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Corrigendum to "Surjectivity of partial differential operators on ultradistributions of Beurling type in two dimensions"

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by

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Abstract. As was pointed out by G. Hoepfner and D. Jornet, the proof of Lemma 3.3 in the paper of the title is incorrect. Nevertheless, the statement of the lemma holds true, as is shown in the present corrigendum.

In [3, Lemma 3.3] it is claimed that

$$\text{(1)} \qquad \textit{If } Q \in L_{\omega,N}(P) \textit{ then } N \in \Lambda(Q).$$

Here, $L_{\omega,N}(P)$ stands for the set of ω-localizations of a polynomial P at infinity, as defined in [3, p. 91], and $\Lambda(Q)$ is the set defined by Hörmander in [1, Section 10.2] (see also [3, p. 93]) by

$$\Lambda(Q) := \{\eta \in \mathbb{R}^d;\ \forall\, \xi \in \mathbb{R}^d, t \in \mathbb{R} :\ Q(\xi + t\eta) = Q(\xi)\}.$$

Unfortunately, the proof of (1) presented in [3] is not correct. A correct proof is given below. We first need an auxiliary result.

PROPOSITION 1. *Let $Q \in L_{\omega,N}(P)$. Then there are polynomials*

$$\alpha(\tau) = \sum_{j=0}^{n} \alpha_j \tau^j, \qquad \xi(\tau) = \sum_{j=0}^{m} \theta_j \tau^j$$

with $m \geq n+2$, $\alpha_j \in \mathbb{R}$, $\theta_j \in \mathbb{R}^d$, and $\theta_m = N$ such that

$$Q(x) = \lim_{\tau\to\infty} \frac{1}{(\sum_\beta |P^{(\beta)}(\xi(\tau))|^2 \alpha(\tau)^{2|\beta|})^{1/2}} P(\alpha(\tau)x + \xi(\tau)),$$

where the limit is taken in the unique Hausdorff vector space topology on the finite-dimensional vector space of polynomials of degree not exceeding the degree of P.

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Proof. For $t, s > 0$ we define

$$c(t,s) := \inf\Big\{|N - b\eta|^2 + \sum_\alpha |Q^{(\alpha)}(0) - as^{|\alpha|}P^{(\alpha)}(\eta)|^2;\ \eta \in \mathbb{R}^d,\ |\eta|^2 = t^2,\\ bt = 1,\ a > 0,\ a^2\Big(\sum_\beta |P^{(\beta)}(\eta)|^2 s^{2|\beta|}\Big) = 1\Big\}.$$

For fixed $t, s > 0$ the set

$$\partial B(0,t) \times \{1/t\} \times \Big\{a > 0;\ a^2\Big(\sum_\beta |P^{(\beta)}(\eta)|^2 s^{2|\beta|}\Big) = 1 \text{ for some } |\eta| = t\Big\}$$

is compact so that the above infimum is in fact a minimum. Moreover, by the Tarski–Seidenberg Theorem (see e.g. [1, Section A.2]), c is a semi-algebraic function. Since $Q \in L_{\omega,N}(P)$, we have $\liminf_{t\to\infty} c(t, \omega(t)) = 0$.

Since c is semi-algebraic, the set

$$E := \{(\tau, t, s) \in \mathbb{R}^3;\ \tau, t, s > 0,\ t > \tau^2 s,\ s > \tau,\ \tau c(t,s) < 1\}$$

is semi-algebraic. Because ω is increasing, and $\lim_{t\to\infty}\omega(t) = \infty$, $\omega = o(t)$ as $t \to \infty$, and $\liminf_{t\to\infty} c(t,\omega(t)) = 0$, for arbitrary $\tau > 0$ there is t such that $t > \tau^2\omega(t)$, $\omega(t) > \tau$, and $\tau c(t, \omega(t)) < 1$, i.e. $(\tau, t, \omega(t)) \in E$. Thus, the image of the projection $E \to \mathbb{R}$, $(\tau, t, s) \mapsto \tau$, equals $(0, \infty)$. By [1, Theorem A 2.8 and its proof], for large τ there are algebraic functions $t(\tau), s(\tau)$ given by convergent Puiseux series such that $(\tau, t(\tau), s(\tau)) \in E$.

In particular, with $t(\tau) = \sum_{j=-\infty}^k t_j\tau^{j/m}, s(\tau) = \sum_{j=-\infty}^l s_j\tau^{j/n}$ it follows from $t(\tau) > \tau^2 s(\tau) > \tau^3$ that $k, l \in \mathbb{N}$, $t_k > 0$, $s_l > 0$ as well as $k/m \geq l/n + 2$. Moreover, $c(t(\tau), s(\tau)) < 1/\tau$. Since $t(\tau), s(\tau)$ are algebraic, applying the Tarski–Seidenberg Theorem once more, it follows that for sufficiently large τ the function

$$d(\tau) = \inf\Big\{|N - b\eta|^2 + \sum_\alpha |Q^{(\alpha)}(0) - as(\tau)^{|\alpha|}P^{(\alpha)}(\eta)|^2;\ \eta \in \mathbb{R}^d,\ |\eta|^2 = t(\tau)^2,\\ bt(\tau) = 1,\ a > 0,\ a^2\Big(\sum_\beta |P^{(\beta)}(\eta)|^2 s(\tau)^{2|\beta|}\Big) = 1\Big\} = c(t(\tau), s(\tau))$$

is semi-algebraic. Again, by compactness, the infimum is attained and by [1, Theorem A 2.8] for sufficiently large τ it is attained with η equal to an algebraic function of τ with convergent Puiseux series $\eta(\tau) = \sum_{j=-\infty}^q \theta_j\tau^{j/r}$.

Since $d(\tau) = c(t(\tau), s(\tau)) < 1/\tau$, by the definition of d we have

$$0 = \lim_{\tau\to\infty} d(\tau) \geq \lim_{\tau\to\infty}\Big|N - \frac{1}{t(\tau)}\eta(\tau)\Big|^2 \\ = \lim_{\tau\to\infty}\Big|N - \frac{1}{\sum_{j=-\infty}^k t_j\tau^{j/m}}\sum_{j=-\infty}^q \theta_j\tau^{j/r}\Big|^2.$$

Since $|N| = 1$, this implies $q/r = k/m$ and $\theta_q = \frac{1}{t_k}N$.

Again since $d(\tau) = c(t(\tau), s(\tau)) < 1/\tau$ and by the definition of d it follows that

$$0 = \lim_{\tau\to\infty} \sum_\alpha \left| Q^{(\alpha)}(0) - \frac{1}{(\sum_\beta s(\tau)^{2|\beta|}|P^{(\beta)}(\eta(\tau))|^2)^{1/2}} s(\tau)^{|\alpha|} P^{(\alpha)}(\eta(\tau)) \right|^2.$$

Defining

$$\eta_0(\tau) := \sum_{j=0}^{q} \theta_j \tau^{j/r}, \qquad s_0(\tau) := \sum_{j=0}^{l} s_j \tau^{j/n}$$

we have $\eta(\tau) - \eta_0(\tau) = \mathcal{O}(1/\tau)$, $s(\tau) - s_0(\tau) = \mathcal{O}(1/\tau)$ as $\tau \to \infty$ so that also

$$0 = \lim_{\tau\to\infty} \sum_\alpha |Q^{(\alpha)}(0) - \frac{1}{(\sum_\beta s_0(\tau)^{2|\beta|}|P^{(\beta)}(\eta_0(\tau))|^2)^{1/2}} s_0(\tau)^{|\alpha|} P^{(\alpha)}(\eta_0(\tau))|^2.$$

Replacing τ by $t_k^{r/q}\tau^{rn}$ and having in mind that $q/r = k/m \geq l/n + 2$ yields the assertion with $\alpha(\tau) = s_0(t^{r/q}\tau^{rn})$ and $\xi(\tau) = \eta_0(t_k^{r/q}\tau^{rn})$. ∎

Proof of claim (1). Let $\alpha(\tau)$ and $\xi(\tau)$ be as in Proposition 1, so that for some $a, \sigma > 0$,

$$Q(x) = \lim_{\tau\to\infty} \frac{1}{a\tau^\sigma} P(\alpha(\tau)x + \xi(\tau))$$

uniformly for x from any fixed compact set, $\alpha(\tau) = \sum_{j=0}^{n} \alpha_j \tau^j$ and $\xi(\tau) = N\tau^m + \sum_{j=0}^{m-1} \theta_j \tau^j$ with $n + 1 - m \leq -1$.

Fix $s \in \mathbb{R}$. Because $n+1-m \leq -1$ it follows that $\frac{s}{m}\alpha(\tau)\tau^{1-m} = \mathcal{O}(1/\tau)$ for $\tau \to \infty$ as well as

$$\xi\left(\tau + \frac{s}{m}\alpha(\tau)\tau^{1-m}\right) = \xi(\tau) + \alpha(\tau)sN + \mathcal{O}(1/\tau), \qquad \tau \to \infty,$$

and

$$\begin{aligned} \frac{\alpha\big(\tau + \frac{s}{m}\alpha(\tau)\tau^{1-m}\big)}{\alpha(\tau)} &= 1 + \sum_{j=1}^{n} \frac{\alpha^{(j)}(\tau)}{j!\alpha(\tau)} \left(\frac{s}{m}\alpha(\tau)\tau^{1-m}\right)^j \\ &= 1 + \mathcal{O}(1/\tau), \qquad \tau \to \infty. \end{aligned}$$

Thus, for every $x \in \mathbb{R}^d$,

$$\begin{aligned} Q(x) &= \lim_{\tau\to\infty} \frac{1}{a\big(\tau + \frac{s}{m}\alpha(\tau)\tau^{1-m}\big)^\sigma} \\ &\qquad \times P\left(\alpha\left(\tau + \frac{s}{m}\alpha(\tau)\tau^{1-m}\right)x + \xi\left(\tau + \frac{s}{m}\alpha(\tau)\tau^{1-m}\right)\right) \\ &= \lim_{\tau\to\infty} \frac{1}{a(\tau + \mathcal{O}(1/\tau))^\sigma} P\big(\alpha(\tau)(x + sN + \mathcal{O}(1/\tau)) + \xi(\tau) + \mathcal{O}(1/\tau)\big) \\ &= Q(x + sN), \end{aligned}$$

which proves the claim. ∎

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References

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