomorphic function \( f \) with a universal Taylor series about the origin in the sense of [13], we construct \( f \in H(\mathbb{D}) \) having a universal Taylor series in 0 for \( H(c + \mathbb{D}) \) for some \( c \in \mathbb{C} \), and we investigate how fast the elements of a given normal family \( \mathcal{M} \) in \( H(c + \mathbb{D}) \) can be approximated by the Taylor polynomials of \( f \).
Also in this situation the \( N_k \) grow slower, as we will see later, since the sets \( L_k \) and the functions \( f_k \) to approximate on \( L_k \) can be chosen closer to their predecessors \( L_{k-1} \) and \( f_{k-1} \). Indeed, by (3.7), \( T_{N_{k-1}}^{(c_l)} f \) is close to \( f_{k-1} \) on \( L_{k-1} \).

If additionally \( L_k \) is close to \( L_{k-1} \) and \( f_k \) is close to \( f_{k-1} \), then \( \| f_k - T_{N_{k-1}}^{(c_l)} f \|_{V_K} \) remains rather small.

We consider the standard compact exhaustions of \( D \) and \( c + D \), respectively, that is \( K = (K_n)_{n \in \mathbb{N}} \) and \( K_c = (K_{c,n})_{n \in \mathbb{N}} \), where

\[
K_n := \frac{n}{n + 1} \mathbb{D}, \quad K_{c,n} := c + K_n, \quad n \in \mathbb{N}.
\]  

(3.8)

Since we are now dealing with disks, we have the following approximation result at our disposal, which will replace the use of Lemma 13.

**Lemma 15.** Let \( L = L_1 \cup L_2 := \overline{D}(a_1, R_1) \cup \overline{D}(a_2, R_2) \) be the union of two disjoint closed disks. Let \( K = K_1 \cup K_2 := \overline{D}(a_1, r_1) \cup \overline{D}(a_2, r_2) \), where \( 0 < r_j < R_j \) \( (j = 1, 2) \). Given \( f \) holomorphic on a neighborhood of \( L \) and \( n \geq 1 \), there exists a polynomial \( p \) such that \( \deg p < 2n \) and

\[
\| f - p \|_K \leq \| f \|_L \cdot \frac{2\alpha^n}{(1 - \alpha)} \left( \frac{\text{diam}(K)}{\text{dist}(L_1, L_2)} \right)^n,
\]

where \( \alpha := \max\{r_1, r_2\} / \min\{R_1, R_2\} \).

**Proof.** Set \( q(z) := (z - a_1)^n(z - a_2)^n \). We consider the special kind of Hermite interpolation polynomial

\[
p(w) := \frac{1}{2\pi i} \int_{\partial L} \frac{f(z) q(z) - q(w)}{q(z)} \frac{dz}{z - w},
\]

and we shall show that this works.

Since \( (q(z) - q(w))/(z - w) \) is a polynomial in \( z, w \) of degree at most \( 2n - 1 \) in each variable, it follows that \( p(w) \) is a polynomial of degree at most \( 2n - 1 \).

Also, by Cauchy’s integral formula, if \( w \in K \), then

\[
f(w) = \frac{1}{2\pi i} \int_{\partial L} \frac{f(z)}{z - w} \frac{dz}{z - w},
\]

and so

\[
f(w) - p(w) = \frac{1}{2\pi i} \int_{\partial L} \frac{f(z)}{z - w} \frac{q(w)}{q(z)} \frac{dz}{z - w}.
\]

It follows that

\[
\| f - p \|_K \leq \frac{\| f \|_L \| q \|_K}{2\pi \text{dist}(\partial L, K)} \int_{\partial L} \frac{|dz|}{|q(z)|}.
\]

Now, if \( w \in K_1 \), then \( |q(w)| \leq r_1^n (\text{diam } K)^n \). An analogous estimate holds for \( w \in K_2 \). Hence

\[
|q|_K \leq \max\{r_1, r_2\}^n (\text{diam } K)^n.
\]

Also, we clearly have

\[
\text{dist}(\partial L, K) \geq \min\{R_1 - r_1, R_2 - r_2\}.
\]

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Further, if $z \in \partial L_1$, then $|q(z)| \geq R_1^n \text{dist}(L_1, L_2)^n$. Hence
$$
\int_{\partial L_1} \frac{|dz|}{|q(z)|} \leq \frac{2\pi R_1}{R_1^n \text{dist}(L_1, L_2)^n}.
$$

An analogous estimate holds for the integral over $\partial L_2$. Putting together these estimates, we get
$$
\|f - p\|_{K} \leq \|f\|_{L} \frac{\max\{r_1, r_2\}^n (\text{diam } K)^n}{\min\{R_1 - r_1, R_2 - r_2\} \text{dist}(L_1, L_2)^n} \left( \frac{1}{R_1^{n-1}} + \frac{1}{R_2^{n-1}} \right).
$$

If we set $r := \max\{r_1, r_2\}$ and $R := \min\{R_1, R_2\}$, then we obtain
$$
\|f - p\|_{K} \leq \|f\|_{L} \frac{r^n (\text{diam } K)^n}{(R - r) \text{dist}(L_1, L_2)^n} \left( \frac{2}{R^{n-1}} \right).
$$

The result follows from this.

In the situation of the above lemma, let us choose $K = K_k \cup K_{c,k}$, that is $a_1 = 0$, $a_2 = c \in \mathbb{C}$, $r_1 = r_2 = \frac{k}{n+1} < 1$, and let $R_1 = R_2 := R > 1$. The disks $L_1, L_2$ are disjoint as long as $\text{dist}(L_1, L_2) = |c| - 2R > 0$. Further, we obtain
$$
\frac{\alpha \text{diam}(K)}{\text{dist}(L_1, L_2)} < \frac{1}{\frac{|c|}{R} - 2R} =: q(R, c),
$$
and the inequality inLemma 15 reads
$$
\|f - p\|_{K} \leq \|f\|_{L} \frac{2R}{R - 1} q(R, c)^n. \tag{3.9}
$$

**Lemma 16.** Let $R > 1$ and $c \in \mathbb{C}$ with $|c| > 2R$ and $q(R, c) < 1$. For any sequence of polynomials $(f_k)_{k \in \mathbb{N}}$, there is $f \in H(\mathbb{D})$ having a uniformly universal Taylor series in the origin for $(f_k, K_{c,k})_{k \in \mathbb{N}}$ such that the corresponding sequence $(N_k)_{k \in \mathbb{N}}$ is strictly increasing, $T_{N_k} f$ and $N_k$ depend only on $f_1, \ldots, f_{k-1}$ and, for $k \geq 7$, we have
$$
N_k \leq N_{k-1} + 1 + \frac{2 \log^+(A)}{\log(q(R, c)^{-1})}, \tag{3.10}
$$
where
$$
A := \frac{2R}{R - 1} (2R)^{\max\{\deg f_{k,N_{k-1}}\}} \left( \frac{2|c| + 2}{|c| - R} \right)^{N_{k-1}} \left( \|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2} \right).
$$

**Proof.** The proof is very similar to that of Proposition 14, only we use Lemma 15 instead of Lemma 13.

As in the proof of Proposition 14, we construct recursively a sequence of polynomials $(P_k)_{k \in \mathbb{N}_0}$ and a strictly increasing sequence of integers $(N_k)_{k \in \mathbb{N}_0}$ such that the degree of $P_k$ satisfies $\deg P_k = N_k$, the origin is a zero of $P_k$ of multiplicity at least $N_{k-1}$, and
$$
\forall k \geq 1: \quad \|P_k\|_{K_k} < \frac{1}{k^2} \text{ and } \left\| \sum_{\nu=0}^{k} P_{\nu} - f_{k} \right\|_{K_{c,k}} < \frac{1}{k^2}. \tag{3.11}
$$
Let $P_0(z) \equiv 1$, and $N_0 := 0$. If, for some $k \in \mathbb{N}$, the polynomials $P_0, \ldots, P_{k-1}$ and the integers $N_0, \ldots, N_{k-1}$ have already been constructed, then we consider

$$h_k : B(0, R) \cup B(c, R) \to \mathbb{C}, \quad z \mapsto \begin{cases} 0, & \text{if } z \in B(0, R) \\ f_k(z) - \sum_{\nu=0}^{k-1} P_\nu(z) / z^{N_{k-1}}, & \text{if } z \in B(c, R), \end{cases}$$

which is well-defined and holomorphic in a neighborhood of $B(0, R) \cup B(c, R)$, since $|c| > 2R$. Lemma 15 and inequality (3.9) yield, for any $n \in \mathbb{N}$, the existence of a polynomial $\Pi_n$ of degree not exceeding $2n - 1$, with

$$\|h_k - \Pi_n\|_{K \cup K_{c,k}} \leq \|h_k\|_{B(0, R) \cup B(c, R)} \frac{2R}{R - 1} q(R, c)^n = \|h_k\|_{B(c, R)} \frac{2R}{R - 1} q(R, c)^n,$$

where we used $h_k|_{B(0, R)} = 0$. As $\max_{K \cup K_{c,k}} |z| = \max_{K_{c,k}} |z| = |c| + \frac{k}{k+1} < |c| + 1$, as well as $\min_{B(c, R)} |z| = |c| - R$, as in the proof of Proposition 14 we obtain that, in order to have

$$\|h_k - \Pi_n\|_{K \cup K_{c,k}} < \frac{1}{k^2 \max_{K \cup K_{c,k}} |z|^{N_{k-1}}},$$

it is sufficient to have

$$k^2 \left\| f_k - \sum_{\nu=0}^{k-1} P_\nu \right\|_{B(c, R)} \left( \frac{|c| + 1}{|c| - R} \right)^{N_{k-1}} \frac{2R}{R - 1} < q(R, c)^{-n}. \quad (3.12)$$

An application of Bernstein’s lemma, cf. [14, Theorem 5.5.7], and (3.11) yields

$$\left\| f_k - \sum_{\nu=0}^{k-1} P_\nu \right\|_{B(c, R)} \leq \left\| f_k - \sum_{\nu=0}^{k-1} P_\nu \right\|_{K_{c,k-1}} \left( \frac{Rk}{k-1} \right)^{\max\{\deg f_k, N_{k-1}\}}$$

$$\leq \left( \|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2} \right) (2R)^{\max\{\deg f_k, N_{k-1}\}}.$$ 

Therefore, $n \in \mathbb{N}$ satisfies (3.12) if $n > \alpha(k)$, where $\alpha(k)$ is given by

$$\log^+ \left( \frac{2R^2}{R - 1} \frac{\max\{\deg f_k, N_{k-1}\} \left( \frac{|c| + 1}{|c| - R} \right)^{N_{k-1}}}{\log q(R, c)^{-1}} \right).$$

(3.13)

Fixing $n \in \mathbb{N} \cap [\alpha(k), 1 + \alpha(k)]$, we continue as in the proof of Proposition 14, to construct $P_k$ and $N_k := \deg P_k \leq N_{k-1} + 2n - 1$.

As in the proof of Proposition 14, it follows that $f : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \sum_{k=0}^{\infty} P_k(z)$ is holomorphic and has a uniformly universal Taylor series in 0 for $(f_k, K_{c,k})_{k \in \mathbb{N}}$. For the corresponding sequence $(N_k)_{k \in \mathbb{N}}$, we have

$$N_k = \deg P_k \leq N_{k-1} + 2n - 1 \leq N_{k-1} + 1 + 2\alpha(k).$$

If $k \geq 7$, then, because $k^2 \leq 2^{k-1} (\leq 2^{N_{k-1}})$, we obtain from (3.13) that $\alpha(k)$ is majorized by

$$\log^+ \left( \frac{2R}{R - 1} \frac{\max\{\deg f_k, N_{k-1}\} \left( \frac{2|c| + 2}{|c| - R} \right)^{N_{k-1}}}{\log q(R, c)^{-1}} \right).$$

(3.14)
This completes the proof.

Remark 17. Proposition 14 and Lemma 16 show how small the values of the sequence \((N_k)_{k \in \mathbb{N}}\) can be chosen. Nevertheless, inspection of their proofs gives that, at each step, \(N_k\) can be chosen arbitrarily large.

Lemma 18. Suppose that \(\mathcal{M}\) be a normal family in \(H(c + D)\). Let \(M_n := 1 + \sup_{f \in \mathcal{M}} \|f\|_{K_{c,n}}\) and let \((\lambda_n)_{n \in \mathbb{N}}\) be the covering numbers of \(\mathcal{M}\), i.e., there are functions \(f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in H(c + D)\) whose \(\frac{1}{n}\)-neighborhoods cover \(\mathcal{M}\). If

\[
m \geq \tau_{n+1}(\mathcal{M}) := (n + 1)^2 \log \left((n + 1)^3 M_{n+1}\right),
\]

then

\[
\|g - T_m^{(c)}g\|_{K_{c,n}} < \frac{1}{n} \quad \forall g \in \mathcal{M} \cup \{f_1^{(n+1)}, \ldots, f_{\lambda_{n+1}}^{(n+1)}\}.
\]

Proof. Let \(g = f_j^{(n+1)}\) for some \(1 \leq j \leq \lambda_{n+1}\). Then there is \(f \in \mathcal{M}\) such that \(\|g - f\|_{K_{c,n+1}} \leq \frac{1}{n+1}\), which implies

\[
\|g\|_{K_{c,n+1}} \leq \|f\|_{K_{c,n+1}} + \frac{1}{n+1} \leq M_{n+1}.
\]

This last estimate obviously also holds for \(g \in \mathcal{M}\). By Cauchy’s estimate, we get

\[
\|g - T_m^{(c)}g\|_{K_{c,n}} \leq \sum_{\nu = m+1}^{\infty} \left(\frac{n(n + 2)}{(n + 1)^2}\right)^\nu = M_{n+1} \frac{n(n + 2)}{(n + 1)^2} \cdot \left(\frac{n(n + 2)}{(n + 1)^2}\right)^m.
\]

The last term is less than \(1/n\) provided that

\[
m \geq (n + 1)^2 \log \left((n + 1)^3 M_{n+1}\right) \geq \frac{\log \left(n^2(n + 2) M_{n+1}\right)}{\log \left(\frac{n^2(n + 2)}{(n+1)^2}\right)}.
\]

This completes the proof.

By choosing an appropriate sequence of polynomials \((f_k)\) in Lemma 16, we can construct a function \(f \in H(D)\) which has a universal Taylor series in 0 for \(H(c + D)\) for some center \(c \in \mathbb{C}\), which is fast approximating for a normal family \(\mathcal{M} \subset H(c + D)\), and such that the corresponding sequence \((N_k)\) has bounded quotients \(N_k/N_{k-1}\) (compare with Remark 11).

Theorem 19. Let \(c \in \mathbb{C}\) with \(|c| > 10\). Let \(\mathcal{M}\) be a normal family in \(H(c + D)\) with covering numbers \((\lambda_n)\), and suppose that \(M_n = O(\exp(n^l))\) for some \(l \in \mathbb{N}\). Then there exists a function \(f \in H(D)\) which has a universal Taylor series for \(H(c + D)\), which is fast approximating for \(\mathcal{M}\) in the sense that

\[
F(f, \mathcal{M}, 2/n) \leq N_{(n+1)(\lambda_{n+1}+1)},
\]

and which has bounded quotients \(N_k/N_{k-1}\).
Proof. For $R = 2$, we have $|\alpha| > 10 > 2R$ and $g(2, c) < 1$. Let $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in H(c + D)$ be functions whose $\frac{1}{n}$-neighborhoods cover $\mathcal{M}$. Let $(q_n)$ be a sequence of polynomials which is dense in $H(c + D)$, and consider the sequence $(g_k)$ given by

$$f_1^{(1)}, \ldots, f_{\lambda_1}^{(1)}, q_1, f_1^{(2)}, \ldots, f_{\lambda_2}^{(2)}, q_2, \ldots, f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)}, q_n, \ldots$$

If $g_k := f_j^{(n)}$ for some $j, n$, then $n$ is uniquely determined by $k$, and we have $n \leq k$. In this case, we set $f_k := T_{\tau_n(\mathcal{M})} g_k$. Using Lemma 18, we deduce

$$\deg f_k = n^2 \log (n^3 M_n) \quad \text{and} \quad \|g_k - f_k\|_{K_{c,n-1}} \leq \frac{1}{n - 1}. \quad (3.14)$$

If, on the other hand, $g_k$ is one of the $(q_n)$, then we define $f_k := g_k$. Without loss of generality $\deg f_k \leq k$ and $\|f_k\|_{K_{c,k}} \leq k$.

Now let $f \in H(D)$ be the function constructed in Lemma 16, that is, $f$ has a uniformly universal Taylor series in the origin for $(f_k, K_{c,k})$ with the corresponding sequence $(N_k)$ satisfying (3.10). In view of (3.10), we can estimate the degree of $f_k$ in case $g_k = f_j^{(n)}$:

$$\deg f_k = n^2 \log (n^3 M_n) \leq k^2 \log (k^3 M_k) = O(k^{l+3}),$$

since $n \leq k$ and $M_k = O(\exp(k^l))$. Hence, by Bernstein’s lemma, cf. [14, Theorem 5.5.7],

$$\|f_k\|_{K_{c,k}} \leq \|f_k\|_{K_{c,n}} \left( \frac{k + 1}{k + n} \right)^{\deg f_k} \leq M_n e^{n^2 \log(n^3 M_n)}.$$

Thus, independently of whether $g_k = f_j^{(n)}$ or $g_k = q_n$, we obtain

$$\log \left( \|f_k - f_{k-1}\|_{K_{c,k-1}} + \frac{1}{(k-1)^2} \right) = O(k^{l+3}).$$

Hence, using the fact that $\max\{\deg f_k, N_{k-1}\} \leq \deg f_k + N_{k-1}$, inequality (3.10) reads

$$N_k \leq \alpha_1 N_{k-1} + \alpha_2 k^{l+3},$$

where $\alpha_1, \alpha_2$ are constants. As mentioned in Remark 17, it is possible to increase $N_k$ at each step. So, let us choose

$$N_k := \max \{\alpha_1 N_{k-1} + \alpha_2 k^{l+3}, (k+1)^{l+3}\}, \quad (3.15)$$

which guarantees $N_{k-1} \geq k^{l+3}$ for every $k \in \mathbb{N}$. Depending on where the maximum in (3.15) is attained, $N_k/N_{k-1}$ is either bounded by the constant $\alpha_1 + \alpha_2$ or by $\left( \frac{k+1}{k} \right)^{l+3}$. Either way, $N_k/N_{k-1}$ remains bounded as $k \to \infty$.

To each $g \in \mathcal{M}$ is associated $g_k = f_j^{(n+1)}$ with $\|g_k - g\|_{K_{c,n+1}} \leq \frac{1}{n+1}$. Using (3.14), we have $\|f_k - g_k\|_{K_{c,n}} < 1/n$, which implies $\|f_k - g\|_{K_{c,n}} \leq 2/n$. By the choice of $f$ and the sequences $(f_k), (g_k)$, we obtain $k \leq (n+1)(\lambda_{n+1} + 1)$ and hence $F(f, \mathcal{M}, 2/n) \leq N_k$.

Since $(q_n)$ is dense in $H(c + D)$, by construction so are the Taylor polynomials of $f$ about 0, and hence $f$ has a universal Taylor series for $H(c + D)$.

In the next section we shall encounter several examples of normal families $\mathcal{M}$ for which $M_n = O(1)$, and so Theorem 19 is applicable with $l = 0$.
4 Normal families and covering numbers

In order to get some impression of the speed of approximation, we conclude with some examples of normal families in $H(\mathbb{D})$. Throughout this section, we suppose $\varepsilon$ to be an arbitrarily small positive number, and consider the standard compact exhaustion (3.8) of $\mathbb{D}$ to define the seminorms in (1.2) and hence the natural metric on $H(\mathbb{D})$ in (1.1).

Let $E := \{f_1, \ldots, f_k\}$ be a finite subset of $H(\mathbb{D})$, let $B^\infty := \{f \in H(\mathbb{D}) : \sup_{\mathbb{D}} |f| \leq 1\}$, and let

$$ S := \{f \in H(\mathbb{D}) : f \text{ one-to-one}, f(0) = 0, f'(0) = 1\}. $$

For each of these three normal families, we obtain the existence of $C$- or $D$-universal functions $f$ with some examples of normal families in $H(\mathbb{D})$.

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For each of these three normal families, we obtain the existence of $C$- or $D$-universal functions $f$ with rates of approximation as follows:

<table>
<thead>
<tr>
<th>$\mathcal{M}$</th>
<th>$F(f, \mathcal{M}, 1/n)$ (for $\mathcal{C}$)</th>
<th>$F(f, \mathcal{M}, 1/n)$ (for $\mathcal{D}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$O(n)$</td>
<td>$O(n^2 \log(n \max{1, M_{12kn+1}}))$</td>
</tr>
<tr>
<td>$B^\infty$</td>
<td>$O(n\lambda_{2n})$</td>
<td>$O(n(n\lambda_{3n})^{1+\varepsilon})$</td>
</tr>
<tr>
<td>$S$</td>
<td>$O(n\lambda_{2n})$</td>
<td>$O(n^2\lambda_{3n} \log(n\lambda_{3n}))$</td>
</tr>
</tbody>
</table>

where $M_n := \sup_{f \in \mathcal{M}} \|f\|_{K_n}$ and $\lambda_n$ denotes the $n$-th covering number of $\mathcal{M}$.

Furthermore, $\mathcal{C}$ is $8$- and $D$ is $(9 + \varepsilon)$-polynomial universal for the automorphism group

$$ \text{Aut}(\mathbb{D}) = \left\{ f_{\gamma, a}(z) := e^{i\gamma} \frac{z - a}{1 - az} : \gamma \in [0, 2\pi), a \in \mathbb{D} \right\}, $$

and $\mathcal{C}$ is $(2 + \varepsilon)$- and $D$ $(4 + \varepsilon)$-polynomial universal for the set of all Koebe extremal functions

$$ K := \left\{ f_n = e^{-i\alpha} f_0(e^{i\alpha} z) : \alpha \in [0, 2\pi) \right\} \subseteq S, \quad f_0(z) = \frac{z}{(1 - z)^2}. $$

For all this and further details, see [10].

In this context, the question arises, interesting in its own right, to estimate the $n$-th covering number $\lambda_n$ for $S$, or, going back one step, to estimate the minimal number $N(\delta)$ of balls of radius $\delta$ required to cover $S$. The following theorem provides upper and lower bounds for $N(\delta)$.

**Theorem 20.** There exist constants $c, C > 0$ such that

$$ e^{c/\sqrt{\delta}} \leq N(\delta) \leq e^{(C/\delta) \log^2(1/\delta)}. $$

In particular, $N(\delta)$ grows faster than any power of $1/\delta$ as $\delta \to 0$. The proof of the upper bound is given in [10]. We give here the proof of the lower bound. It is based on an elementary lemma.

**Lemma 21.** Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, where $\sum_{k=2}^{\infty} k|a_k| < 1$. Then $f \in S$.

**Proof.** Let $z, w \in \mathbb{D}$. Then $|z^k - w^k| \leq k|z - w|$ for all $k$, so

$$ |f(z) - f(w)| \geq |z - w| - \sum_{k=2}^{\infty} |a_k| |z^k - w^k| \geq |z - w| \left( 1 - \sum_{k=2}^{\infty} k|a_k| \right). $$

It follows that $f$ is injective. Thus $f \in S$. \qed

21
Proof of the lower bound. Let $f, g \in H(D)$, say $f(z) = \sum_{k=0}^{\infty} a_k z^k$, and $g(z) = \sum_{k=0}^{\infty} b_k z^k$. By the maximum principle and the standard Cauchy estimates, for each $k \in \mathbb{N}$ we have
\[
\|f - g\|_{K_k} = \max_{|z|=k/(k+1)} |f(z) - g(z)| \geq |a_k - b_k| \left( \frac{k}{k+1} \right)^k \geq \frac{|a_k - b_k|}{4},
\]
and consequently
\[
d(f, g) \geq \sup_{k \in \mathbb{N}} \min_{k \in \mathbb{N}} \left( \frac{|a_k - b_k|}{4}, \frac{1}{k} \right).
\] (4.1)

Now let $n \geq 2$, and consider the family $F_n$ of polynomials $f$ of the form
\[
f(z) := z + \frac{1}{n^2} \sum_{k=2}^{n} \varepsilon_k z^k, \quad (\varepsilon_k \in \{-1, 1\}, \ k = 2, \ldots, n).
\]
By Lemma 21, we clearly have $F_n \subset S$. Also, by (4.1), the distance between distinct polynomials $f, g \in F_n$ is at least $1/(2n^2)$. Thus, in any covering by balls of radius $1/(4n^2)$, each of the polynomials of $F_n$ must belong to a different ball. There are $2^n - 1$ elements in the family $F_n$. Therefore
\[
N(1/(4n^2)) \geq 2^{n-1}.
\]
As this holds for each $n \geq 2$, the lower bound follows. \hfill \square

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References


