Composition and Differentiation Operators and Fast Approximation*

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Abstract

Let $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ and $\mathcal{D} = (D^n)_{n\in\mathbb{N}}$ be families of composition and differentiation operators, respectively, i.e.,

$$C_n f = f \circ \varphi_n, \quad Df = f',$$

where f is holomorphic on some domain $\Omega \subseteq \mathbb{C}$. Our main question is: How fast can a totally bounded set $\mathcal{M}$ of holomorphic functions, in other words a normal family, be approximated by the "orbit" $\{C_n f\colon n \in \mathbb{N}\}$ or $\{D^n f\colon n \in \mathbb{N}\}$ respectively, of one suitably constructed function f? Our answer consists of upper bounds for the numbers

$$F(f, 1/n) := \inf\{N \in \mathbb{N}\colon \text{Any } g \in \mathcal{M} \text{ is approximable with error } < 1/n \text{ by the first } N \text{ elements of the orbit of } f\},\ n \in \mathbb{N}.$$

In particular, we calculate such bounds for well-known classical normal families, like the biholomorphisms of the unit disk $\mathbb{D}$, or the set

$$S := \{f \text{ biholomorphic on } \mathbb{D}\colon f(0) = 0, f'(0) = 1\}.$$

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1 Introduction and notation

Let $(\mathcal{X}, d)$ be a complete metric space, $(\mathcal{Y}, d)$ a separable metric space, $\mathcal{M} \subseteq \mathcal{Y}$, and $\mathcal{L} = (L_n)_{n\in\mathbb{N}}$ be a sequence of continuous mappings $L_n : \mathcal{X} \to \mathcal{Y}$. The sequence $\mathcal{L}$ is called *universal for* $\mathcal{M}$, if there is $x \in \mathcal{X}$ such that $\mathcal{M}$ is contained in the closure of the orbit of x under $\mathcal{L}$, that is

$$\mathcal{M} \subseteq \overline{\{L_n x : n \in \mathbb{N}\}},$$

i.e., for every $y \in \mathcal{M}$ and for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $d(y, L_N x) < \varepsilon$. Such x are called $\mathcal{L}$*-universal for* $\mathcal{M}$ and we denote the set of all $\mathcal{L}$-universal elements for $\mathcal{M}$

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by $\mathcal{U}(\mathcal{L},\mathcal{M})$. In case of $\mathcal{M}=\mathcal{Y}$, we simply speak of $\mathcal{L}$-universality etc., and write $\mathcal{U}(\mathcal{L})$ instead of $\mathcal{U}(\mathcal{L},\mathcal{M})$.

We consider the question, how fast certain given elements $y\in\mathcal{Y}$ can be approximated by $(L_nx)_{n\in\mathbb{N}}$ for some $x\in\mathcal{U}(\mathcal{L})$. With this in mind, given $x\in\mathcal{X}$ and $\mathcal{M}\subseteq\mathcal{Y}$, we define

$$F(x,\varepsilon):=F(x,\mathcal{L},\mathcal{M},d,\varepsilon):=\sup_{y\in\mathcal{M}}\inf\big\{N\in\mathbb{N}\colon d(y,L_Nx)<\varepsilon\big\}.$$

For $x\in\mathcal{U}(\mathcal{L})$, we clearly have that $F(x,\varepsilon)$ is finite for every $\varepsilon>0$ if and only if $\mathcal{M}$ is totally bounded (pre-compact), that is, $\mathcal{M}$ can be covered by a finite number of ε-balls for every $\varepsilon>0$. If the metric space $\mathcal{Y}$ is complete, then, $\mathcal{M}$ is totally bounded if and only if $\mathcal{M}$ is relatively compact, cf. [14, Corollary 4.10]. Moreover, if $\mathcal{M}\subseteq\mathcal{Y}$ is totally bounded and $y_1^{(n)},\ldots,y_{\lambda_n}^{(n)}\in\mathcal{Y}$ satisfy

$$\mathcal{M}\subseteq\bigcup_{j=1}^{\lambda_n}B(y_j^{(n)},\frac{1}{n}),$$

where $B(z,r)=\{y\in\mathcal{Y}:d(y,z)<r\}$ is the open ball with center z and radius r, then, for each $x\in\mathcal{U}(\mathcal{L})$, there is $k_n\in\mathbb{N}$ satisfying

$$\forall\,1\le j\le\lambda_n\ \exists\,1\le N\le k_n:\ d(L_Nx,y_j^{(n)})<\frac{1}{n}.$$

In particular, if $\mathcal{L}$ is universal, then, for any totally bounded set $\mathcal{M}\subseteq\mathcal{Y}$, there is a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that

$$\big\{x\in\mathcal{U}(\mathcal{L})\colon\ F(x,\mathcal{L},\mathcal{M},2/n)\le k_n\ \forall\,n\in\mathbb{N}\big\}$$

containing

$$\mathcal{U}(\mathcal{L})\,\cap\,\bigcap_{n\in\mathbb{N}}\bigcap_{j=1}^{\lambda_n}\bigcup_{N=1}^{k_n}L_N^{-1}\big(B(y_j^{(n)},\frac{1}{n})\big)\tag{1}$$

is not empty. We are interested in upper bounds for k_n depending on $\mathcal{M}$. Therefore, we introduce the following notation. For a given totally bounded subset $\mathcal{M}$ of $\mathcal{Y}$ and $n\in\mathbb{N}$, we define

$$\lambda_n:=\lambda_n(\mathcal{M}):=\min\left\{l\in\mathbb{N}:\exists\,y_1,\ldots,y_l\in\mathcal{Y}\ \text{ with }\ \mathcal{M}\subseteq\bigcup_{j=1}^{l}B(y_j,1/n)\right\},$$

to be the *n-th covering number of* $\mathcal{M}$. Since $\mathcal{M}$ is totally bounded, λ_n is well-defined and the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is obviously increasing. It should be noted that λ_n depends on the given metric d on $\mathcal{Y}$! For each $x\in\mathcal{X}$, we obviously have

$$\forall\,n\in\mathbb{N}:\ \lambda_n\le F(x,\mathcal{L},\mathcal{M},d,1/n).$$

In this paper, we investigate special sequences of continuous linear operators between spaces of holomorphic functions $H(\Omega)$ on an open subset Ω of $\mathbb{C}$. As usual, we endow $H(\Omega)$ with the compact-open topology, that is, the locally convex topology on $H(\Omega)$ induced by the increasing sequence of seminorms $\|f\|_{K_n}=\sup\{|f(z)|:z\in K_n\}$, $n\in\mathbb{N}$, where $\mathcal{K}=(K_n)_{n\in\mathbb{N}}$ is a compact exhaustion of Ω, i.e., $K_n\subseteq\Omega$ compact, K_n is contained in the interior of K_{n+1} for each $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}} K_n = \Omega$. This makes $H(\Omega)$ a Fréchet space; a metric defining the topology is given by

$$d_{\mathcal{K}}(f,g) := \sup_{n\in\mathbb{N}} \min\left\{ \|f-g\|_{K_n}, \frac{1}{n} \right\}. \tag{2}$$

It should be noted at this point that $d_{\mathcal{K}}(f,g) < 1/n$ if (and only if) $\|f-g\|_{K_n} < 1/n$.

In particular, we consider $\Omega = \mathbb{D}$, the open unit disk. For this special situation, we will always choose the natural standard compact exhaustion

$$\mathcal{K}_{\mathbb{D}} := (K_n)_{n\in\mathbb{N}}, \text{ where } K_n := \frac{n}{n+1}\bar{\mathbb{D}}. \tag{3}$$

Recall, a subset $\mathcal{M}$ of $H(\Omega)$ is bounded, by definition, if $\sup_{f\in\mathcal{M}} \|f\|_{K_n} < \infty$ for each $n \in \mathbb{N}$, i.e., if and only if $\mathcal{M}$ is locally bounded. By Montel's Theorem, every bounded subset $\mathcal{M}$ of $H(\Omega)$ is relatively compact. Obviously, the converse is always true. Therefore, the bounded subsets of $H(\Omega)$ are precisely the totally bounded subsets, which are also called *normal families* in this context. Examples will be given in Section 4.

2 Composition Operators and Fast Approximation

In this section, we consider composition operators on spaces of holomorphic functions, that is, for a given sequence $(\varphi_n)_{n\in\mathbb{N}}$ of injective holomorphic mappings $\varphi_n\colon \Omega_1 \to \Omega_2$ between open sets Ω_1, Ω_2 in $\mathbb{C}$, we consider the sequence $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ of linear operators

$$C_n : H(\Omega_2) \to H(\Omega_1),\ f \mapsto f \circ \varphi_n.$$

Universality of such composition operators has been investigated by several authors, e.g. Bernal and Montes [4], followed by many others and also on different function spaces, see e.g. [2], [3], [5], [7], [6], [10], [11]. Recall, (φ_n) is called *run away*, if for every pair of compact sets $K \subseteq \Omega_1, L \subseteq \Omega_2$, there exists an $N \in \mathbb{N}$ with

$$\varphi_N(K) \cap L = \emptyset.$$

This property characterizes the existence of a $\mathcal{C}$-universal element if $\Omega_1 = \Omega_2$ is not conformally equivalent to $\mathbb{C}\backslash\{0\}$, cf. [4]. In view of the following theorem, it is important to have run away sequences tending in a "controlled" manner towards the boundary of Ω_2. Thoughout this section, we assume the open sets Ω_1, Ω_2 to consist of simply connected components, and every compact exhaustion $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of them should also have only simply connected components, see e.g. [16, Theorem 13.3].

If Ω is a domain in $\mathbb{C}$, a sequence of sets $(L_n)_{n\in\mathbb{N}}$ is said to *tend to infinity* provided that, given a compact set $L \subseteq \Omega$, there is $n_0 \in \mathbb{N}$ such that $L_n \cap L = \emptyset$ for all $n \geq n_0$. Observe that, if $\Omega^\star = \Omega \cup \{\omega\}$ denotes the one-point compactification of Ω, then $(L_n)_{n\in\mathbb{N}}$ tends to infinity if and only if $\lim_{n\to\infty} \max\{\chi(z,\omega)\colon z \in L_n\} = 0$, where χ is any distance on $\Omega^\star$ defining its topology.

Proposition 1. *Let $\varphi_n : \Omega_1 \to \Omega_2$, $n \in \mathbb{N}$, be a sequence of injective holomorphic mappings which is run away. Then, for each compact exhaustion $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of Ω_1, there is a sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers such that $\varphi_{m_n}(K_n)$ $(n \in \mathbb{N})$ is pairwise disjoint and tends to infinity.*

Note, the image $\varphi(G)$ of a simply connected domain G under an injective holomorphic mapping φ is also simply connected. Thus, the sets $\varphi_{m_n}(K_n)$ ($n \in \mathbb{N}$) above have also connected complements.

Proof. Fix any compact exhaustion $(L_n)_{n\in\mathbb{N}}$ of Ω_2. Set $m_1 := 1$. Since $(\varphi_n)_{n\in\mathbb{N}}$ is run away, there is $m_2 \in \mathbb{N}$ such that

$$\varphi_{m_2}(K_2) \cap \big(\varphi_{m_1}(K_1) \cup L_1\big) = \emptyset.$$

If $m_1, m_2, \ldots, m_n$ have been found, there is, by hypothesis, $m_{n+1} \in \mathbb{N}$ such that

$$\varphi_{m_{n+1}}(K_{n+1}) \cap \left(\bigcup_{j=1}^{n} \varphi_{m_j}(K_j) \cup L_n \right) = \emptyset.$$

Clearly $\varphi_{m_n}(K_n)$ ($n \in \mathbb{N}$) fulfills the requirements of the assertion. □

For the following we abbreviate $\mathcal{C} := (C_{m_n})_{n\in\mathbb{N}}$. Before stating our first main result, we provide an approximation lemma based on Arakelian's Approximation Theorem, cf. [1], [9].

Lemma 2. *Let Ω be a domain, $(K_n)_{n\in\mathbb{N}}$ a sequence of pairwise disjoint compact sets in Ω, whose complements are connected. Assume that $(K_n)_{n\in\mathbb{N}}$ tends to infinity and that $f_n \in A(K_n)$, i.e., f_n is continuous on K_n and holomorphic in the interior of K_n. Then, there exists $f \in H(\Omega)$ with*

$$\forall\, n \in \mathbb{N}\colon\ \max_{z\in K_n} \big|f(z) - f_n(z)\big| < \frac{1}{n}.$$

Proof. Define

$$\delta(z) := -\ln n, \quad q(z) := f_n(z), \quad z \in K_n.$$

The union $U := \bigcup_{n\in\mathbb{N}} K_n$ is closed in Ω and obviously satisfies that $\Omega^\star \setminus U$ is connected and locally connected at ω. Thus, by Arakelian's Theorem, there exist $g, h \in H(\Omega)$ with

$$\big|\delta(z) - g(z)\big| < 1, \quad \left|\frac{q(z)}{e^{g(z)-1}} - h(z)\right| < 1, \quad z \in U.$$

For $f(z) := h(z) \cdot e^{g(z)-1}$ and $z \in K_n$, we obtain

$$\big|f(z) - f_n(z)\big| = \big|f(z) - q(z)\big| < e^{\mathrm{Re}\ g(z)-1} \leq e^{|g(z)-\delta(z)|-1+\delta(z)} < e^{\delta(z)} = \frac{1}{n}.$$

□

Theorem 3. *Let $\varphi_n\colon \Omega_1 \to \Omega_2$, $n \in \mathbb{N}$, be a sequence of injective holomorphic mappings which is run away and let $\mathcal{K}$ be a compact exhaustion of Ω_1. Then, there is a subsequence $(\varphi_{m_n})_{n\in\mathbb{N}}$ of $(\varphi_n)_{n\in\mathbb{N}}$ and a universal function $f \in \mathcal{U}(\mathcal{C})$ such that for each normal family $\mathcal{M}$ in $H(\Omega_1)$ with covering numbers $(\lambda_n)_{n\in\mathbb{N}} = \big(\lambda_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, we have*

$$\forall\, n \in \mathbb{N}\colon\ F(f, \mathcal{C}, \mathcal{M}, d_{\mathcal{K}}, \frac{2}{n}) \leq n(\lambda_n + 1).$$

Proof. 1. Let $(m_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers corresponding to the compact exhaustion $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$, as in Proposition 1. Then, the sets $\varphi_{m_n}(K_n)$ $(n \in \mathbb{N})$ are pairwise disjoint, have connected complements and tend to infinity.

2. According to Mergelian's Theorem, the set of polynomials with coefficients in $\mathbb{Q}+i\mathbb{Q}$ is dense in $(H(\Omega_1), d_{\mathcal{K}})$. Let (q_n) be an enumeration of them, and let $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in H(\Omega_1)$ be those functions whose $\frac{1}{n}$-neighborhoods cover $\mathcal{M}$. We define (f_N) as the following sequence

$$f_1^{(1)}, f_2^{(1)}, \ldots, f_{\lambda_1}^{(1)}, q_1, f_1^{(2)}, f_2^{(2)}, \ldots, f_{\lambda_2}^{(2)}, q_2, f_1^{(3)}, f_2^{(3)}, \ldots, f_{\lambda_3}^{(3)}, q_3, \ldots$$

3. According to Lemma 2, there exists a function $f \in H(\Omega_2)$, such that

$$\max_{\varphi_{m_N}(K_N)} \left| f(z) - f_N(\varphi_{m_N}^{-1}(z)) \right| < \frac{1}{N}, \quad N \in \mathbb{N},$$

or equivalently,

$$\left\| C_{m_N} f - f_N \right\|_{K_N} = \left\| (f \circ \varphi_{m_N}) - f_N \right\|_{K_N} < \frac{1}{N}, \quad N \in \mathbb{N}.$$

By definition of the metric $d_{\mathcal{K}}$ this implies

$$d_{\mathcal{K}}(C_{m_N} f, f_N) < \frac{1}{N}, \quad N \in \mathbb{N}.$$

4. Fix $g \in \mathcal{M}$ and $n \in \mathbb{N}$. According to the second step, we find a function f_N with

$$n \le N \le \sum_{j=1}^{n-1} (\lambda_j + 1) + \lambda_n \le n(\lambda_n + 1) \quad \text{and} \quad d_{\mathcal{K}}(f_N, g) < \frac{1}{n}.$$

Together with the third step, we have

$$d_{\mathcal{K}}(C_{m_N} f, g) < \frac{1}{n} + \frac{1}{N} \le \frac{2}{n}.$$

Moreover,

$$d_{\mathcal{K}}(C_{m_k} f, q_n) < \frac{1}{k}, \quad n \in \mathbb{N},$$

with $k = \sum_{j=1}^{n} (\lambda_j + 1)$ showing that $f \in \mathcal{U}(\mathcal{C})$ satisfies the desired property.

□

Remark 4.

(i) Roughly speaking, for a sequence of composition operators between spaces of holomorphic functions, the speed of approximating the elements of a normal family $\mathcal{M}$ by a universal function is only governed by the size of $\mathcal{M}$, measured by the covering numbers $(\lambda_n)_{n\in\mathbb{N}}$.

(ii) In [4], it is proved that, in case of $\Omega_1 = \Omega_2$ not being conformally equivalent to $\mathbb{C}\backslash\{0\}$, the set $\mathcal{U}(\mathcal{C})$ is a dense G_δ-set, if non-empty. The above theorem states that there is

$$f \in \mathcal{U}(\mathcal{C}) \cap \bigcap_{n\in\mathbb{N}} \bigcap_{j=1}^{\lambda_n} \bigcup_{N=1}^{n(\lambda_n+1)} C_{m_N}^{-1}\big(B(f_j^{(n)}, \frac{1}{n})\big),$$

where $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)}$ are the centers of open $1/n$-balls covering the normal family $\mathcal{M}$. The continuity of the operators C_{m_N} implies that the above set is a G_δ-set. But in general it is not dense.

To see this, let $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ be the compact exhaustion of Ω_1 giving the metric $d_\mathcal{K}$ and let $\mathcal{M} = \{0\}$. Then, one has $\lambda_n = 1$ and one can take $f_1^{(n)} = 0$, $n \in \mathbb{N}$. Assume, there is a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that

$$\bigcap_{n\in\mathbb{N}} \bigcup_{N=1}^{k_n} C_{m_N}^{-1}\big(B(0, \frac{1}{n})\big) = \Big\{ f \in H(\Omega_2) : \forall\, n \in \mathbb{N}\; \exists\, 1 \le N \le k_n \text{ with } \sup_{z\in K_n} \big|f(\varphi_{m_N}(z))\big| < \frac{1}{n} \Big\}$$

is dense in $H(\Omega_2)$. Let $K \subseteq \Omega_2$ be compact such that $\bigcup_{N=1}^{k_1} \varphi_{m_N}(K_1) \subseteq K$. By assumption, there is

$$g \in \Big\{ f \in H(\Omega_2)\colon \big\|f - 2\big\|_K < 1 \Big\} \cap \bigcap_{n\in\mathbb{N}} \bigcup_{N=1}^{k_n} C_{m_N}^{-1}\big(B(0, \frac{1}{n})\big).$$

Hence, there exists an $1 \le N \le k_1$ with

$$\big\|g - 0\big\|_{\varphi_{m_N}(K_1)} = \big\|C_{m_N} g - 0\big\|_{K_1} < 1,$$

which gives a contradiction to $\big\|g - 2\big\|_K < 1$.

Let $\mathcal{X}, \mathcal{Y}$ be metric spaces and $\mathcal{L} = (L_N)_{N\in\mathbb{N}}$ a universal sequence of continuous mappings from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{M} \subseteq \mathcal{Y}$ is totally bounded, we have just seen that for any sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ the G_δ-set in (1) need not be dense in $\mathcal{X}$ although there is always *some* sequence $(k_n)_{n\in\mathbb{N}}$ such that the above set is non-empty, cf. the introduction. However, if one weakens the requirement

$$\forall\, n \in \mathbb{N}\colon\; F(x, \mathcal{L}, \mathcal{M}, 2/n) \le k_n$$

to (we use the standard Landau notations)

$$\big(F(x, \mathcal{L}, \mathcal{M}, 2/n)\big)_{n\in\mathbb{N}} \in O\big((k_n)_{n\in\mathbb{N}}\big), \qquad \text{shortly } F(x, \mathcal{L}, \mathcal{M}, 2/n) \in O(k_n),$$

then the corresponding set is dense, see the next result. Whenever the index, mostly $n \in \mathbb{N}$, is clear, we will shorten the Landau notation from $(a_n)_{n\in\mathbb{N}} \in O\big((b_n)_{n\in\mathbb{N}}\big)$ to $a_n \in O(b_n)$.

Theorem 5. *Let $\varphi_n\colon \Omega_1 \to \Omega_2$, $n \in \mathbb{N}$, be a sequence of injective holomorphic mappings which is run away, and let $\mathcal{K}$ be a compact exhaustion of Ω_1. Then, there is a subsequence $(\varphi_{m_n})_{n\in\mathbb{N}}$ of $(\varphi_n)_{n\in\mathbb{N}}$ and a dense set of universal functions $f \in \mathcal{U}(\mathcal{C})$ in $H(\Omega_2)$, such that* *for every choice of countably many normal families $\mathcal{M}_i$ in $H(\Omega_1)$, $i \in \mathbb{N}$, with covering numbers $(\lambda_{n,i})_{n\in\mathbb{N}} = \big(\lambda_n(\mathcal{M}_i)\big)_{n\in\mathbb{N}}$, we have*

$$\forall\, i \in \mathbb{N}\colon\; F(f, \mathcal{C}, \mathcal{M}_i, d_{\mathcal{K}}, \frac{2}{n}) \in O(n\lambda_{n,i}). \tag{4}$$

Proof. 1. Let $(m_n)_{n\in\mathbb{N}}$ be again a strictly increasing sequence of natural numbers corresponding to the compact exhaustion $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$, as in Proposition 1. Then, the sets $\varphi_{m_n}(K_n)$ $(n \in \mathbb{N})$ are pairwise disjoint, have connected complements and tend to infinity. We have to show that for given $h \in H(\Omega_2)$, $K \subseteq \Omega_2$ compact and $\varepsilon > 0$, there exists a universal function $f \in \mathcal{U}(\mathcal{C})$ with the desired property and

$$\|f - h\|_K < \varepsilon.$$

Since $\varphi_{m_n}(K_n)$ $(n \in \mathbb{N})$ tends to infinity, there is some $M \in \mathbb{N}$ such that $K \cap \varphi_{m_n}(K_n) = \emptyset$ for all $n > M$.

2. Also, let (q_n) be as in the proof of Theorem 3, and let $f_1^{(n,i)}, \ldots, f_{\lambda_{n,i}}^{(n,i)} \in H(\Omega_1)$ be those functions whose $\frac{1}{n}$-neighborhoods cover $\mathcal{M}_i$, merged in sequences $(f_n^{(i)})_{n\in\mathbb{N}}$ defined as

$$f_1^{(1,i)}, f_2^{(1,i)}, \ldots, f_{\lambda_1}^{(1,i)}, f_1^{(2,i)}, f_2^{(2,i)}, \ldots, f_{\lambda_2}^{(2,i)} f_1^{(3,i)}, f_2^{(3,i)}, \ldots, f_{\lambda_3}^{(3,i)}, \ldots$$

With these sequences we build (f_N) as follows: Every $(2j-1)$-st element of (f_N) is q_j, $j \in \mathbb{N}$. From the remaining elements every $(2j-1)$-st element is $f_j^{(1)}$, $j \in \mathbb{N}$. Again, from the remaining every $(2j-1)$-st element is $f_j^{(2)}$, $j \in \mathbb{N}$, and so on.

3. According to Lemma 2, there exists a function $f \in H(\Omega_2)$, such that

$$\|f - h\|_K < \varepsilon \quad\text{and}\quad \max_{\varphi_{m_{M+N}}(K_{M+N})} \left|f(z) - f_N(\varphi_{m_{M+N}}^{-1}(z))\right| < \frac{1}{M+N}, \quad N \in \mathbb{N},$$

or equivalently,

$$\|C_{m_{M+N}} f - f_N\|_{K_{M+N}} = \|(f \circ \varphi_{m_{M+N}}) - f_N\|_{K_{M+N}} < \frac{1}{M+N}, \quad N \in \mathbb{N}.$$

By definition of the metric $d_{\mathcal{K}}$, this implies

$$d_{\mathcal{K}}(C_{m_{M+N}} f, f_{M+N}) < \frac{1}{M+N}, \quad N \in \mathbb{N}.$$

4. Fix $g \in \mathcal{M}_i$ and $n \in \mathbb{N}$. According to the second step, we find a function f_N with

$$n \le M + N \le \tilde{c}_i \cdot n(\lambda_{n,i} + 1) \le c_i\, n\lambda_{n,i}, \tag{5}$$

for appropriately chosen constants $\tilde{c}_i$, c_i, and

$$d_{\mathcal{K}}(f_N, g) < \frac{1}{n}.$$

Together with the third step, we have

$$d_{\mathcal{K}}(C_{m_{M+N}} f, g) < \frac{1}{n} + \frac{1}{M+N} \le \frac{2}{n}.$$

Moreover,

$$d_{\mathcal{K}}(C_{m_{M+2n-1}}f, q_n) < \frac{1}{M+2n-1}, \quad n \in \mathbb{N},$$

showing that $f \in \mathcal{U}(\mathcal{C})$ satisfies the desired property.

□

In equation (4), we have seen

$$\forall\, i, n \in \mathbb{N}\colon\; F(f, \mathcal{C}, \mathcal{M}_i, d_{\mathcal{K}}, \frac{2}{n}) \le c_i\, n\lambda_{n,i},$$

where the constants c_i as given in (5) grow exponentially in i, more precisely $(c_i)_{i\in\mathbb{N}} \in \Theta\big((2^i)_{i\in\mathbb{N}}\big)$, i.e., $(c_i)_{i\in\mathbb{N}} \in O\big((2^i)_{i\in\mathbb{N}}\big)$ and $(2^i)_{i\in\mathbb{N}} \in O\big((c_i)_{i\in\mathbb{N}}\big)$, as we see from the second step of the above proof.

3 Differentiation Operators and Fast Approximation

In this section, we consider the differentiation operator

$$D\colon H(\Omega) \to H(\Omega),\; f \mapsto f',$$

on spaces of holomorphic functions on a simply connected bounded domain $\Omega \subseteq \mathbb{C}$, as well as the sequence $\mathcal{D} := (D^n)_{n\in\mathbb{N}}$. It is known that the existence of $f \in \mathcal{U}(\mathcal{D})$ is equivalent to Ω being simply connected, cf. [18]. Therefore, without loss of generality, we may and will assume Ω to be simply connected throughout the whole paragraph. Since differentiation commutes with translations, we can assume $0 \in \Omega$ without loss of generality. More precisely, we may assume that 0 is contained in the interior of K_1 for a compact exhaustion $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of Ω.
Moreover, there is a compact exhaustion $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of Ω such that K_n is connected and simply connected for every $n \in \mathbb{N}$, see e.g. [16, Theorem 13.3]. Therefore, we assume without loss of generality that for the metric $d_{\mathcal{K}}$ inducing the compact-open topology on $H(\Omega)$, cf. (2), we have K_n connected and simply connected.
Furthermore, we denote the m-th Faber polynomial for K_n by $F_{n,m}$, $m \in \mathbb{N}_0$. Then, $F_{n,m}$ is a polynomial of degree m which is obtained in the following way, see e.g. [9] or [13].
By the Riemann Mapping Theorem, there is a unique conformal mapping $\varphi_n : \mathbb{C}\backslash K_n \to \mathbb{C}\backslash\overline{\mathbb{D}}$ with $\varphi_n(\infty) = \infty$ and $\varphi_n'(\infty) > 0$. Hence, for some $c > 0$, we have for $|z|$ sufficiently large

$$\varphi_n(z) = \frac{1}{c}z + c_0 + \sum_{\nu=1}^{\infty} c_\nu z^{-\nu}.$$

Moreover, for $|z|$ sufficiently large and every $m \in \mathbb{N}$, we have

$$\varphi_n^m(z) = F_{n,m}(z) + \sum_{\nu=1}^{\infty} \alpha_\nu z^{-\nu},$$

that is, $F_{n,m}$ is the analytic part of the Laurent expansion of φ_n^m. With $\psi_n := \varphi_n^{-1} : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash K_n$, we have

$$\psi_n(w) = cw + d_0 + \sum_{\nu=1}^{\infty} d_\nu w^{-\nu}, |w| > 1.$$

For $R > 1$, we set $\Gamma_{n,R} := \{\psi_n(w)\colon |w| = R\}$. Then, $\Gamma_{n,R}$ is a closed Jordan curve, and for each $n \in \mathbb{N}$, there is $R_n > 1$, such that $\Gamma_{n,R} \subseteq \Omega$ for all $1 < R < R_n$. Denoting by $I_{n,R}$ the bounded (open) component of $\mathbb{C}\backslash\Gamma_{n,R}$, we obtain $K_n \subseteq I_{n,R} \subseteq \Omega$ for every $n \in \mathbb{N}$ and $1 < R < R_n$.
If f is a complex function holomorphic in a neighborhood $I_{n,R}$ of K_n, we define for $\nu \in \mathbb{N}_0$

$$a_\nu(f, K_n) := \frac{1}{2\pi i} \int_{|w|=r} \frac{f(\psi_n(w))}{w^{\nu+1}} dw,$$

which is independent of $r \in (1, R)$. Then

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu(f, K_n) F_{n,\nu}(z),$$

where the series converges uniformly and absolutely on $I_{n,R}$, in particular, on K_n. Thus, this expansion is valid in I_{n,R_n} for every $f \in H(\Omega)$. Moreover, the above so-called Faber expansion of f is unique, see again e.g. [9] or [13]. From this, and the fact that $F_{n,m}$ is a polynomial of degree m, it follows that for every polynomial p, we have $p = \sum_{\nu=0}^{m} a_\nu(p, K_n) F_{n,\nu}$, whenever $m \geq \deg(p)$. In case of $K_n = \{z \in \mathbb{C} : |z - z_0| \leq \rho\}$, the above expansion of f is nothing but the Taylor expansion of f about z_0.
From [13, Lemma preceding Theorem 3.16], it follows

$$\frac{1}{2} R^\nu < \big|F_{n,\nu}(z)\big| < \frac{3}{2} R^\nu, \tag{6}$$

for all $1 < R < R_n$, for every $z \in \Gamma_{n,R}$, and $\nu \in \mathbb{N}_0$.
For $f \in H(\Omega)$ and $n, m \in \mathbb{N}$, we define

$$T_{n,m} f : \mathbb{C} \to \mathbb{C}, \quad T_{n,m} f(z) := \sum_{\nu=0}^{m} a_\nu(f, K_n) F_{n,\nu}(z),$$

that is, $T_{n,m} f$ is a polynomial of degree $\leq m$.
Moreover, we denote by $f^{(-j)}$ the j-th anti-derivative of f, i.e.,

$$f^{(0)}(z) := f(z), \quad f^{(-j)}(z) := \int_0^z f^{(-j+1)}(\zeta) d\zeta, \quad j \in \mathbb{N}, z \in \Omega.$$

Recall, we assume without restriction $0 \in \Omega$. It is very well-known that for every $f \in H(\Omega)$ the sequence $(I_j f)_{j \in \mathbb{N}_0}$ converges to zero in $H(\Omega)$, where $I_j\colon H(\Omega) \to H(\Omega)$, $I_j f := f^{(-j)}$, $j \in \mathbb{N}_0$, see e.g. [12, Lemma 1].
The next Lemma is rather technical. Its conclusions simplify in case of $\Omega = \mathbb{D}$, which will be stated separately as Corollary 7 below.

Lemma 6. *Let $\mathcal{K}$ be a compact exhaustion of Ω and $\mathcal{M} \subseteq H(\Omega)$ a normal family. For $n \in \mathbb{N}$, let*

$$M_n := M_n(\mathcal{M}) := \sup_{f \in \mathcal{M}} \max_{|w| = \frac{1}{2}(1+R_n)} \big|f\big(\psi_n(w)\big)\big|.$$

1. *There is an increasing sequence*

$$\gamma_n(\mathcal{M}) \in O\left(\frac{R_n+1}{R_n-1}\ln\big(n\,\frac{R_n+1}{R_n-1}M_n\big)\right),$$

of natural numbers tending to infinity such that, for every $f \in \mathcal{M}$, we have

$$\big\|T_{n,\gamma_n}f - f\big\|_{K_n} < \frac{1}{n}.$$

Moreover, if there is $k \in \mathbb{N}_0$ such that $M_n(\mathcal{M}) \in O(n^k)$, then,

$$\gamma_n(\mathcal{M}) \in O\left(\frac{R_n+1}{R_n-1}\ln\big(n\,\frac{R_n+1}{R_n-1}\big)\right).$$

2. *There is a sequence $\big(\sigma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$ of natural numbers tending to infinity, such that for every $f \in \mathcal{M}$, $n \in \mathbb{N}$, and $m \in \mathbb{N}_0$, we have*

$$\big\|(T_{n,m}f)^{(-j)}\big\|_{K_n} < \frac{1}{n^2},$$

whenever $j \geq \sigma_n(\mathcal{M})$.

We point out that the above sequences $\big(\gamma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$ and $\big(\sigma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$ depend on the compact exhaustion $\mathcal{K}$ of Ω!

Proof. Let $n \in \mathbb{N}$ be arbitrary. Note, $M_n < \infty$ by the total boundedness of $\mathcal{M}$.

1. For $f \in \mathcal{M}$ and $1 < R < R_n$, we have by the maximum principle

$$\big\|T_{n,m}f - f\big\|_{K_n} \leq \sum_{\nu=m+1}^{\infty} \big|a_\nu(f,K_n)\big|\,\big\|F_{n,\nu}\big\|_{K_n} \underset{(6)}{\leq} \frac{3}{2}\sum_{\nu=m+1}^{\infty}\big|a_\nu(f,K_n)\big|R^\nu.$$

Moreover, for the Faber coefficients we obtain

$$\big|a_\nu(f,K_n)\big| = \frac{1}{2\pi}\left|\int_{|w|=\frac{1}{2}(1+R_n)} \frac{f(\psi_n(w))}{w^{\nu+1}}dw\right| \leq \left(\frac{2}{1+R_n}\right)^\nu M_n,$$

so, for $1 < R < \frac{2}{3} + \frac{1}{3}R_n = \frac{1}{3}(2+R_n)$,

$$\begin{aligned}
\frac{3}{2}\sum_{\nu=m+1}^{\infty}\big|a_\nu(f,K_n)\big|R^\nu &\leq \frac{3}{2}M_n\sum_{\nu=m+1}^{\infty}\left(\frac{2R}{1+R_n}\right)^\nu \\
&= \frac{3}{2}M_n\left(\frac{2R}{1+R_n}\right)^{m+1}\frac{1}{1-\frac{2R}{1+R_n}} \\
&\leq \frac{3}{2}M_n\left(\frac{4+2R_n}{3+3R_n}\right)^{m+1} 3\frac{1+R_n}{R_n-1} \\
&\leq 5\,\frac{R_n+1}{R_n-1}M_n\left(\frac{4+2R_n}{3+3R_n}\right)^{m+1}.
\end{aligned}$$

Thus, in order that $\left\|T_{n,m}f - f\right\|_{K_n} < \frac{1}{n}$, it suffices

$$\ln\left(5n\,\frac{R_n+1}{R_n-1}M_n\right) < (m+1)\ln\left(\frac{3+3R_n}{4+2R_n}\right) = (m+1)\ln\left(1+\frac{R_n-1}{2(2+R_n)}\right).$$

Using the elementary inequality

$$\forall\, x \geq 0: \quad \frac{x}{1+x} \leq \ln(1+x),$$

the above inequality is surely satisfied if

$$\ln\left(5n\,\frac{R_n+1}{R_n-1}M_n\right) < (m+1)\frac{\frac{R_n-1}{2(2+R_n)}}{1+\frac{R_n-1}{2(2+R_n)}} = (m+1)\frac{R_n-1}{3(R_n+1)}.$$

Taking all this together, we conclude

$$\sum_{\nu=m+1}^{\infty} \left|a_\nu(f,K_n)\right| \left\|F_{n,\nu}\right\|_{K_n} < \frac{1}{n}$$

for $n \in \mathbb{N}$, and for all $f \in \mathcal{M}$, provided that

$$m \geq 3\,\frac{R_n+1}{R_n-1}\ln\left(5n\,\frac{R_n+1}{R_n-1}M_n\right). \tag{7}$$

2. (i) Now, we consider $T_{n,m}$ as a continuous linear operator from $H(\Omega)$ into $H(I_{n,R_n})$ and, first, we show that $\mathcal{N} := \bigcup_{m\in\mathbb{N}_0} T_{n,m}(\mathcal{M})$ is a normal family in $H(I_{n,R_n})$:

   From the above mentioned properties of the Faber expansion, it follows that for every $f \in H(\Omega)$ the sequence $(T_{n,m}f)_{m\in\mathbb{N}_0}$ converges in $H(I_{n,R_n})$ to $f|_{I_{n,R_n}}$. Since $H(\Omega)$ is a Fréchet space, the equicontinuity of the sequence of operators $(T_{n,m})_{m\in\mathbb{N}_0}$ follows from the Uniform Boundedness Principle.

   Next, let U be an absolutely convex zero neighborhood in $H(I_{n,R_n})$. By the equicontinuity of $(T_{n,m})_{m\in\mathbb{N}_0}$, there is an absolutely convex zero neighborhood V in $H(\Omega)$ such that $T_{n,m}(V) \subseteq U$ for every $m \in \mathbb{N}_0$. Since $\mathcal{M}$ is a normal family, hence, bounded in $H(\Omega)$, there is $\rho > 0$ with $\mathcal{M} \subseteq \rho V$, implying $\mathcal{N} := \bigcup_{m\in\mathbb{N}_0} T_{n,m}(\mathcal{M}) \subseteq \rho U$. Since U was arbitrary this gives the boundedness of $\mathcal{N}$ in $H(I_{n,R_n})$. Thus, $\mathcal{N}$ is relatively compact, i.e., a normal family.

   (ii) Since we assumed $0 \in K_1$, the above-explained mappings $I_j : H(I_{n,R_n}) \to H(I_{n,R_n})$ are well-defined, continuous and linear. Moreover, for each $f \in H(I_{n,R_n})$ the sequence $(I_j f)_{j\in\mathbb{N}_0}$ tends to zero in $H(I_{n,R_n})$. The Uniform Boundedness Principle implies, again, the equicontinuity of $(I_j)_{j\in\mathbb{N}_0}$. Because $K_n \subseteq I_{n,R_n}$, we can find a zero neighborhood V such that $\left\|I_j f\right\|_{K_n} < \frac{1}{2n^2}$ for every $f \in V$ and every $j \in \mathbb{N}_0$. Since for every $f \in H(I_{n,R_n})$ there is $j(f) \in \mathbb{N}$ with $\left\|I_j f\right\|_{K_n} < \frac{1}{2n^2}$ for each $j \geq j(f)$,

$$\left\|I_j g\right\|_{K_n} \leq \left\|I_j(g-f)\right\|_{K_n} + \left\|I_j f\right\|_{K_n} < \frac{1}{n^2}$$

   holds for every $g \in f + V$ and $j \geq j(f)$.

Because $\mathcal{N} \subseteq \bigcup_{f\in\mathcal{N}}(f+V)$ is totally bounded, there are $f_1,\ldots,f_k \in \mathcal{N}$ such that

$$\bigcup_{m\in\mathbb{N}_0} T_{n,m}(\mathcal{M}) = \mathcal{N} \subseteq \bigcup_{l=1}^{k}(f_l+V).$$

Setting $\sigma_n := \max\{j(f_1),\ldots,j(f_k)\}$, we finally obtain $\|(T_{n,m}f)^{(-j)}\|_{K_n} < \frac{1}{n^2}$ for each $f \in \mathcal{M}$, $m \in \mathbb{N}_0$, and $j \geq \sigma_n$.

□

Corollary 7. *Let $\mathcal{M} \subseteq H(\mathbb{D})$ be a normal family and $\mathcal{K}_{\mathbb{D}}$ be the standard compact exhaustion of $\mathbb{D}$, cf.* (3).

1. *For each $n \in \mathbb{N}$, we have $M_n = M_n(\mathcal{M}) = \sup_{f\in\mathcal{M}} \|f\|_{K_{2n+1}}$.*
2. *For the sequence $\big(\gamma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, we have*

$$\gamma_n(\mathcal{M}) \in O\big(n\ln(nM_n)\big),$$

and if $M_n(\mathcal{M}) \in O(n^k)$ for some $k \in \mathbb{N}_0$, then,

$$\gamma_n(\mathcal{M}) \in O\big(n\ln(n)\big).$$

3. *For the sequence $\big(\sigma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, we can assume without restriction*

$$\sigma_n(\mathcal{M}) \in O\big(\ln(n^2 M_n)\big).$$

Proof. For the compact set $K_n = \frac{n}{n+1}\overline{\mathbb{D}}$, we have $\varphi_n : \mathbb{C}\backslash K_n \to \mathbb{C}\backslash\overline{\mathbb{D}}$, $\varphi_n(z) = \frac{n+1}{n}z$. Thus, $\psi_n(z) = \frac{n}{n+1}z$ and $R_n = \frac{n+1}{n}$. Moreover, because $\varphi_n^\nu(z) = (\frac{n+1}{n})^\nu z^\nu$, we have $F_{n,\nu}(z) = (\frac{n+1}{n})^\nu z^\nu$. So, we obtain for sufficiently small $1 < r$

$$\begin{aligned}
a_\nu(f,K_n) &= \frac{1}{2\pi i}\int_{|w|=r}\frac{f(\psi_n(w))}{w^{\nu+1}}dw \\
&= \left(\frac{n}{n+1}\right)^\nu \frac{1}{2\pi i}\int_{|w|=\frac{n}{n+1}r}\frac{f(w)}{w^{\nu+1}}dw \\
&= \left(\frac{n}{n+1}\right)^\nu a_\nu(f),
\end{aligned}$$

where $a_\nu(f)$ denotes the ν-th Taylor coefficient of f expanded about the origin. Therefore, for every $f \in H(\mathbb{D})$, $n \in \mathbb{N}$, and $\nu \in \mathbb{N}_0$, we have

$$\forall\, z \in K_n : \; a_\nu(f,K_n)F_{n,\nu}(z) = a_\nu(f)z^\nu. \tag{8}$$

1. For each $f \in \mathcal{M}$, we have

$$\max_{|w|=\frac{1}{2}(1+R_n)} \big|f\big(\psi_n(w)\big)\big| = \max_{|z|=\frac{2n+1}{2n+2}} \big|f(z)\big| = \|f\|_{K_{2n+1}}.$$

2. From inequality (7), equation (8), and $\frac{R_n+1}{R_n-1} = 2n+1$, we obtain for every $f \in \mathcal{M}$ and each $n \in \mathbb{N}$ that

$$\sum_{\nu=m+1}^{\infty} \big|a_\nu(f)\big| \left(\frac{n}{n+1}\right)^{\nu} < \frac{1}{n},$$

whenever

$$m \geq 3(2n+1)\ln(5n(2n+1)M_n(\mathcal{M})). \tag{9}$$

3. As shown above, the m-th partial sums $T_{n,m}$ of the Faber expansions are independent of n and coincide with the m-th Taylor polynomials expanded about the origin. Because $K_n = \frac{n}{n+1}\bar{\mathbb{D}}$, it follows

$$\big|a_\nu(f)\big| = \left|\frac{1}{2\pi i}\int_{|z|=\frac{2n+1}{2n+2}} \frac{f(z)}{z^{\nu+1}}dz\right| \leq \left(\frac{2n+2}{2n+1}\right)^{\nu} \cdot \|f\|_{K_{2n+1}}, \tag{10}$$

which leads, for every $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $j \geq 2$, and $f \in \mathcal{M}$, to

$$\begin{aligned} \big\|(T_{n,m}f)^{(-j)}\big\|_{K_n} &= \left\|\sum_{\nu=0}^{m} \frac{a_\nu(f)}{(\nu+1)\cdots(\nu+j)} z^{\nu+j}\right\|_{K_n} \leq \frac{1}{j!}\sum_{\nu=0}^{\infty}\big|a_\nu(f)\big|\left(\frac{n}{n+1}\right)^{\nu+j} \\ &\underset{(10)}{\leq} \frac{1}{j!}\left(\frac{n}{n+1}\right)^{j}\|f\|_{K_{2n+1}} \cdot \sum_{\nu=0}^{\infty}\left(\frac{n(2n+2)}{(n+1)(2n+1)}\right)^{\nu} \leq \frac{(2n+1)M_n}{j!} \\ &\leq \frac{3nM_n}{j!}. \end{aligned}$$

If j satisfies $j! > 3n^2 M_n$, we get $\big\|(T_{n,m}f)^{(-j)}\big\|_{K_n} < \frac{1}{n^2}$ for all $f \in \mathcal{M}$. In particular, by applying Stirling's Formula, we can choose $\sigma_n(\mathcal{M}) \in O\big(\ln(n^2 M_n)\big)$.

□

Theorem 8. *Let $\mathcal{K}$ be a compact exhaustion of Ω and $\mathcal{M}$ be a normal family in $H(\Omega)$ with covering numbers $(\lambda_n)_{n\in\mathbb{N}} = \big(\lambda_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, as well as the sequences $(\gamma_n)_{n\in\mathbb{N}} = \big(\gamma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}} = \big(\sigma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$ from Lemma 6. Then, there exists a universal function $f \in \mathcal{U}(\mathcal{D})$ such that*

$$\forall\, n \in \mathbb{N}\colon\; F(f, \mathcal{D}, \mathcal{M}, d_{\mathcal{K}}, \frac{3}{n}) \leq n(\lambda_n+1)\big(\gamma_n + \sigma_{n(\lambda_n+1)}\big).$$

Proof. 1. Let $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in H(\Omega)$ be those functions whose $\frac{1}{n}$-neighborhoods cover $\mathcal{M}$. Moreover, let $\mathcal{Q} = \{q_n := f_{\lambda_n+1}^{(n)} : n \in \mathbb{N}\}$ be a dense set of polynomials in $H(\Omega)$, which exists by Mergelian's Theorem and our general assumption that Ω is simply connected. Without restriction, we may assume $\deg(q_n) \leq \gamma_n$, as well as $\big\|q_n^{(-j)}\big\|_{K_n} < 1/n^2$ for every $j \geq \sigma_n$, holds for every $n \in \mathbb{N}$. Otherwise, we elongate the sequence (q_n) by adding the zero polynomial several times, noticing (γ_n), (σ_n) may be chosen to tend to ∞, as $n \to \infty$.

Now, we define $(f_k)_{k\in\mathbb{N}}$ as the following sequence

$$f_1^{(1)}, f_2^{(1)}, \ldots, f_{\lambda_1+1}^{(1)}, f_1^{(2)}, f_2^{(2)}, \ldots, f_{\lambda_2+1}^{(2)}, \ldots, f_1^{(n)}, f_2^{(n)}, \ldots, f_{\lambda_n+1}^{(n)}, \ldots$$

For every $k \in \mathbb{N}$, there are unique $n = n(k) \in \mathbb{N}$, $n \leq k$, and $1 \leq j \leq \lambda_n + 1$ such that $f_k = f_j^{(n)}$. According to Lemma 6 and the fact that the degree of q_n does not exceed γ_n, it holds for $P_k := T_{n,\gamma_n} f_k = T_{n(k),\gamma_{n(k)}} f_k$ that

$$\|P_k - f_k\|_{K_n} = \|T_{n,\gamma_n} f_k - f_k\|_{K_n} < \frac{1}{n}.$$

Therefore, by the definition of our metric, this implies

$$d_{\mathcal{K}}(f_k, P_k) < \frac{1}{n} \tag{11}$$

for every $k \in \mathbb{N}$. Note, in case of $f_k = f_{\lambda_n+1}^{(n)} = q_n$, we have $P_k = T_{n,\gamma_n} q_n = q_n$, because q_n is a polynomial of degree not exceeding γ_n.

Next, we define

$$N_1 := \sigma_1 + 1, \quad N_k := \gamma_{n(k)} + \sigma_k + N_{k-1},\ k \geq 2,$$

and the function f as

$$f(z) := \sum_{j=1}^{\infty} P_j^{(-N_j)}(z).$$

Since, for every $n \leq l$, we have

$$\sum_{j=l}^{l+m} \|P_j^{(-N_j)}\|_{K_n} \leq \sum_{j=l}^{l+m} \|P_j^{(-N_j)}\|_{K_j} \leq \sum_{j=l}^{l+m} \frac{1}{j^2}$$

by Lemma 6 and the choice of $\mathcal{Q}$, f is a well-defined holomorphic function in Ω.

2. Let $k \in \mathbb{N}$. For all $1 \leq j < k$, we have $N_k - N_j > \gamma_{n(k)} \geq \gamma_{n(j)}$. It follows

$$f^{(N_k)}(z) = P_k(z) + \sum_{j=k+1}^{\infty} P_j^{(-N_j+N_k)}(z)$$

For $j \geq k+1$, we have $N_j - N_k \geq \sigma_j$. Since $k \geq n$, we estimate

$$\begin{aligned}
\|f^{(N_k)} - P_k\|_{K_n} &= \left\| \sum_{j=k+1}^{\infty} P_j^{(-N_j+N_k)} \right\|_{K_n} \leq \sum_{j=k+1}^{\infty} \|P_j^{(-N_j+N_k)}\|_{K_j} \\
&= \sum_{j=k+1}^{\infty} \|(T_{n(j),\gamma_{n(j)}} f_j)^{(-N_j+N_k)}\|_{K_j} \\
&\leq \sum_{j=k+1}^{\infty} \frac{1}{j^2} < \frac{1}{k} < \frac{1}{n}.
\end{aligned}$$

Therefore, by the definition of our metric $d_{\mathcal{K}}$, we obtain

$$d_{\mathcal{K}}(D^{N_k} f, P_k) < \frac{1}{n}. \tag{12}$$

3. Let given an arbitrary function $g \in \mathcal{M}$. Hence, there exists a function f_k with $k \leq n \cdot (\lambda_n + 1)$ and
$$d_{\mathcal{K}}(f_k, g) < \frac{1}{n}.$$
Together with (11) and (12), it follows
$$d_{\mathcal{K}}(D^{N_k} f, g) \leq d_{\mathcal{K}}(D^{N_k} f, P_k) + d_{\mathcal{K}}(P_k, f_k) + d_{\mathcal{K}}(f_k, g) < \frac{3}{n}.$$
We calculate
$$N_k = \sum_{j=1}^{k} \gamma_{n(j)} + \sigma_j \leq k\big(\gamma_{n(k)} + \sigma_k\big) \leq n(\lambda_n + 1)\big(\gamma_n + \sigma_{n(\lambda_n+1)}\big),$$
as proposed.

4. Moreover, by construction, we have $P_k = q_n$ for every $k \in \mathbb{N}$ with $f_k = q_n$. From (12), we conclude
$$d_{\mathcal{K}}(D^{N_k} f, q_n) < \frac{1}{n}$$
for such k, which finally shows $f \in \mathcal{U}(\mathcal{D})$.

□

Combining the above Theorem 8 with Corollary 7, we immediately get the following.

Corollary 9. *Let $\mathcal{K}_{\mathbb{D}}$ be the standard exhaustion of $\mathbb{D}$ and $\mathcal{M} \subseteq H(\mathbb{D})$ be a normal family with covering numbers $(\lambda_n)_{n\in\mathbb{N}}$. Then, there is a universal function $f \in \mathcal{U}(\mathcal{D})$ such that*
$$F\big(f, \frac{3}{n}\big) \in O\big(n^2 \lambda_n \ln(n\lambda_n \max\{1, M_{2n\lambda_n}\})\big)$$
or equivalently
$$F\big(f, \frac{1}{n}\big) \in O\big(n^2 \lambda_{3n} \ln(n\lambda_{3n} \max\{1, M_{6n\lambda_{3n}}\})\big),$$
where $M_n = M_n(\mathcal{M}) = \sup_{f\in\mathcal{M}} \left\|f\right\|_{K_{2n+1}}$.

Remark 10. In contrast to sequences of composition operators, the speed of approximating elements of a normal family $\mathcal{M}$ by universal functions for the differentiation operator is not only governed by the size of $\mathcal{M}$, measured by the covering numbers $\big(\lambda_n\big)_{n\in\mathbb{N}}$.
In case of $\Omega = \mathbb{D}$, also the growth of the members of $\mathcal{M}$, given by the sequence $(M_n)_{n\in\mathbb{N}}$, comes into play. In the general case, the sequences $\big(\gamma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, quantizing the approximative behavior of the Faber expansion, and $\big(\sigma_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, giving the speed of convergence towards zero of the anti-derivatives, are relevant.

4 Examples of normal families

We conclude with some examples of normal families in $H(\mathbb{D})$ and apply our results from the previous sections. Throughout, we choose the standard compact exhaustion $\mathcal{K}_{\mathbb{D}}$ of $\mathbb{D}$, cf. (3). Therefore, we omit the reference to the fixed metric $d_{\mathcal{K}_{\mathbb{D}}}$ in the notation of this section.

Trivially, every finite subset $E = \{f_1, \ldots, f_k\}$ of $H(\mathbb{D})$ is a normal family with eventually constant sequence $\lambda_n(E) = k$. Applying Theorem 3 and Corollary 9 respectively yields the following result.

Corollary 11. *Let $\mathcal{C}$ be a sequence of composition operators as in Theorem 3, $\mathcal{D}$ the sequence of differentiation operators. For every finite subset $E = \{f_1, \ldots, f_k\}$ of $H(\mathbb{D})$, there are $f \in \mathcal{U}(\mathcal{C})$ and $g \in \mathcal{U}(\mathcal{D})$ such that*

$$F(f, \mathcal{C}, E, \frac{1}{n}) \in O(n)$$

and

$$F\big(g, \mathcal{D}, E, \frac{1}{n}\big) \in O\big(n^2 \ln(n \max\{1, M_{6kn}(E)\})\big)$$

respectively.

Moreover, the unit ball

$$B^\infty := \big\{f \in H(\mathbb{D}) : \big|f(z)\big| \leq 1 \text{ for all } z \in \mathbb{D}\big\}$$

of $H^\infty(\mathbb{D})$ is a normal family in $H(\mathbb{D})$ because, obviously, it is locally bounded. It is immediately seen that the corresponding sequence $\big(M_n(B^\infty)\big)_{n\in\mathbb{N}}$ is constantly equal to one. Hence, taking $\ln(n) \in O\big(n^\varepsilon\big)$ for each $\varepsilon > 0$ into account, another application of Theorem 3 and Corollary 9 gives the next corollary.

Corollary 12. *Let $\mathcal{C}$ be a sequence of composition operators as in Theorem 3, $\mathcal{D}$ the sequence of differentiation operators. There is $f \in \mathcal{U}(\mathcal{C})$ with*

$$F(f, \mathcal{C}, B^\infty, \frac{1}{n}) \in O\big(n\lambda_{2n}(B^\infty)\big).$$

Moreover, there is $g \in \mathcal{U}(\mathcal{D})$ such that

$$F(g, \mathcal{D}, B^\infty, \frac{1}{n}) \in O\big(n^{2+\varepsilon}(\lambda_{3n}(B^\infty))^{1+\varepsilon}\big),$$

for every $\varepsilon > 0$.

By Corollary 9 the covering numbers $\big(\lambda_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, as well as the sequence $\big(M_n(\mathcal{M})\big)_{n\in\mathbb{N}}$, determine how fast the approximation of a normal family $\mathcal{M} \subseteq H(\mathbb{D})$ by a universal function may be.
In order to get a better impression of the concrete error terms involved, we shall consider the following example. It is very well-known, cf. [16, Theorem 12.6], that the set of holomorphic one-to-one mappings of $\mathbb{D}$ onto itself, $Aut(\mathbb{D})$, is given by

$$Aut(\mathbb{D}) = \left\{ f_{\gamma,a}(z) = e^{i\gamma}\frac{z-a}{1-\bar{a}z} : \gamma \in [0, 2\pi), a \in \mathbb{D} \right\}.$$

Since $f_{\gamma,a}(\mathbb{D}) = \mathbb{D}$, $Aut(\mathbb{D})$ is bounded in $H(\mathbb{D})$, so a normal family, and $M_n(Aut(\mathbb{D})) = 1$ for every $n \in \mathbb{N}$. Next, we give bounds for $\lambda_n(Aut(\mathbb{D}))$.

Lemma 13. *For the normal family $Aut(\mathbb{D})$ in $H(\mathbb{D})$, we have $\lambda_n(Aut(\mathbb{D})) \in O(n^7)$.*

Proof. Fix two functions $f_{\gamma_j,a_j} \in Aut(\mathbb{D})$ $(j = 1, 2)$. Because $|f_{0,a_2}(z)| \le 1$, we have for every $z \in K_n$ that

$$\begin{aligned}
|f_{\gamma_1,a_1}(z) - f_{\gamma_2,a_2}(z)| &= \left| e^{i\gamma_1} \frac{z - a_1}{1 - \bar{a}_1 z} - e^{i\gamma_2} \frac{z - a_2}{1 - \bar{a}_2 z} \right| \\
&= \left| e^{i\gamma_1} \left(f_{0,a_1}(z) - f_{0,a_2}(z)\right) + (e^{i\gamma_1} - e^{i\gamma_2}) f_{0,a_2}(z) \right| \\
&\le \left| \frac{(z - a_1)(1 - \bar{a}_2 z) - (z - a_2)(1 - \bar{a}_1 z)}{(1 - \bar{a}_1 z)(1 - \bar{a}_2 z)} \right| + \left| e^{i(\gamma_1 - \gamma_2)} - 1 \right| \\
&\le \frac{1}{(1 - (\frac{n}{n+1}))^2} \left| a_2 - a_1 + (a_1 \bar{a}_2 - a_2 \bar{a}_1) z + (\bar{a}_1 - \bar{a}_2) z^2 \right| \\
&\quad + \left| i \int_0^{\gamma_1 - \gamma_2} e^{it} dt \right| \\
&\le (n+1)^2 \left(2|a_1 - a_2| + |a_1 \bar{a}_2 - a_2 \bar{a}_1| \right) + |\gamma_1 - \gamma_2| \\
&\le 4(n+1)^2 |a_1 - a_2| + |\gamma_1 - \gamma_2|
\end{aligned}$$

Thus, for $\|f_{\gamma_1,a_1} - f_{\gamma_2,a_2}\|_{K_n} < 1/n$ to hold, only $O(n)$ different γ and $O(n^6)$ different $a \in \mathbb{D}$ are needed. Since, by the definition of the metric $d_{\mathcal{K}}$, the inequality $\|f_{\gamma_1,a_1} - f_{\gamma_2,a_2}\|_{K_n} < 1/n$ implies $d_{\mathcal{K}}(f_{\gamma_1,a_1}, f_{\gamma_2,a_2}) < 1/n$, we obtain $\lambda_n \in O(n^7)$.

□

Remark 14. If one considers, instead of $Aut(\mathbb{D})$, the smaller set

$$\begin{aligned}
\mathcal{M} &:= \left\{ f \in Aut(\mathbb{D}) \colon \text{ the only zero } z_0 \text{ of } f \text{ satisfies } |z_0| \le r \right\} \\
&= \left\{ f_{\gamma,a} \colon |a| \le r, \gamma \in [0, 2\pi) \right\}
\end{aligned}$$

for fixed $r \in (0, 1)$, a similar calculation as in the proof of Lemma 13 gives $\lambda_n(\mathcal{M}) \in O(n^3)$.

These growth estimations motivate to introduce the following notion.

Definition 15. *Let $(\mathcal{X}, d)$ be a complete metric space, $(\mathcal{Y}, d)$ a separable metric space, $\mathcal{M} \subseteq \mathcal{Y}$ be totally bounded, and $\mathcal{L} = (L_n)_{n \in \mathbb{N}}$ be a sequence of continuous mappings $L_n : \mathcal{X} \to \mathcal{Y}$. We say that an element $x \in \mathcal{X}$ is m-polynomial $\mathcal{L}$-universal for $\mathcal{M}$ if $x \in \mathcal{U}(\mathcal{L})$ and*

$$F(x, \mathcal{L}, \mathcal{M}, 1/n) \in O(n^m).$$

We abbreviate the set of all such x by $\mathcal{U}_m(\mathcal{L}, \mathcal{M})$. $\mathcal{L}$ is called m-polynomial universal for $\mathcal{M}$ if $\mathcal{U}_m(\mathcal{L}, \mathcal{M}) \neq \emptyset$.

Again, taking $\ln(n) \in O(n^\varepsilon)$ for each $\varepsilon > 0$ into account, Theorem 3, Corollary 9 and Lemma 13 immediately give us

Corollary 16. *Let $\mathcal{C}$ be a sequence of composition operators as in Theorem 3, $\mathcal{D}$ the sequence of differentiation operators. Consider the normal family $Aut(\mathbb{D})$ in $H(\mathbb{D})$. Then, there exist*

(i) 8-polynomial $\mathcal{C}$-universal functions for $Aut(\mathbb{D})$,

(ii) $(9{+}\varepsilon)$-polynomial $\mathcal{D}$-universal functions for $Aut(\mathbb{D})$ for each $\varepsilon > 0$.

Remark 17.

(i) If the covering numbers $\lambda_n = \lambda_n(\mathcal{M})$ of a totally bounded subset $\mathcal{M}$ satisfy $\lambda_n \in O(n^m)$, the number m is related to the so-called box-counting dimension of $\mathcal{M}$.

(ii) Let $\mathcal{M} \subseteq \mathcal{Y}$ be totally bounded with covering numbers (λ_n). Hence, for every $n \in \mathbb{N}$, there are $f_1^{(n)}, \ldots, f_{\lambda_n}^{(n)} \in \mathcal{Y}$ which cover $\mathcal{M}$ with their $\frac{1}{n}$-neighborhoods. Then, we have

$$\mathcal{U}_m(\mathcal{L}, \mathcal{M}) = \bigcup_{c\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \bigcap_{j=1}^{\lambda_n} \bigcup_{N=1}^{c\cdot n^m} L_N^{-1}\big(U_{1/n}(f_j^{(n)})\big). \tag{13}$$

From the description (13), we deduce that the polynomial universal elements form a countable union of G_δ-sets, which is called a $G_{\delta\sigma}$-set in the literature. A natural question is: Is it also G_δ?

A very prominent example of a normal family in $H(\mathbb{D})$ is the set

$$S = \{f \in H(\mathbb{D})\colon f \text{ one-to-one}, f(0) = 0, f'(0) = 1\}.$$

From the well-known inequality due to Koebe, see e.g. [15, Satz 15.15]:

$$\forall\, f \in S, z \in \mathbb{D}: \ \big|f(z)\big| \leq \frac{|z|}{(1 - |z|)^2}, \tag{14}$$

follows the boundedness of S in $H(\mathbb{D})$, in fact, S is a normal family and

$$M_n(S) = (2n + 1)(2n + 2) \in O(n^2). \tag{15}$$

A special subset of S is given by

$$K := \big\{f_\alpha : \ \alpha \in [0, 2\pi)\big\} \subseteq S, \quad f_0(z) = \frac{z}{(1 - z)^2}, \quad f_\alpha(z) = e^{-i\alpha} f_0(e^{i\alpha} z),$$

the so-called *Koebe extremal functions*. Obviously, K is a normal family also with $M_n(K) \in O(n^2)$. As Taylor expansions about the origin, one gets

$$f_0(z) = \sum_{\nu=1}^{\infty} \nu z^\nu, \quad f_\alpha(z) = \sum_{\nu=1}^{\infty} \nu e^{i(\nu-1)\alpha} z^\nu.$$

Lemma 18. *For the normal family K in $H(\mathbb{D})$, we have $\lambda_n(K) \in O\big(n^2 \ln(n)\big)$.*

Proof. Consider

$$T_m f_\beta(z) = \sum_{\nu=1}^{m} \nu e^{i(\nu-1)\beta} z^\nu, \ m \in \mathbb{N}, \ \beta \in [0.2\pi),$$

where $T_m f$ denotes, again, the m-th Taylor polynomial of f expanded about the origin. By Corollary 7, there is a sequence $\gamma_n \in O\big(n \ln(n)\big)$ with $\big\|T_{\gamma_{2n}} f - f\big\|_{K_n} < \frac{1}{2n}$ for all $f \in S$. Using the simple estimate

$$\big|e^{i(\nu-1)\alpha} - e^{i(\nu-1)\beta}\big| = \left|\int_\beta^\alpha \frac{1}{i(\nu - 1)} e^{i(\nu-1)t} dt\right| \leq \frac{1}{\nu - 1}\big|\alpha - \beta\big|, \tag{16}$$

we obtain, for $f \in K$ and $z \in K_n$,

$$\begin{aligned}\left|f_\alpha(z) - T_{\gamma_{2n}} f_\beta(z)\right| \quad &\leq \quad \sum_{\nu=2}^{\gamma_{2n}} \nu \left|e^{i(\nu-1)\alpha} - e^{i(\nu-1)\beta}\right| + \left\|f_\alpha - T_{\gamma_{2n}} f_\alpha\right\|_{K_n} \\ &\underset{(16)}{<} \quad \sum_{\nu=2}^{\gamma_{2n}} \frac{\nu}{\nu-1}\left|\alpha - \beta\right| + \frac{1}{2n} \;<\; 2\gamma_{2n}\left|\alpha - \beta\right| + \frac{1}{2n}.\end{aligned}$$

Thus, for $\left\|f_\alpha - T_{\gamma_{2n}} f_\beta\right\|_{K_n} < 1/n$ to hold for some $\beta \in [0, 2\pi)$ only $O(n^2 \ln(n))$ values of β are needed. □

As above, we deduce from the results of the previous section and Lemma 18:

Corollary 19. *Let $\mathcal{C}$ be a sequence of composition operators as in Theorem 3, $\mathcal{D}$ the sequence of differentiation operators. Consider the normal family K of Koebe extremal functions in $H(\mathbb{D})$. Then, there exist*

(i) $(2{+}\varepsilon)$-polynomial $\mathcal{C}$-universal functions for K for each $\varepsilon > 0$,

(ii) $(4{+}\varepsilon)$-polynomial $\mathcal{D}$-universal functions for K for each $\varepsilon > 0$.

Before we give (what we think to be rather coarse) bounds for the growth of $\big(\lambda_n(S)\big)_{n\in\mathbb{N}}$, we apply Theorem 3 and Corollary 9 to S.

Corollary 20. *Let $\mathcal{C}$ be a sequence of composition operators as in Theorem 3, $\mathcal{D}$ the sequence of differentiation operators, and $(\lambda_n)_{n\in\mathbb{N}} = \big(\lambda_n(S)\big)_{n\in\mathbb{N}}$. Then, there are some $f \in \mathcal{U}(\mathcal{C})$ and $g \in \mathcal{U}(\mathcal{D})$ with*

$$F(f, \mathcal{C}, S, \frac{1}{n}) \in O\big(n\lambda_{2n}\big),$$

respectively

$$F(g, \mathcal{D}, S, \frac{1}{n}) \in O\big(n^2 \lambda_{3n} \ln(n\lambda_{3n})\big).$$

The next result gives bounds for $\big(\lambda_n(S)\big)_{n\in\mathbb{N}}$.

Lemma 21. *We have*

$$\lambda_n(S) \in O\big(\exp(n^{1+\varepsilon})\big),$$

for every $\varepsilon > 0$.

Proof. 1. Let $n \in \mathbb{N}$ be fixed. Consider for $f \in S$ its Taylor expansion $f(z) = z + \sum_{\nu=2}^{\infty} a_\nu(f) z^\nu$ about 0. By de Branges' famous proof of Bieberbach's Conjecture [8], we know $a_\nu(f) \in \nu\bar{\mathbb{D}}$ for all $\nu \geq 2$. In (9), we obtained

$$\sum_{\nu=m+1}^{\infty} \left|a_\nu(f)\right| \left(\frac{n}{n+1}\right)^\nu < \frac{1}{n},$$

whenever

$$m \geq m_n := 3(2n+1)\lceil \ln(5n(2n+1)M_n(S)) \rceil; \tag{17}$$

as mentioned earlier $M_n(S) = (2n+1)(2n+2)$.

2. As we will see from the following estimate, any function

$$g(z) := z + \sum_{\nu=2}^{m_{2n}} b_\nu z^\nu,$$

whose coefficients b_ν fulfill $\big|a_\nu(f) - b_\nu\big| \le \frac{1}{2n^2}$ for each $2 \le \nu \le m_{2n}$, satisfies $d_{\mathcal{K}}(f,g) < \frac{1}{n}$. Counting how many of these functions g are at most needed, so that for any $f \in S$ there is at least one such g with $d_{\mathcal{K}}(f,g) < \frac{1}{n}$, will give us an upper bound for $\lambda_n(S)$ in the next step. But before, we estimate

$$\begin{aligned} \big\|f - g\big\|_{K_n} &\le \sum_{\nu=2}^{m_{2n}} \big|a_\nu(f) - b_\nu\big| \left(\frac{n}{n+1}\right)^\nu + \sum_{\nu=m_{2n}+1}^{\infty} \nu \left(\frac{n}{n+1}\right)^\nu \\ &< \frac{1}{2n^2} \sum_{\nu=1}^{\infty} \left(\frac{n}{n+1}\right)^\nu + \frac{1}{2n} = \frac{1}{n}. \end{aligned}$$

By the definition of our metric $d_{\mathcal{K}}$, this implies $d_{\mathcal{K}}(f,g) < \frac{1}{n}$.

3. For fixed $\nu \in [2, m_{2n}] \cap \mathbb{N}$, we set a grid of points b_ν, spaced at intervals of $\frac{1}{2n^2}$ parallel to the real and imaginary axes, on the disk $\nu\bar{\mathbb{D}}$. This shows that there are at most $16n^4(\nu+1)^2$ points b_ν needed, so that for any $f \in S$ there is at least one b_ν with $\big|a_\nu(f) - b_\nu\big| \le \frac{1}{2n^2}$. Hence,

$$\lambda_n(S) \le \prod_{\nu=2}^{m_{2n}} 16n^4(\nu+1)^2 \le 16^{m_{2n}}\, n^{4m_{2n}}\, \left((m_{2n}+1)!\right)^2. \tag{18}$$

Using $(m_{2n}+1)! = \Gamma(m_{2n}+2)$, as well as

$$\lim_{z\to\infty} \frac{\Gamma(z+2)}{z\sqrt{2\pi z}(\frac{z}{e})^z} = 1,$$

cf. [15, page 59], there is $C > 1$ such that

$$\forall\, n \in \mathbb{N}:\ \left((m_{2n}+1)!\right)^2 \le C m_{2n}^3 \left(\frac{m_{2n}}{e}\right)^{2m_{2n}} < C m_{2n}^{2m_{2n}+3} < C m_{2n}^{3m_{2n}}. \tag{19}$$

Combining equations (18) and (19), we obtain

$$\lambda_n(S) \le C(16n)^{m_{2n}}\, (n m_{2n})^{3m_{2n}}. \tag{20}$$

From (17), it follows $m_{2n} \in O\big(n\ln(n)\big)$ Together with (20), we conclude

$$\lambda_n(S) \in O\big(\exp(n\ln^2(n))\big).$$

Since $\lim_{x\to\infty} \frac{\ln^2(x)}{x^\varepsilon} = 0$ for every $\varepsilon > 0$, this finally implies the lemma.

□

For further examples of normal families one may consult [17].

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