Composition and Differentiation Operators and Fast Approximation

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Abstract

Let \( C = (C_n)_{n \in \mathbb{N}} \) and \( D = (D^n)_{n \in \mathbb{N}} \) be families of composition and differentiation operators, respectively, i.e.,

\[ C_nf = f \circ \varphi_n, \quad Df = f', \]

where \( f \) is holomorphic on some domain \( \Omega \subseteq \mathbb{C} \). Our main question is: How fast can a totally bounded set \( M \) of holomorphic functions, in other words a normal family, be approximated by the “orbit” \( \{C_nf : n \in \mathbb{N}\} \) or \( \{D^n f : n \in \mathbb{N}\} \) respectively, of one suitably constructed function \( f \)? Our answer consists of upper bounds for the numbers

\[ F(f, 1/n) := \inf\{N \in \mathbb{N} : \text{Any } g \in M \text{ is approximable with error } < 1/n \}\]

by the first \( N \) elements of the orbit of \( f \), \( n \in \mathbb{N} \).

In particular, we calculate such bounds for well-known classical normal families, like the biholomorphisms of the unit disk \( \mathbb{D} \), or the set

\[ S := \{f \text{ biholomorphic on } \mathbb{D} : f(0) = 0, f'(0) = 1\}. \]

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1 Introduction and notation

Let \( (X, d) \) be a complete metric space, \( (Y, d) \) a separable metric space, \( M \subseteq Y \), and \( L = (L_n)_{n \in \mathbb{N}} \) be a sequence of continuous mappings \( L_n : X \to Y \). The sequence \( L \) is called universal for \( M \), if there is \( x \in X \) such that \( M \) is contained in the closure of the orbit of \( x \) under \( L \), that is

\[ M \subseteq \{L_n x : n \in \mathbb{N}\}, \]

i.e., for every \( y \in M \) and for every \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) with \( d(y, L_N x) < \varepsilon \). Such \( x \) are called \( L \)-universal for \( M \) and we denote the set of all \( L \)-universal elements for \( M \)
by \( \mathcal{U}(\mathcal{L}, \mathcal{M}) \). In case of \( \mathcal{M} = \mathcal{Y} \), we simply speak of \( \mathcal{L} \)-universality etc., and write \( \mathcal{U}(\mathcal{L}) \) instead of \( \mathcal{U}(\mathcal{L}, \mathcal{M}) \).

We consider the question, how fast certain given elements \( y \in \mathcal{Y} \) can be approximated by \((L_n x)_{n \in \mathbb{N}}\) for some \( x \in \mathcal{U}(\mathcal{L}) \). With this in mind, given \( x \in \mathcal{X} \) and \( \mathcal{M} \subseteq \mathcal{Y} \), we define

\[
F(x, \varepsilon) := F(x, \mathcal{L}, \mathcal{M}, d, \varepsilon) := \sup_{y \in \mathcal{M}} \{ N \in \mathbb{N} : d(y, L_N x) < \varepsilon \}.
\]

For \( x \in \mathcal{U}(\mathcal{L}) \), we clearly have that \( F(x, \varepsilon) \) is finite for every \( \varepsilon > 0 \) if and only if \( \mathcal{M} \) is totally bounded (pre-compact), that is, \( \mathcal{M} \) can be covered by a finite number of \( \varepsilon \)-balls for every \( \varepsilon > 0 \). If the metric space \( \mathcal{Y} \) is complete, then, \( \mathcal{M} \) is totally bounded if and only if \( \mathcal{M} \) is relatively compact, cf. [14, Corollary 4.10]. Moreover, if \( \mathcal{M} \subseteq \mathcal{Y} \) is totally bounded and \( y_1^{(n)}, \ldots, y_{\lambda_n}^{(n)} \in \mathcal{Y} \) satisfy

\[
\mathcal{M} \subseteq \bigcup_{j=1}^{\lambda_n} B(y_j^{(n)}, \frac{1}{n}),
\]

where \( B(z, r) = \{ y \in \mathcal{Y} : d(y, z) < r \} \) is the open ball with center \( z \) and radius \( r \), then, for each \( x \in \mathcal{U}(\mathcal{L}) \), there is \( k_n \in \mathbb{N} \) satisfying

\[
\forall 1 \leq j \leq \lambda_n \exists 1 \leq N \leq k_n : d(L_N x, y_j^{(n)}) < \frac{1}{n}.
\]

In particular, if \( \mathcal{L} \) is universal, then, for any totally bounded set \( \mathcal{M} \subseteq \mathcal{Y} \), there is a sequence \( (k_n)_{n \in \mathbb{N}} \) of natural numbers such that

\[
\{ x \in \mathcal{U}(\mathcal{L}) : F(x, \mathcal{L}, \mathcal{M}, 2/n) \leq k_n \ \forall \ n \in \mathbb{N} \}
\]

containing

\[
\mathcal{U}(\mathcal{L}) \cap \bigcap_{n \in \mathbb{N}} \bigcap_{j=1}^{\lambda_n} \bigcup_{N=1}^{k_n} L_N^{-1}(B(y_j^{(n)}, \frac{1}{n}))
\]

is not empty. We are interested in upper bounds for \( k_n \) depending on \( \mathcal{M} \). Therefore, we introduce the following notation. For a given totally bounded subset \( \mathcal{M} \) of \( \mathcal{Y} \) and \( n \in \mathbb{N} \), we define

\[
\lambda_n := \lambda_n(\mathcal{M}) := \min \left\{ l \in \mathbb{N} : \exists y_1, \ldots, y_l \in \mathcal{Y} \text{ with } \mathcal{M} \subseteq \bigcup_{j=1}^{l} B(y_j, 1/n) \right\}.
\]

to be the \( n \)-th covering number of \( \mathcal{M} \). Since \( \mathcal{M} \) is totally bounded, \( \lambda_n \) is well-defined and the sequence \((\lambda_n)_{n \in \mathbb{N}}\) is obviously increasing. It should be noted that \( \lambda_n \) depends on the given metric \( d \) on \( \mathcal{Y} \)!

For each \( x \in \mathcal{X} \), we obviously have

\[
\forall n \in \mathbb{N} : \lambda_n \leq F(x, \mathcal{L}, \mathcal{M}, d, 1/n).
\]

In this paper, we investigate special sequences of continuous linear operators between spaces of holomorphic functions \( H(\Omega) \) on an open subset \( \Omega \) of \( \mathbb{C} \). As usual, we endow \( H(\Omega) \) with the compact-open topology, that is, the locally convex topology on \( H(\Omega) \) induced by the increasing sequence of seminorms \( \| f \|_{K_n} = \sup \{ |f(z)| : z \in K_n \} \), \( n \in \mathbb{N} \), where \( K = (K_n)_{n \in \mathbb{N}} \) is a compact exhaustion of \( \Omega \), i.e., \( \tilde{K}_n \subseteq \Omega \) compact, \( K_n \) is contained
in the interior of \( K_{n+1} \) for each \( n \in \mathbb{N} \), and \( \bigcup_{n \in \mathbb{N}} K_n = \Omega \). This makes \( H(\Omega) \) a Fréchet space; a metric defining the topology is given by

\[
d_K(f, g) := \sup_{n \in \mathbb{N}} \min \left\{ \| f - g \|_{K_n^1} : \frac{1}{n} \right\}.
\]

(2)

It should be noted at this point that \( d_K(f, g) < 1/n \) if (and only if) \( \| f - g \|_{K_n^1} < 1/n \).

In particular, we consider \( \Omega = \mathbb{D} \), the open unit disk. For this special situation, we will always choose the natural standard compact exhaustion

\[ K_{\mathbb{D}} := (K_n)_{n \in \mathbb{N}}, \text{ where } K_n := \frac{n}{n+1} \mathbb{D}. \]

(3)

Recall, a subset \( \mathcal{M} \) of \( H(\Omega) \) is bounded, by definition, if \( \sup_{f \in \mathcal{M}} \| f \|_{K_n^1} < \infty \) for each \( n \in \mathbb{N} \), i.e., if and only if \( \mathcal{M} \) is locally bounded. By Montel’s Theorem, every bounded subset \( \mathcal{M} \) of \( H(\Omega) \) is relatively compact. Obviously, the converse is always true. Therefore, the bounded subsets of \( H(\Omega) \) are precisely the totally bounded subsets, which are also called normal families in this context. Examples will be given in Section 4.

2 Composition Operators and Fast Approximation

In this section, we consider composition operators on spaces of holomorphic functions, that is, for a given sequence \( (\varphi_n)_{n \in \mathbb{N}} \) of injective holomorphic mappings \( \varphi_n : \Omega_1 \to \Omega_2 \) between open sets \( \Omega_1, \Omega_2 \) in \( \mathbb{C} \), we consider the sequence \( \mathcal{C} = (C_n)_{n \in \mathbb{N}} \) of linear operators

\[ C_n : H(\Omega_2) \to H(\Omega_1), \quad f \mapsto f \circ \varphi_n. \]

Universality of such composition operators has been investigated by several authors, e.g. Bernal and Montes [1], followed by many others and also on different function spaces, see e.g. [2], [3], [5], [7], [6], [10], [11]. Recall, \( (\varphi_n) \) is called run away, if for every pair of compact sets \( K \subseteq \Omega_1, L \subseteq \Omega_2 \), there exists an \( N \in \mathbb{N} \) with

\[ \varphi_N(K) \cap L = \emptyset. \]

This property characterizes the existence of a \( C \)-universal element if \( \Omega_1 = \Omega_2 \) is not conformally equivalent to \( \mathbb{C} \setminus \{0\} \), cf. [3]. In view of the following theorem, it is important to have run away sequences tending in a “controlled” manner towards the boundary of \( \Omega_2 \). Throughout this section, we assume the open sets \( \Omega_1, \Omega_2 \) to consist of simply connected components, and every compact exhaustion \( \mathcal{K} = (K_n)_{n \in \mathbb{N}} \) of them should also have only simply connected components, see e.g. [16] Theorem 13.3).

If \( \Omega \) is a domain in \( \mathbb{C} \), a sequence of sets \( (L_n)_{n \in \mathbb{N}} \) is said to tend to infinity provided that, given a compact set \( L \subseteq \Omega \), there is \( n_0 \in \mathbb{N} \) such that \( L_n \cap L = \emptyset \) for all \( n \geq n_0 \). Observe that, if \( \Omega^* = \Omega \cup \{\omega\} \) denotes the one-point compactification of \( \Omega \), then \( (L_n)_{n \in \mathbb{N}} \) tends to infinity if and only if \( \lim_{n \to \infty} \max \{ \chi(z, \omega) : z \in L_n \} = 0 \), where \( \chi \) is any distance on \( \Omega^* \) defining its topology.

**Proposition 1.** Let \( \varphi_n : \Omega_1 \to \Omega_2, \ n \in \mathbb{N} \), be a sequence of injective holomorphic mappings which is run away. Then, for each compact exhaustion \( \mathcal{K} = (K_n)_{n \in \mathbb{N}} \) of \( \Omega_1 \), there is a sequence \( (m_n)_{n \in \mathbb{N}} \) of natural numbers such that \( \varphi_{m_n}(K_n) \ (n \in \mathbb{N}) \) is pairwise disjoint and tends to infinity.
Note, the image $\varphi(G)$ of a simply connected domain $G$ under an injective holomorphic mapping $\varphi$ is also simply connected. Thus, the sets $\varphi_{m_n}(K_n)$ ($n \in \mathbb{N}$) above have also connected complements.

**Proof.** Fix any compact exhaustion $(L_n)_{n \in \mathbb{N}}$ of $\Omega_2$. Set $m_1 := 1$. Since $(\varphi_n)_{n \in \mathbb{N}}$ is run away, there is $m_2 \in \mathbb{N}$ such that

$$\varphi_{m_2}(K_2) \cap (\varphi_{m_1}(K_1) \cup L_1) = \emptyset.$$ 

If $m_1, m_2, \ldots, m_n$ have been found, there is, by hypothesis, $m_{n+1} \in \mathbb{N}$ such that

$$\varphi_{m_{n+1}}(K_{n+1}) \cap (\bigcup_{j=1}^{n} \varphi_{m_j}(K_j) \cup L_n) = \emptyset.$$ 

Clearly $\varphi_{m_n}(K_n)$ ($n \in \mathbb{N}$) fulfills the requirements of the assertion. 

For the following we abbreviate $C := (C_{m_n})_{n \in \mathbb{N}}$. Before stating our first main result, we provide an approximation lemma based on Arakelian’s Approximation Theorem, cf. [1], [9].

**Lemma 2.** Let $\Omega$ be a domain, $(K_n)_{n \in \mathbb{N}}$ a sequence of pairwise disjoint compact sets in $\Omega$, whose complements are connected. Assume that $(K_n)_{n \in \mathbb{N}}$ tends to infinity and that $f_n \in A(K_n)$, i.e., $f_n$ is continuous on $K_n$ and holomorphic in the interior of $K_n$. Then, there exists $f \in H(\Omega)$ with

$$\forall n \in \mathbb{N}: \max_{z \in K_n} |f(z) - f_n(z)| < \frac{1}{n}.$$ 

**Proof.** Define

$$\delta(z) := -\ln n, \quad q(z) := f_n(z), \quad z \in K_n.$$ 

The union $U := \bigcup_{n \in \mathbb{N}} K_n$ is closed in $\Omega$ and obviously satisfies that $\Omega^\ast \setminus U$ is connected and locally connected at $\omega$. Thus, by Arakelian’s Theorem, there exist $g, h \in H(\Omega)$ with

$$|\delta(z) - g(z)| < 1, \quad \left| \frac{q(z)}{e^{g(z) - 1}} - h(z) \right| < 1, \quad z \in U.$$

For $f(z) := h(z) \cdot e^{g(z)-1}$ and $z \in K_n$, we obtain

$$|f(z) - f_n(z)| = |f(z) - q(z)| < e^{\text{Re } g(z)-1} \leq e|g(z) - \delta(z)|^{-1 + \delta(z)} < e^{\delta(z)} = \frac{1}{n}.$$ 

**Theorem 3.** Let $\varphi_n: \Omega_1 \to \Omega_2$, $n \in \mathbb{N}$, be a sequence of injective holomorphic mappings which is run away and let $K$ be a compact exhaustion of $\Omega_1$. Then, there is a subsequence $(\varphi_{m_n})_{n \in \mathbb{N}}$ of $(\varphi_n)_{n \in \mathbb{N}}$ and a universal function $f \in U(C)$ such that for each normal family $\mathcal{M}$ in $H(\Omega_1)$ with covering numbers $(\lambda_n)_{n \in \mathbb{N}} = (\lambda_n(\mathcal{M}))_{n \in \mathbb{N}}$, we have

$$\forall n \in \mathbb{N}: F(f, C, \mathcal{M}, d_K, \frac{2}{n}) \leq n(\lambda_n + 1).$$
Proof. 1. Let \((m_n)_{n \in \mathbb{N}}\) be a strictly increasing sequence of natural numbers corresponding to the compact exhaustion \(K = (K_n)_{n \in \mathbb{N}},\) as in Proposition 1. Then, the sets \(\varphi_{m_n}(K_n) \ (n \in \mathbb{N})\) are pairwise disjoint, have connected complements and tend to infinity.

2. According to Mergelian’s Theorem, the set of polynomials with coefficients in \(\mathbb{Q} + i\mathbb{Q}\) is dense in \((H(\Omega_1), d_K)\). Let \((q_n)\) be an enumeration of them, and let \(f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in H(\Omega_1)\) be those functions whose \(\frac{1}{n}\) neighborhoods cover \(M\). We define \((f_N)\) as the following sequence

\[
f_1^{(1)}, f_2^{(1)}, \ldots, f_{\lambda_1}^{(1)}, q_1, f_1^{(2)}, f_2^{(2)}, \ldots, f_{\lambda_2}^{(2)}, q_2, f_1^{(3)}, f_2^{(3)}, \ldots, f_{\lambda_3}^{(3)}, q_3, \ldots
\]

3. According to Lemma 2, there exists a function \(f \in H(\Omega_2),\) such that

\[
\max_{\varphi_{m_N}(K_N)} |f(z) - f_N(\varphi_{m_N}^{-1}(z))| < \frac{1}{N}, \quad N \in \mathbb{N},
\]

or equivalently,

\[
\|C_{m_N}f - f_N\|_{K_N} = \|(f \circ \varphi_{m_N}) - f_N\|_{K_N} < \frac{1}{N}, \quad N \in \mathbb{N}.
\]

By definition of the metric \(d_K\) this implies

\[
d_K(C_{m_N}f, f_N) < \frac{1}{N}, \quad N \in \mathbb{N}.
\]

4. Fix \(g \in M\) and \(n \in \mathbb{N}\). According to the second step, we find a function \(f_N\) with

\[
n \leq N \leq \sum_{j=1}^{n-1} (\lambda_j + 1) + \lambda_n \leq n(\lambda_n + 1) \quad \text{and} \quad d_K(f_N, g) < \frac{1}{n}.
\]

Together with the third step, we have

\[
d_K(C_{m_N}f, g) < \frac{1}{n} + \frac{1}{N} \leq \frac{2}{n}.
\]

Moreover,

\[
d_K(C_{m_k}f, q_n) < \frac{1}{k}, \quad n \in \mathbb{N},
\]

with \(k = \sum_{j=1}^{n}(\lambda_j + 1)\) showing that \(f \in \mathcal{U}(\mathcal{C})\) satisfies the desired property. \(\square\)

Remark 4.

(i) Roughly speaking, for a sequence of composition operators between spaces of holomorphic functions, the speed of approximating the elements of a normal family \(M\) by a universal function is only governed by the size of \(M\), measured by the covering numbers \((\lambda_n)_{n \in \mathbb{N}}\).
(ii) In [4], it is proved that, in case of $\Omega_1 = \Omega_2$ not being conformally equivalent to $\mathbb{C}\setminus\{0\}$, the set $U(C)$ is a dense $G_\delta$-set, if non-empty. The above theorem states that there is

$$f \in U(C) \cap \bigcap_{n \in \mathbb{N}} \bigcap_{j=1}^{\lambda_n} \bigcup_{N=1}^{n(\lambda_n+1)} C_{mN}^{-1}(B(\phi^{(n)}_j, \frac{1}{n})),$$

where $\phi^{(n)}_1, \ldots, \phi^{(n)}_{\lambda_n}$ are the centers of open $1/n$-balls covering the normal family $M$. The continuity of the operators $C_{mN}$ implies that the above set is a $G_\delta$-set. But in general it is not dense.

To see this, let $K = (K_n)_{n \in \mathbb{N}}$ be the compact exhaustion of $\Omega_1$ giving the metric $d_K$ and let $M = \{0\}$. Then, one has $\lambda_n = 1$ and one can take $\phi^{(n)}_1 = 0$, $n \in \mathbb{N}$. Assume, there is a sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that

$$\bigcap_{n \in \mathbb{N}} \bigcup_{N=1}^{k_n} C_{mN}^{-1}(B(0, \frac{1}{n}))$$

is dense in $H(\Omega_2)$. Let $K \subseteq \Omega_2$ be compact such that $\bigcup_{N=1}^{k_1} \phi_{m_N}(K_1) \subseteq K$. By assumption, there is

$$g \in \left\{ f \in H(\Omega_2) : \| f - 2 \|_K < 1 \right\} \cap \bigcap_{n \in \mathbb{N}} \bigcup_{N=1}^{k_n} C_{mN}^{-1}(B(0, \frac{1}{n})).$$

Hence, there exists an $1 \leq N \leq k_1$ with

$$\| g - 0 \|_{\phi_{m_N}(K_1)} = \| C_{m_N} g - 0 \|_{K_1} < 1,$$

which gives a contradiction to $\| g - 2 \|_K < 1$.

Let $\mathcal{X}, \mathcal{Y}$ be metric spaces and $L = (L_N)_{N \in \mathbb{N}}$ a universal sequence of continuous mappings from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{M} \subseteq \mathcal{Y}$ is totally bounded, we have just seen that for any sequence of natural numbers $(k_n)_{n \in \mathbb{N}}$ the $G_\delta$-set in $\mathcal{X}$ need not be dense in $\mathcal{X}$ although there is always some sequence $(k_n)_{n \in \mathbb{N}}$ such that the above set is non-empty, cf. the introduction.

However, if one weakens the requirement

$$\forall \, n \in \mathbb{N} : \| F(x, L, M, 2/n) \| \leq k_n$$

to (we use the standard Landau notations)

$$(F(x, L, M, 2/n))_{n \in \mathbb{N}} \in O((k_n)_{n \in \mathbb{N}}),$$

then the corresponding set is dense, see the next result. Whenever the index, mostly $n \in \mathbb{N}$, is clear, we will shorten the Landau notation from $(a_n)_{n \in \mathbb{N}} \in O((b_n)_{n \in \mathbb{N}})$ to $a_n \in O(b_n)$.

**Theorem 5.** Let $\varphi_n : \Omega_1 \to \Omega_2$, $n \in \mathbb{N}$, be a sequence of injective holomorphic mappings which is run away, and let $K$ be a compact exhaustion of $\Omega_1$. Then, there is a subsequence $(\varphi_{m_n})_{n \in \mathbb{N}}$ of $(\varphi_n)_{n \in \mathbb{N}}$ and a dense set of universal functions $f \in U(C)$ in $H(\Omega_2)$, such that
for every choice of countably many normal families \( \mathcal{M}_i \) in \( H(\Omega_1) \), \( i \in \mathbb{N} \), with covering numbers \( (\lambda_{n,i})_{n \in \mathbb{N}} = (\lambda_n(\mathcal{M}_i))_{n \in \mathbb{N}} \), we have

\[
\forall i \in \mathbb{N}: \quad F(f, \mathcal{C}, \mathcal{M}_i, d_K, \frac{2}{n}) \in O(n\lambda_{n,i}). \tag{4}
\]

**Proof.** 1. Let \((m_n)_{n \in \mathbb{N}}\) be again a strictly increasing sequence of natural numbers corresponding to the compact exhaustion \( \mathcal{K} = (K_n)_{n \in \mathbb{N}} \), as in Proposition [1]. Then, the sets \( \varphi_{m_n}(K_n) \) \((n \in \mathbb{N})\) are pairwise disjoint, have connected complements and tend to infinity. We have to show that for given \( h \in H(\Omega_2) \), \( K \subseteq \Omega_2 \) compact and \( \varepsilon > 0 \), there exists a universal function \( f \in \mathcal{U}(\mathcal{C}) \) with the desired property and

\[
\|f - h\|_K < \varepsilon.
\]

Since \( \varphi_{m_n}(K_n) \) \((n \in \mathbb{N})\) tends to infinity, there is some \( M \in \mathbb{N} \) such that \( K \cap \varphi_{m_n}(K_n) = \emptyset \) for all \( n > M \).

2. Also, let \((q_n)\) be as in the proof of Theorem [3] and let \( f_{1}^{(n,i)}, \ldots, f_{\lambda_{n,i}}^{(n,i)} \in H(\Omega_1) \) be those functions whose \( \frac{1}{n}\)-neighborhoods cover \( \mathcal{M}_i \), merged in sequences \((f^{(i)}_{n})_{n \in \mathbb{N}}\)

\[
\begin{align*}
&f_1^{(1,i)}, f_2^{(1,i)}, \ldots, f_{\lambda_1}^{(1,i)}, f_1^{(2,i)}, f_2^{(2,i)}, \ldots, f_{\lambda_2}^{(2,i)}, f_1^{(3,i)}, f_2^{(3,i)}, \ldots, f_{\lambda_3}^{(3,i)}, \ldots
\end{align*}
\]

With these sequences we build \( (f_N) \) as follows: Every \((2j - 1)\)-st element of \( (f_N) \) is \( q_j, j \in \mathbb{N} \). From the remaining elements every \((2j - 1)\)-st element is \( f_j^{(1)}, j \in \mathbb{N} \). Again, from the remaining every \((2j - 1)\)-st element is \( f_j^{(2)}, j \in \mathbb{N} \), and so on.

3. According to Lemma [2] there exists a function \( f \in H(\Omega_2) \), such that

\[
\|f - h\|_K < \varepsilon \quad \text{and} \quad \max_{\varphi_{m_{M+N}}(K_{M+N})} |f(z) - f_{N}(\varphi_{m_{M+N}}^{-1}(z))| < \frac{1}{M + N}, \quad N \in \mathbb{N},
\]

or equivalently,

\[
\|C_{m_{M+N}} f - f_N\|_{K_{M+N}} = \|(f \circ \varphi_{m_{M+N}}) - f_N\|_{K_{M+N}} < \frac{1}{M + N}, \quad N \in \mathbb{N}.
\]

By definition of the metric \( d_K \), this implies

\[
d_K(C_{m_{M+N}} f, f_{M+N}) < \frac{1}{M + N}, \quad N \in \mathbb{N}.
\]

4. Fix \( g \in \mathcal{M}_i \) and \( n \in \mathbb{N} \). According to the second step, we find a function \( f_N \) with

\[
n \leq M + N \leq c_i \cdot n(\lambda_{n,i} + 1) \leq c_i n\lambda_{n,i},
\]

for appropriately chosen constants \( c_i, \lambda_{n,i} \), and

\[
d_K(f_N, g) < \frac{1}{n}.
\]

Together with the third step, we have

\[
d_K(C_{m_{M+N}} f, g) < \frac{1}{n} + \frac{1}{M + N} \leq \frac{2}{n}.
\]
Moreover,
\[ d_K(C_{M+2n-1} f, q_n) < \frac{1}{M + 2n - 1}, \quad n \in \mathbb{N}, \]
showing that \( f \in U(C) \) satisfies the desired property.

In equation (4), we have seen
\[ \forall i, n \in \mathbb{N}: F(f, C, M_i, d_K, \frac{2}{n}) \leq c_i n \lambda_{n,i}, \]
where the constants \( c_i \) as given in (5) grow exponentially in \( i \), more precisely \((c_i)_{i \in \mathbb{N}} \in \Theta((2^i)_{i \in \mathbb{N}})\), and \((2^i)_{i \in \mathbb{N}} \in O((c_i)_{i \in \mathbb{N}})\), as we see from the second step of the above proof.

3 Differentiation Operators and Fast Approximation

In this section, we consider the differentiation operator

\[ D: H(\Omega) \to H(\Omega), \quad f \mapsto f', \]
on spaces of holomorphic functions on a simply connected bounded domain \( \Omega \subseteq \mathbb{C} \), as well as the sequence \( D := (D^n)_{n \in \mathbb{N}} \). It is known that the existence of \( f \in U(D) \) is equivalent to \( \Omega \) being simply connected, cf. [18]. Therefore, without loss of generality, we may and will assume \( \Omega \) to be simply connected throughout the whole paragraph. Since differentiation commutes with translations, we can assume \( 0 \in \Omega \) without loss of generality. More precisely, we may assume that \( 0 \) is contained in the interior of \( K_1 \) for a compact exhaustion \( K = (K_n)_{n \in \mathbb{N}} \) of \( \Omega \).

Moreover, there is a compact exhaustion \( K = (K_n)_{n \in \mathbb{N}} \) of \( \Omega \) such that \( K_n \) is connected and simply connected for every \( n \in \mathbb{N} \), see e.g. [16, Theorem 13.3]. Therefore, we assume without loss of generality that for the metric \( d_K \) inducing the compact-open topology on \( H(\Omega) \), cf. (2), we have \( K_n \) connected and simply connected.

Furthermore, we denote the \( m \)-th Faber polynomial for \( K_n \) by \( F_{n,m} \), \( m \in \mathbb{N}_0 \). Then, \( F_{n,m} \) is a polynomial of degree \( m \) which is obtained in the following way, see e.g. [9] or [13]. By the Riemann Mapping Theorem, there is a unique conformal mapping \( \varphi_n: \mathbb{C}\setminus K_n \to \mathbb{C}\setminus D \) with \( \varphi_n(\infty) = \infty \) and \( \varphi_n'(\infty) > 0 \). Hence, for some \( c > 0 \), we have for \( |z| \) sufficiently large
\[ \varphi_n(z) = \frac{1}{c} z + c_0 + \sum_{\nu=1}^{\infty} c_\nu z^{-\nu}. \]
Moreover, for \( |z| \) sufficiently large and every \( m \in \mathbb{N} \), we have
\[ \varphi_{n,m}(z) = F_{n,m}(z) + \sum_{\nu=1}^{\infty} \alpha_\nu z^{-\nu}, \]
that is, \( F_{n,m} \) is the analytic part of the Laurent expansion of \( \varphi_{n,m} \). With \( \psi_n := \varphi_n^{-1}: \mathbb{C}\setminus D \to \mathbb{C}\setminus K_n \), we have
\[ \psi_n(w) = cw + \sum_{\nu=1}^{\infty} d_\nu w^{-\nu}, \quad |w| > 1. \]
For $R > 1$, we set $\Gamma_{n,R} := \{\psi_n(w) : |w| = R\}$. Then, $\Gamma_{n,R}$ is a closed Jordan curve, and for each $n \in \mathbb{N}$, there is $R_n > 1$, such that $\Gamma_{n,R} \subseteq \Omega$ for all $1 < R < R_n$. Denoting by $I_{n,R}$ the bounded (open) component of $\mathbb{C} \setminus \Gamma_{n,R}$, we obtain $K_n \subseteq I_{n,R} \subseteq \Omega$ for every $n \in \mathbb{N}$ and $1 < R < R_n$.

If $f$ is a complex function holomorphic in a neighborhood $I_{n,R}$ of $K_n$, we define for $\nu \in \mathbb{N}_0$

$$a_\nu(f, K_n) := \frac{1}{2\pi i} \int_{|w|=r} \frac{f(\psi_n(w))}{w^{\nu+1}} dw,$$

which is independent of $r \in (1, R)$. Then

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu(f, K_n) F_{n,\nu}(z),$$

where the series converges uniformly and absolutely on $I_{n,R}$, in particular, on $K_n$. Thus, this expansion is valid in $I_{n,R}$ for every $f \in H(\Omega)$. Moreover, the above so-called Faber expansion of $f$ is unique, see again e.g. [9] or [13]. From this, and the fact that $F_{n,m}$ is a polynomial of degree $m$, it follows that for every polynomial $p$, we have

$$p = \sum_{\nu=0}^{m} a_\nu(p, K_n) F_{n,\nu},$$

whenever $m \geq \deg(p)$. In case of $K_n = \{z \in \mathbb{C} : |z - z_0| \leq \rho\}$, the above expansion of $f$ is nothing but the Taylor expansion of $f$ about $z_0$.

From [13, Lemma preceding Theorem 3.16], it follows

$$\frac{1}{2} R^{\nu} < |F_{n,\nu}(z)| < \frac{3}{2} R^{\nu},$$

(6)

for all $1 < R < R_n$, for every $z \in \Gamma_{n,R}$, and $\nu \in \mathbb{N}_0$.

For $f \in H(\Omega)$ and $n, m \in \mathbb{N}$, we define

$$T_{n,m}f : \mathbb{C} \to \mathbb{C}, \quad T_{n,m}f(z) := \sum_{\nu=0}^{m} a_\nu(f, K_n) F_{n,\nu}(z),$$

that is, $T_{n,m}f$ is a polynomial of degree $\leq m$.

Moreover, we denote by $f^{(-j)}$ the $j$-th anti-derivative of $f$, i.e.,

$$f^{(0)}(z) := f(z), \quad f^{(-j)}(z) := \int_0^z f^{(-j+1)}(\zeta)d\zeta, \quad j \in \mathbb{N}, z \in \Omega.$$

Recall, we assume without restriction $0 \in \Omega$. It is very well-known that for every $f \in H(\Omega)$ the sequence $(I_jf)_{j \in \mathbb{N}_0}$ converges to zero in $H(\Omega)$, where $I_j : H(\Omega) \to H(\Omega)$, $I_j f := f^{(-j)}$, $j \in \mathbb{N}_0$, see e.g. [12] Lemma 1].

The next Lemma is rather technical. Its conclusions simplify in case of $\Omega = \mathbb{D}$, which will be stated separately as Corollary 7 below.

**Lemma 6.** Let $\mathcal{K}$ be a compact exhaustion of $\Omega$ and $\mathcal{M} \subseteq H(\Omega)$ a normal family. For $n \in \mathbb{N}$, let

$$M_n := M_n(\mathcal{M}) := \sup_{f \in \mathcal{M}} \max_{|w|=\frac{1}{2}(1+R_n)} |f(\psi_n(w))|.$$
1. There is an increasing sequence

\[ \gamma_n(M) \in O \left( \frac{R_n + 1}{R_n - 1} \ln \left( n \frac{R_n + 1}{R_n - 1} M_n \right) \right), \]

of natural numbers tending to infinity such that, for every \( f \in M \), we have

\[ \| T_{n, \gamma_n} f - f \|_{K_n} < \frac{1}{n}. \]

Moreover, if there is \( k \in \mathbb{N}_0 \) such that \( M_n(M) \in O(n^k) \), then,

\[ \gamma_n(M) \in O \left( R_n + 1 \ln \left( n R_n + 1 \right) \right). \]

2. There is a sequence \( (\sigma_n(M))_{n \in \mathbb{N}} \) of natural numbers tending to infinity, such that for every \( f \in M \), \( n \in \mathbb{N} \), and \( m \in \mathbb{N}_0 \), we have

\[ \| (T_{n,m} f)^{-j} \|_{K_n} < \frac{1}{n^2}, \]

whenever \( j \geq \sigma_n(M) \).

We point out that the above sequences \( (\gamma_n(M))_{n \in \mathbb{N}} \) and \( (\sigma_n(M))_{n \in \mathbb{N}} \) depend on the compact exhaustion \( K \) of \( \Omega! \)

**Proof.** Let \( n \in \mathbb{N} \) be arbitrary. Note, \( M_n < \infty \) by the total boundedness of \( M \).

1. For \( f \in M \) and \( 1 < R < R_n \), we have by the maximum principle

\[ \| T_{n,m} f - f \|_{K_n} \leq \sum_{\nu = m+1}^{\infty} |a_{\nu}(f, K_n)| \| F_{n,\nu} \|_{K_n} \leq \frac{3}{2} \sum_{\nu = m+1}^{\infty} |a_{\nu}(f, K_n)| R^\nu. \]

Moreover, for the Faber coefficients we obtain

\[ |a_{\nu}(f, K_n)| = \frac{1}{2\pi} \left| \int_{|w| = \frac{1}{2}(1 + R_n)} f(\psi_n(w)) \frac{1}{w^{\nu+1}} dw \right| \leq \left( \frac{2}{1 + R_n} \right)^\nu M_n, \]

so, for \( 1 < R < \frac{2}{3} + \frac{1}{3}R_n = \frac{1}{3}(2 + R_n) \),

\[ \frac{3}{2} \sum_{\nu = m+1}^{\infty} |a_{\nu}(f, K_n)| R^\nu \leq \frac{3}{2} M_n \sum_{\nu = m+1}^{\infty} \left( \frac{2R}{1 + R_n} \right)^\nu \]

\[ = \frac{3}{2} M_n \left( \frac{2R}{1 + R_n} \right)^{m+1} \frac{1}{1 - \frac{2R}{1 + R_n}} \]

\[ \leq \frac{3}{2} M_n \left( \frac{4 + 2R_n}{3 + 3R_n} \right)^{m+1} \frac{1 + R_n}{R_n - 1} \]

\[ \leq \frac{5 R_n + 1}{R_n - 1} M_n \left( \frac{4 + 2R_n}{3 + 3R_n} \right)^{m+1}. \]
Thus, in order that $\|T_{n,m}f - f\|_{K_n} < \frac{1}{n}$, it suffices
\[
\ln \left( 5n \frac{R_n + 1}{R_n - 1} M_n \right) < (m + 1) \ln \left( \frac{3 + 3R_n}{4 + 2R_n} \right) = (m + 1) \ln \left( 1 + \frac{R_n - 1}{2(2 + R_n)} \right).
\]

Using the elementary inequality
\[
\forall x \geq 0 : \frac{x}{1 + x} \leq \ln(1 + x),
\]
the above inequality is surely satisfied if
\[
\ln \left( 5n \frac{R_n + 1}{R_n - 1} M_n \right) < (m + 1) \frac{R_n - 1}{2(2 + R_n)} = (m + 1) \frac{R_n - 1}{3(R_n + 1)}.
\]

Taking all this together, we conclude
\[
\sum_{\nu=m+1}^{\infty} |a_{\nu}(f, K_n)| \|F_{n,\nu}\|_{K_n} < \frac{1}{n}
\]
for $n \in \mathbb{N}$, and for all $f \in \mathcal{M}$, provided that
\[
m \geq 3 \frac{R_n + 1}{R_n - 1} \ln \left( 5n \frac{R_n + 1}{R_n - 1} M_n \right).
\]  

(7)

2. (i) Now, we consider $T_{n,m}$ as a continuous linear operator from $H(\Omega)$ into $H(I_{n,R_n})$ and, first, we show that $\mathcal{N} := \bigcup_{m \in \mathbb{N}_0} T_{n,m}(\mathcal{M})$ is a normal family in $H(I_{n,R_n})$:

From the above mentioned properties of the Faber expansion, it follows that for every $f \in H(\Omega)$ the sequence $(T_{n,m}f)_{m \in \mathbb{N}_0}$ converges in $H(I_{n,R_n})$ to $f|_{I_{n,R_n}}$. Since $H(\Omega)$ is a Fréchet space, the equicontinuity of the sequence of operators $(T_{n,m})_{m \in \mathbb{N}_0}$ follows from the Uniform Boundedness Principle.

Next, let $U$ be an absolutely convex zero neighborhood in $H(I_{n,R_n})$. By the equicontinuity of $(T_{n,m})_{m \in \mathbb{N}_0}$, there is an absolutely convex zero neighborhood $V$ in $H(\Omega)$ such that $T_{n,m}(V) \subseteq U$ for every $m \in \mathbb{N}_0$. Since $\mathcal{M}$ is a normal family, hence, bounded in $H(\Omega)$, there is $\rho > 0$ with $\mathcal{M} \subseteq \rho V$, implying $\mathcal{N} := \bigcup_{m \in \mathbb{N}_0} T_{n,m}(\mathcal{M}) \subseteq \rho U$. Since $U$ was arbitrary this gives the boundedness of $\mathcal{N}$ in $H(I_{n,R_n})$. Thus, $\mathcal{N}$ is relatively compact, i.e., a normal family.

(ii) Since we assumed $0 \in K_1$, the above-explained mappings $I_j : H(I_{n,R_n}) \to H(I_{n,R_n})$ are well-defined, continuous and linear. Moreover, for each $f \in H(I_{n,R_n})$ the sequence $(I_j f)_{j \in \mathbb{N}_0}$ tends to zero in $H(I_{n,R_n})$. The Uniform Boundedness Principle implies, again, the equicontinuity of $(I_j)_{j \in \mathbb{N}_0}$. Because $K_n \subseteq I_{n,R_n}$, we can find a zero neighborhood $V$ such that $\|I_j f\|_{K_n} < \frac{1}{2n^2}$ for every $f \in V$ and every $j \in \mathbb{N}_0$. Since for every $f \in H(I_{n,R_n})$ there is $j(f) \in \mathbb{N}$ with $\|I_j f\|_{K_n} < \frac{1}{2n^2}$ for each $j \geq j(f)$,
\[
\|I_j g\|_{K_n} \leq \|I_j (g - f)\|_{K_n} + \|I_j f\|_{K_n} < \frac{1}{n^2}
\]
holds for every $g \in f + V$ and $j \geq j(f)$.  

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Because $\mathcal{N} \subseteq \bigcup_{f \in \mathcal{N}} (f + V)$ is totally bounded, there are $f_1, \ldots, f_k \in \mathcal{N}$ such that

$$\bigcup_{m \in \mathbb{N}_0} T_{n,m}(\mathcal{M}) = \mathcal{N} \subseteq \bigcup_{l=1}^k (f_l + V).$$

Setting $\sigma_n := \max\{j(f_1), \ldots, j(f_k)\}$, we finally obtain $\|(T_{n,m}f)^{(-j)}\|_{K_n} < \frac{1}{m^2}$ for each $f \in \mathcal{M}$, $m \in \mathbb{N}_0$, and $j \geq \sigma_n$.

\[\text{Corollary 7. Let } \mathcal{M} \subseteq H(\mathbb{D}) \text{ be a normal family and } K_\mathbb{D} \text{ be the standard compact exhaustion of } \mathbb{D}, \text{ cf. } [3].\]

1. For each $n \in \mathbb{N}$, we have $M_n = M_n(\mathcal{M}) = \sup_{f \in \mathcal{M}} \|f\|_{K_{2n+1}}.$

2. For the sequence $(\gamma_n(\mathcal{M}))_{n \in \mathbb{N}}$, we have

$$\gamma_n(\mathcal{M}) \in O(n \ln(nM_n)),$$

and if $M_n(\mathcal{M}) \in O(n^k)$ for some $k \in \mathbb{N}_0$, then,

$$\gamma_n(\mathcal{M}) \in O(n \ln(n)).$$

3. For the sequence $(\sigma_n(\mathcal{M}))_{n \in \mathbb{N}}$, we can assume without restriction

$$\sigma_n(\mathcal{M}) \in O(\ln(n^2M_n)).$$

\[\text{Proof. For the compact set } K_n = \frac{n}{n+1} \mathbb{D}, \text{ we have } \varphi_n : \mathbb{C} \setminus K_n \to \mathbb{C} \setminus \mathbb{D}, \varphi_n(z) = \frac{n+1}{n} z.\]

Thus, $\psi_n(z) = \frac{n}{n+1} z$ and $R_n = \frac{n+1}{n}$. Moreover, because $\varphi_n'(z) = (\frac{n+1}{n})^\nu z^\nu$, we have $F_{n,\nu}(z) = (\frac{n+1}{n})^\nu z^\nu$. So, we obtain for sufficiently small $1 < r$

$$a_\nu(f, K_n) = \frac{1}{2\pi i} \int_{|w| = r} \frac{f(\psi_n(w))}{w^{\nu+1}} dw$$

$$= \left( \frac{n}{n+1} \right)^\nu \frac{1}{2\pi i} \int_{|w| = \frac{n}{n+1} r} \frac{f(w)}{w^{\nu+1}} dw$$

$$= \left( \frac{n}{n+1} \right)^\nu a_\nu(f),$$

where $a_\nu(f)$ denotes the $\nu$-th Taylor coefficient of $f$ expanded about the origin. Therefore, for every $f \in H(\mathbb{D})$, $n \in \mathbb{N}$, and $\nu \in \mathbb{N}_0$, we have

$$\forall z \in K_n : a_\nu(f, K_n) F_{n,\nu}(z) = a_\nu(f) z^\nu. \quad (8)$$

1. For each $f \in \mathcal{M}$, we have

$$\max_{|w| = \frac{1}{2} (1 + R_n)} |f(\psi_n(w))| = \max_{|z| = \frac{n+1}{n+2}} |f(z)| = \|f\|_{K_{2n+1}}.$$
2. From inequality \( \Box \), equation \( \Box \), and \( \frac{2n+1}{n-1} = 2n + 1 \), we obtain for every \( f \in \mathcal{M} \) and each \( n \in \mathbb{N} \) that
\[
\sum_{\nu=m+1}^{\infty} |a_\nu(f)| \left( \frac{n}{n+1} \right)^\nu < \frac{1}{n},
\]
whenever
\[
m \geq 3(2n + 1) \ln(5n(2n + 1)M_n(\mathcal{M})).
\]
(9)

3. As shown above, the \( m \)-th partial sums \( T_{n,m} \) of the Faber expansions are independent of \( n \) and coincide with the \( m \)-th Taylor polynomials expanded about the origin. Because \( K_n = \frac{n}{n+1} \mathbb{D} \), it follows
\[
|a_\nu(f)| = \left| \frac{1}{2\pi i} \int_{|z|=\frac{2n+1}{n+1}} f(z) z^{-\nu-1} dz \right| \leq \left( \frac{2n+2}{2n+1} \right)^\nu \|f\|_{K_{2n+1}^1},
\]
which leads, for every \( n \in \mathbb{N}, m \in \mathbb{N}_0, j \geq 2 \), and \( f \in \mathcal{M} \), to
\[
\|(T_{n,m}f)^{(\nu-j)}\|_{K_n} = \left\| \sum_{\nu=0}^{m} a_\nu(f) \left( \frac{n}{n+1} \right)^\nu \right\|_{K_n} \leq 1 \left( \frac{n}{n+1} \right)^{j} \|f\|_{K_{2n+1}^1} \sum_{\nu=0}^{\infty} \frac{\nu^j}{j!} \left( \frac{n(2n+2)}{(n+1)(2n+1)} \right)^\nu \leq \frac{(2n+1)M_n}{j!}
\]
\[
\leq \frac{3nM_n}{j!}.
\]
If \( j \) satisfies \( j! > 3n^2 M_n \), we get \( \|(T_{n,m}f)^{(\nu-j)}\|_{K_n} < \frac{1}{n^2} \) for all \( f \in \mathcal{M} \). In particular, by applying Stirling’s Formula, we can choose \( \sigma_n(\mathcal{M}) \in O(\ln(n^2 M_n)) \).

Theorem 8. Let \( K \) be a compact exhaustion of \( \Omega \) and \( \mathcal{M} \) be a normal family in \( H(\Omega) \) with covering numbers \( (\lambda_n)_{n \in \mathbb{N}} = (\lambda_n(\mathcal{M}))_{n \in \mathbb{N}} \), as well as the sequences \( (\gamma_n)_{n \in \mathbb{N}} = (\gamma_n(\mathcal{M}))_{n \in \mathbb{N}} \) and \( (\sigma_n)_{n \in \mathbb{N}} = (\sigma_n(\mathcal{M}))_{n \in \mathbb{N}} \) from Lemma \( \Box \). Then, there exists a universal function \( f \in \mathcal{U}(\mathbb{D}) \) such that
\[
\forall n \in \mathbb{N}: F(f, \mathcal{D}, \mathcal{M}, d_{K}, \frac{3}{n}) \leq n(\lambda_n + 1)(\gamma_n + \sigma_n(\lambda_n + 1)).
\]

Proof. 1. Let \( f^{(n)}_{1}, \ldots, f^{(n)}_{\lambda_n+1} \in H(\Omega) \) be those functions whose \( \frac{1}{n} \)-neighborhoods cover \( \mathcal{M} \). Moreover, let \( Q = \{q_n := f^{(n)}_{\lambda_n+1} : n \in \mathbb{N}\} \) be a dense set of polynomials in \( H(\Omega) \), which exists by Mergelyan’s Theorem and our general assumption that \( \Omega \) is simply connected. Without restriction, we may assume deg\(q_n) \leq \gamma_n \), as well as \( \|q_n^{(\nu-j)}\|_{K_n} < 1/n^2 \) for every \( j \geq \sigma_n \), holds for every \( n \in \mathbb{N} \). Otherwise, we elongate the sequence \( (q_n) \) by adding the zero polynomial several times, noticing \( (\gamma_n), (\sigma_n) \) may be chosen to tend to \( \infty \), as \( n \to \infty \).

Now, we define \( (f_k)_{k \in \mathbb{N}} \) as the following sequence
\[
f^{(1)}_1, f^{(1)}_2, \ldots, f^{(1)}_{\lambda_1+1}, f^{(2)}_1, f^{(2)}_2, \ldots, f^{(2)}_{\lambda_2+1}, \ldots, f^{(m)}_1, f^{(m)}_2, \ldots, f^{(m)}_{\lambda_m+1}, \ldots
\]
For every \( k \in \mathbb{N} \), there are unique \( n = n(k) \in \mathbb{N} \), \( n \leq k \), and \( 1 \leq j \leq \lambda_n + 1 \) such that \( f_k = f_j^{(n)} \). According to Lemma 6 and the fact that the degree of \( q_n \) does not exceed \( \gamma_n \), it holds for \( P_k := T_{n,\gamma_n} f_k = T_{n(k),\gamma_n(k)} f_k \) that

\[
\| P_k - f_k \|_{K_n} = \| T_{n,\gamma_n} f_k - f_k \|_{K_n} < \frac{1}{n}.
\]

Therefore, by the definition of our metric, this implies

\[
d_K(f_k, P_k) < \frac{1}{n}
\]

for every \( k \in \mathbb{N} \). Note, in case of \( f_k = f_j^{(n)} = q_n \), we have \( P_k = T_{n,\gamma_n} q_n = q_n \), because \( q_n \) is a polynomial of degree not exceeding \( \gamma_n \).

Next, we define \( N_1 := \sigma_1 + 1 \), \( N_k := \gamma_n(k) + \sigma_k + N_{k-1} \), \( k \geq 2 \), and the function \( f \) as

\[
f(z) := \sum_{j=1}^{\infty} P_j^{(-N_j)}(z).
\]

Since, for every \( n \leq l \), we have

\[
\sum_{j=l}^{l+m} \| P_j^{(-N_j)} \|_{K_n} \leq \sum_{j=l}^{l+m} \| P_j^{(-N_j)} \|_{K_j} \leq \sum_{j=l}^{l+m} \frac{1}{j^2}
\]

by Lemma 6 and the choice of \( Q \), \( f \) is a well-defined holomorphic function in \( \Omega \).

2. Let \( k \in \mathbb{N} \). For all \( 1 \leq j < k \), we have \( N_k - N_j > \gamma_n(k) \geq \gamma_n(j) \). It follows

\[
f^{(N_k)}(z) = P_k(z) + \sum_{j=k+1}^{\infty} P_j^{(-N_j+N_k)}(z)
\]

For \( j \geq k + 1 \), we have \( N_j - N_k \geq \sigma_j \). Since \( k \geq n \), we estimate

\[
\| f^{(N_k)} - P_k \|_{K_n} = \left\| \sum_{j=k+1}^{\infty} P_j^{(-N_j+N_k)} \right\|_{K_n} \leq \sum_{j=k+1}^{\infty} \| P_j^{(-N_j+N_k)} \|_{K_j}
\]

\[
= \sum_{j=k+1}^{\infty} \| (T_{n(j),\gamma_n(j)} f_j)^{(-N_j+N_k)} \|_{K_j}
\]

\[
\leq \sum_{j=k+1}^{\infty} \frac{1}{j^2} \leq \frac{1}{k} < \frac{1}{n}.
\]

Therefore, by the definition of our metric \( d_K \), we obtain

\[
d_K(D^{N_k} f, P_k) < \frac{1}{n}.
\]
3. Let given an arbitrary function $g \in \mathcal{M}$. Hence, there exists a function $f_k$ with $k \leq n \cdot (\lambda_n + 1)$ and

$$d_K(f_k, g) < \frac{1}{n}.$$ 

Together with (11) and (12), it follows

$$d_K(D^N f, g) < \frac{3}{n}.$$ 

We calculate

$$N_k = \sum_{j=1}^{k} \gamma_n(j) + \sigma_j \leq k(\gamma_n(k) + \sigma_k) \leq n(\lambda_n + 1)(\gamma_n + \sigma_n(\lambda_n + 1)),$$

as proposed.

4. Moreover, by construction, we have $P_k = q_n$ for every $k \in \mathbb{N}$ with $f_k = q_n$. From (12), we conclude

$$d_K(D^N f, q_n) < \frac{1}{n}$$

for such $k$, which finally shows $f \in \mathcal{U}(D)$. 

Combining the above Theorem with Corollary we immediately get the following.

**Corollary 9.** Let $K_D$ be the standard exhaustion of $D$ and $\mathcal{M} \subseteq H(D)$ be a normal family with covering numbers $(\lambda_n)_{n \in \mathbb{N}}$. Then, there is a universal function $f \in \mathcal{U}(D)$ such that

$$F(f, \frac{3}{n}) \in O(n^{2} \lambda_n \ln(n\lambda_n \max\{1, M_{2n}\}))$$

or equivalently

$$F(f, \frac{1}{n}) \in O(n^{2} \lambda_{3n} \ln(n\lambda_{3n} \max\{1, M_{6n}\})), $$

where $M_n = M_n(\mathcal{M}) = \sup_{f \in \mathcal{M}} \| f \|_{K_{2n+1}}$.

**Remark 10.** In contrast to sequences of composition operators, the speed of approximating elements of a normal family $\mathcal{M}$ by universal functions for the differentiation operator is not only governed by the size of $\mathcal{M}$, measured by the covering numbers $(\lambda_n)_{n \in \mathbb{N}}$. In case of $\Omega = \mathbb{D}$, also the growth of the members of $\mathcal{M}$, given by the sequence $(M_n)_{n \in \mathbb{N}}$, comes into play. In the general case, the sequences $(\gamma_n(\mathcal{M}))_{n \in \mathbb{N}}$ quantizing the approximative behavior of the Faber expansion, and $(\sigma_n(\mathcal{M}))_{n \in \mathbb{N}}$ giving the speed of convergence towards zero of the anti-derivatives, are relevant.

### 4 Examples of normal families

We conclude with some examples of normal families in $H(D)$ and apply our results from the previous sections. Throughout, we choose the standard compact exhaustion $K_D$ of $\mathbb{D}$, cf. (3). Therefore, we omit the reference to the fixed metric $d_K$ in the notation of this
Trivially, every finite subset \( E = \{f_1, \ldots, f_k\} \) of \( H(D) \) is a normal family with eventually constant sequence \( \lambda_n(E) = k \). Applying Theorem 3 and Corollary 9 respectively yields the following result.

**Corollary 11.** Let \( C \) be a sequence of composition operators as in Theorem 3, \( D \) the sequence of differentiation operators. For every finite subset \( E = \{f_1, \ldots, f_k\} \) of \( H(D) \), there are \( f \in U(C) \) and \( g \in U(D) \) such that

\[
F(f, C, E, \frac{1}{n}) \in O(n)
\]

and

\[
F(g, D, E, \frac{1}{n}) \in O(n^2 \ln(n \max \{1, M_{6kn}(E)\}))
\]

respectively.

Moreover, the unit ball

\[
B^\infty := \{ f \in H(D) : |f(z)| \leq 1 \text{ for all } z \in D \}
\]

of \( H^\infty(D) \) is a normal family in \( H(D) \) because, obviously, it is locally bounded. It is immediately seen that the corresponding sequence \( (M_n(B^\infty))_{n \in \mathbb{N}} \) is constantly equal to one. Hence, taking \( \ln(n) \in O(n^\varepsilon) \) for each \( \varepsilon > 0 \) into account, another application of Theorem 3 and Corollary 9 gives the next corollary.

**Corollary 12.** Let \( C \) be a sequence of composition operators as in Theorem 3, \( D \) the sequence of differentiation operators. There is \( f \in U(C) \) with

\[
F(f, C, B^\infty, \frac{1}{n}) \in O(n\lambda_{2n}(B^\infty)).
\]

Moreover, there is \( g \in U(D) \) such that

\[
F(g, D, B^\infty, \frac{1}{n}) \in O(n^{2+\varepsilon}(\lambda_{3n}(B^\infty))^{1+\varepsilon}),
\]

for every \( \varepsilon > 0 \).

By Corollary 9 the covering numbers \( (\lambda_n(M))_{n \in \mathbb{N}} \), as well as the sequence \( (M_n(M))_{n \in \mathbb{N}} \), determine how fast the approximation of a normal family \( M \subseteq H(D) \) by a universal function may be.

In order to get a better impression of the concrete error terms involved, we shall consider the following example. It is well-known, cf. [16, Theorem 12.6], that the set of holomorphic one-to-one mappings of \( D \) onto itself, \( Aut(D) \), is given by

\[
Aut(D) = \left\{ f_{\gamma,a}(z) = e^{i\gamma} \frac{z-a}{1-\bar{a}z} : \gamma \in [0, 2\pi), a \in D \right\}.
\]

Since \( f_{\gamma,a}(D) = D \), \( Aut(D) \) is bounded in \( H(D) \), so a normal family, and \( M_n(Aut(D)) = 1 \) for every \( n \in \mathbb{N} \). Next, we give bounds for \( \lambda_n(Aut(D)) \).

**Lemma 13.** For the normal family \( Aut(D) \) in \( H(D) \), we have \( \lambda_n(Aut(D)) \in O(n^7) \).
Thus, for 
\[ d \] are needed. Since, by the definition of the metric \( K \) for fixed \( n \) if one considers, instead of \( \gamma \) and \( O(n^5) \) different \( a \in \mathbb{D} \) are needed. Since, by the definition of the metric \( d_K \), the inequality \( \| f_{\gamma_1} - f_{\gamma_2, a_2} \|_{K_n} < 1/n \) implies \( d_K(f_{\gamma_1, a_1}, f_{\gamma_2, a_2}) < 1/n \), we obtain \( \lambda_n \in O(n^7) \).

\[ \square \]

Remark 14. If one considers, instead of \( Aut(\mathbb{D}) \), the smaller set
\[ M := \{ f \in Aut(\mathbb{D}) : \text{the only zero } z_0 \text{ of } f \text{ satisfies } |z_0| \leq r \} = \{ f_{\gamma, a} : |a| \leq r, \gamma \in [0, 2\pi) \} \]
for fixed \( r \in (0, 1) \), a similar calculation as in the proof of Lemma 13 gives \( \lambda_n(M) \in O(n^3) \). These growth estimations motivate to introduce the following notion.

Definition 15. Let \( (X, d) \) be a complete metric space, \( (Y, d) \) a separable metric space, \( M \subseteq Y \) be totally bounded, and \( L = (L_n)_{n \in \mathbb{N}} \) be a sequence of continuous mappings \( L_n : X \to Y \). We say that an element \( x \in X \) is \( m \)-polynomial \( L \)-universal for \( M \) if \( x \in U(L) \) and
\[ F(x, L, M, 1/n) = O(n^m). \]
We abbreviate the set of all such \( x \) by \( U_m(L, M) \). \( L \) is called \( m \)-polynomial universal for \( M \) if \( U_m(L, M) \neq \emptyset \).

Again, taking \( \ln(n) \in O(n^\varepsilon) \) for each \( \varepsilon > 0 \) into account, Theorem 3, Corollary 9, and Lemma 13 immediately give us

Corollary 16. Let \( C \) be a sequence of composition operators as in Theorem 3, \( D \) the sequence of differentiation operators. Consider the normal family \( Aut(\mathbb{D}) \) in \( H(\mathbb{D}) \). Then, there exist

(i) \( 8 \)-polynomial \( C \)-universal functions for \( Aut(\mathbb{D}) \),
(ii) \( (9+\varepsilon) \)-polynomial \( D \)-universal functions for \( Aut(\mathbb{D}) \) for each \( \varepsilon > 0 \).
Remark 17.

(i) If the covering numbers \( \lambda_n = \lambda_n(\mathcal{M}) \) of a totally bounded subset \( \mathcal{M} \) satisfy \( \lambda_n \in O(n^m) \), the number \( m \) is related to the so-called box-counting dimension of \( \mathcal{M} \).

(ii) Let \( \mathcal{M} \subseteq \mathcal{Y} \) be totally bounded with covering numbers \( (\lambda_n) \). Hence, for every \( n \in \mathbb{N} \), there are \( f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in \mathcal{Y} \) which cover \( \mathcal{M} \) with their \( \frac{1}{n} \)-neighborhoods. Then, we have

\[
\mathcal{U}_n(\mathcal{L}, \mathcal{M}) = \bigcup_{c \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcap_{j=1}^{\lambda_n} L_{2n}^{-1}(U_{1/n}(f_j^{(n)})).
\] (13)

From the description (13), we deduce that the polynomial universal elements form a countable union of \( G_\delta \)-sets, which is called a \( G_\delta \sigma \)-set in the literature. A natural question is: Is it also \( G_\delta \)?

A very prominent example of a normal family in \( H(D) \) is the set

\[
S = \{ f \in H(D): f \text{ one-to-one}, f(0) = 0, f'(0) = 1 \}.
\]

From the well-known inequality due to Koebe, see e.g. [15, Satz 15.15]:

\[
\forall f \in S, z \in D : |f(z)| \leq \frac{|z|}{(1-|z|)^2},
\] (14)

follows the boundedness of \( S \) in \( H(D) \), in fact, \( S \) is a normal family and

\[
M_n(S) = (2n+1)(2n+2) \in O(n^2).
\] (15)

A special subset of \( S \) is given by

\[
K := \{ f_\alpha : \alpha \in [0, 2\pi) \} \subseteq S, \quad f_0(z) = \frac{z}{(1-z)^2}, \quad f_\alpha(z) = e^{-i\alpha} f_0(e^{i\alpha} z),
\]

the so-called Koebe extremal functions. Obviously, \( K \) is a normal family also with \( M_n(K) \in O(n^2) \). As Taylor expansions about the origin, one gets

\[
f_0(z) = \sum_{\nu=1}^\infty \nu z^\nu, \quad f_\alpha(z) = \sum_{\nu=1}^\infty \nu e^{i(\nu-1)\alpha} z^\nu.
\]

**Lemma 18.** For the normal family \( K \) in \( H(D) \), we have \( \lambda_n(K) \in O(n^2 \ln(n)) \).

**Proof.** Consider

\[
T_m f_\beta(z) = \sum_{\nu=1}^m \nu e^{i(\nu-1)\beta} z^\nu, \quad m \in \mathbb{N}, \quad \beta \in [0, 2\pi),
\]

where \( T_m f \) denotes, again, the \( m \)-th Taylor polynomial of \( f \) expanded about the origin. By Corollary 7 there is a sequence \( \gamma_n \in O(n \ln(n)) \) with \( \| T_{\gamma_n} f - f \|_{K_n} < \frac{1}{2n} \) for all \( f \in S \). Using the simple estimate

\[
|e^{i(\nu-1)\alpha} - e^{i(\nu-1)\beta}| = \left| \int_\beta^\alpha \frac{1}{2(\nu-1)} e^{i(\nu-1)t} dt \right| \leq \frac{1}{\nu-1} |\alpha - \beta|,
\] (16)
we obtain, for $f \in K$ and $z \in K_n$,

$$
|f_\alpha(z) - T_{\gamma_2 n} f_\beta(z)| \leq \sum_{\nu=2}^{\gamma_2 n} \nu |e^{i(\nu-1)\alpha} - e^{i(\nu-1)\beta}| + \|f_\alpha - T_{\gamma_2 n} f_\alpha\|_{K_n}
$$

$$
\leq \sum_{\nu=2}^{\gamma_2 n} \nu \n |\alpha - \beta| + \frac{1}{2n} < 2\gamma_2 n |\alpha - \beta| + \frac{1}{2n}.
$$

Thus, for $\|f_\alpha - T_{\gamma_2 n} f_\beta\|_{K_n} < 1/n$ to hold for some $\beta \in [0, 2\pi)$ only $O(n^2 \ln(n))$ values of $\beta$ are needed.

As above, we deduce from the results of the previous section and Lemma 18:

**Corollary 19.** Let $C$ be a sequence of composition operators as in Theorem 3, $D$ the sequence of differentiation operators. Consider the normal family $K$ of Koebe extremal functions in $H(D)$. Then, there exist

(i) $(2+\varepsilon)$-polynomial $C$-universal functions for $K$ for each $\varepsilon > 0$,

(ii) $(4+\varepsilon)$-polynomial $D$-universal functions for $K$ for each $\varepsilon > 0$.

Before we give (what we think to be rather coarse) bounds for the growth of $(\lambda_n(S))_{n \in \mathbb{N}}$ we apply Theorem 3 and Corollary 9 to $S$.

**Corollary 20.** Let $C$ be a sequence of composition operators as in Theorem 3, $D$ the sequence of differentiation operators, and $(\lambda_n)_{n \in \mathbb{N}} = (\lambda_n(S))_{n \in \mathbb{N}}$. Then, there are some $f \in U(C)$ and $g \in U(D)$ with

$$
F(f, C, S, \frac{1}{n}) \in O(n \lambda_{2n}),
$$

respectively

$$
F(g, D, S, \frac{1}{n}) \in O(n^2 \lambda_{3n} \ln(n \lambda_{3n})).
$$

The next result gives bounds for $(\lambda_n(S))_{n \in \mathbb{N}}$.

**Lemma 21.** We have

$$
\lambda_n(S) \in O(\exp(n^{1+\varepsilon})),
$$

for every $\varepsilon > 0$.

**Proof.** 1. Let $n \in \mathbb{N}$ be fixed. Consider for $f \in S$ its Taylor expansion $f(z) = z + \sum_{\nu=2}^{\infty} a_{\nu}(f) \nu^\nu$ about 0. By de Branges’ famous proof of Bieberbach’s Conjecture [8], we know $a_{\nu}(f) \in \nu \mathbb{D}$ for all $\nu \geq 2$. In [9], we obtained

$$
\sum_{\nu=m+1}^{\infty} |a_{\nu}(f)| \left(\frac{n}{n+1}\right)^\nu < \frac{1}{n},
$$

whenever

$$
m \geq m_n := 3(2n+1)[\ln(5n(2n+1)M_n(S))];
$$

as mentioned earlier $M_n(S) = (2n+1)(2n+2)$. 

2. As we will see from the following estimate, any function
\[ g(z) := z + \sum_{\nu=2}^{m_{2n}} b_{\nu} z^\nu, \]
whose coefficients \( b_{\nu} \) fulfill \(|a_{\nu}(f) - b_{\nu}| \leq \frac{1}{2n^2} \) for each \( 2 \leq \nu \leq m_{2n} \), satisfies \( d_K(f,g) < \frac{1}{n} \). Counting how many of these functions \( g \) are at most needed, so that for any \( f \in S \) there is at least one such \( g \) with \( d_K(f,g) < \frac{1}{n} \), will give us an upper bound for \( \lambda_n(S) \) in the next step. But before, we estimate

\[ \|f - g\|_{K_n} \leq \sum_{\nu=2}^{m_{2n}} |a_{\nu}(f) - b_{\nu}| \left( \frac{n}{n+1} \right)^\nu + \sum_{\nu=m_{2n}+1}^{\infty} \nu \left( \frac{n}{n+1} \right)^\nu \]

\[ < \frac{1}{2n^2} \sum_{\nu=1}^{\infty} \left( \frac{n}{n+1} \right)^\nu + \frac{1}{2n} = \frac{1}{n}. \]

By the definition of our metric \( d_K \), this implies \( d_K(f,g) < \frac{1}{n} \).

3. For fixed \( \nu \in [2, m_{2n}] \cap \mathbb{N} \), we set a grid of points \( b_{\nu} \), spaced at intervals of \( \frac{1}{2n^2} \) parallel to the real and imaginary axes, on the disk \( \nu \bar{D} \). This shows that there are at most \( 16n^4(\nu + 1)^2 \) points \( b_{\nu} \) needed, so that for any \( f \in S \) there is at least one \( b_{\nu} \) with \(|a_{\nu}(f) - b_{\nu}| \leq \frac{1}{2n^2} \). Hence,

\[ \lambda_n(S) \leq \prod_{\nu=2}^{m_{2n}} 16n^4(\nu + 1)^2 \leq 16^{m_{2n}} n^{4m_{2n}} ((m_{2n} + 1)!)^2. \] (18)

Using \((m_{2n} + 1)! = \Gamma(m_{2n} + 2)\), as well as

\[ \lim_{z \to \infty} \frac{\Gamma(z+2)}{z^2 \sqrt{2\pi} \left( \frac{z}{e} \right)^z} = 1, \]

cf. [15 page 59], there is \( C > 1 \) such that

\[ \forall n \in \mathbb{N} : ((m_{2n} + 1)!)^2 \leq C m_{2n}^3 \left( \frac{m_{2n}}{e} \right)^{2m_{2n}} < C^{m_{2n}+3} < C m_{2n}^{3m_{2n}}. \] (19)

Combining equations (18) and (19), we obtain

\[ \lambda_n(S) \leq C (16n)^{m_{2n}} (nm_{2n})^{3m_{2n}}. \] (20)

From (17), it follows \( m_{2n} \in O(n \ln(n)) \) Together with (20), we conclude

\[ \lambda_n(S) \in O(\exp(n \ln^2(n))). \]

Since \( \lim_{x \to \infty} \frac{\ln^2(x)}{x^x} = 0 \) for every \( \varepsilon > 0 \), this finally implies the lemma.

For further examples of normal families one may consult [17].

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References


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