CHAOTIC $C_0$-SEMIGROUPS INDUCED BY SEMIFLOWS IN
LEBESGUE AND SOBOLEV SPACES

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Abstract. We give characterizations of chaos for $C_0$-semigroups induced by semiflows on $L^p_\rho(\Omega)$ for open $\Omega \subseteq \mathbb{R}$ similar to the characterizations of hypercyclicity and mixing of such $C_0$-semigroups proved in [18]. Moreover, we characterize hypercyclicity, mixing, and chaos for these classes of $C_0$-semigroups on $W^{1,\rho}(I)$ for a bounded interval $I \subset \mathbb{R}$ and prove that these $C_0$-semigroups are never hypercyclic on $W^{1,\rho}(I)$. We apply our results to concrete first order partial differential equations, such as the von Foerster-Lasota equation.

1. Introduction

The aim of this article is to characterize certain dynamical properties of $C_0$-semigroups induced by weighted semiflows on different function spaces. Recall that a $C_0$-semigroup $T$ on a separable Banach space $X$ is called hypercyclic if there is $x \in X$ with its orbit $\{T(t)x; t \geq 0\}$ under $T$ being dense in $X$. $T$ is called chaotic if $T$ is hypercyclic and the set of periodic points, i.e. $\{x \in X; \exists t > 0 : T(t)x = x\}$, is dense in $X$. It is well-known, that a $C_0$-semigroup $T$ on a separable Banach space $X$ is hypercyclic if and only if $T$ is (topologically) transitive, i.e. for any pair of non-empty, open subsets $U, V$ of $X$ there is $t > 0$ such that $T(t)(U) \cap V \neq \emptyset$. If for any pair of non-empty, open subsets $U, V$ of $X$ there is $t_0 > 0$ such that $T(t)(U) \cap V \neq \emptyset$ whenever $t \geq t_0$ then $T$ is called mixing, while $T$ is weakly mixing if the direct sum $C_0$-semigroup $T \oplus T$ is transitive on $X \oplus X$.

The study of chaotic properties for $C_0$-semigroups has attracted the attention of many researchers. We refer the reader to Chapter 7 of the monograph by Grosse-Erdmann and Peris [15] and the references therein. Some recent papers in the topic are [11, 13, 14, 16, 22, 25].

For $\Omega \subseteq \mathbb{R}$ open and a Borel measure $\mu$ over $\Omega$ admitting a strictly positive Lebesgue density $\rho$ we consider $C_0$-semigroups $T$ on $L^p(\Omega, \mu), 1 \leq p < \infty$, of the form

$$T(t)f(x) = h_t(x)f(\varphi(t,x)),$$

where $\varphi$ is the solution semiflow of an ordinary differential equation

$$\dot{x} = F(x)$$

in $\Omega$ and

$$h_t(x) = \exp \left(\int_0^t h(\varphi(s,x))ds\right)$$

with $h \in C(\Omega)$. Such $C_0$-semigroups appear in a natural way when dealing with initial value problems for linear first order partial differential operators. While characterizations of hypercyclicity, (weak) mixing, and chaos of such $C_0$-semigroups where obtained for open $\Omega \subseteq \mathbb{R}^d$ for arbitrary dimension $d$ in [17], evaluation of these...
conditions in concrete examples is sometimes rather involved. In contrast to general dimension the case \( d = 1 \) allows for significantly simplified characterizations. In [13] these were given for hypercyclicity and mixing. In section 2, we give a simplified characterization of chaos for such \( C_0 \)-semigroups. Moreover, we further evaluate and extend the conditions obtained in [13].

In section 3 we investigate the above kind of \( C_0 \)-semigroups on the Sobolev spaces \( W^{1,p}(a,b) \), where \((a, b) \subseteq \mathbb{R}\) is a bounded interval. In case of \( F(a) = 0 \) we show that hypercyclicity never occurs while we characterize hypercyclicity, (weak) mixing, mixing, and chaos on the closed subspace \( W^{1,p}(a,b) = \{ f \in W^{1,p}(a,b); f(a) = 0 \} \).

In order to illustrate our results, we consider concrete initial value problems for first order partial differential operators in section 4. Among them, we consider the linear von Foerster-Lasota equation

\[
\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = h(x)u(x) \quad t \geq 0, \ x \in [0, 1]
\]

with initial condition

\[ u(x, 0) = v(x) \quad x \in [0, 1], \]

where \( v \) belongs to a suitable function space.

(1) is a particular case of the equation

\[
\frac{\partial u}{\partial t} + c(x) \frac{\partial u}{\partial x} = f(x,u) \quad t \geq 0, \ x \in [0, 1]
\]

that was introduced in [20] to describe the reproduction of a population of red blood cells, mainly in connection with studies about anemia. After the paper [19], this problem has already been studied in different function spaces by several authors either with an ergodic theoretical approach (see [25] and the references quoted therein) or by explicitly constructing hypercyclic and periodic solutions (see [11] or by investigating spectral properties of the differential operator associated to the equation (1) (see [27] and the applications in [9]). In the present paper the results will follow from the characterizations proved in section 3.

2. Characterizations of dynamical properties of \( C_0 \)-semigroups on \( L^p(\Omega, \mu) \) induced by weighted semiflows

Let \( \Omega \subseteq \mathbb{R} \) be an open set and \( F: \Omega \rightarrow \mathbb{R} \) be a \( C^1 \)-function. Hence, for every \( x_0 \in \Omega \) there is a unique solution \( \varphi(\cdot, x_0) \) of the initial value problem

\[ \dot{x} = F(x), \ x(0) = x_0. \]

Denoting its maximal domain of definition by \( J(x_0) \) it is well-known that \( J(x_0) \) is an open interval containing 0. Throughout this section we make the general assumption that \([0, \infty) \subseteq J(x_0) \) for every \( x_0 \in \Omega \), i.e. \( \varphi : [0, \infty) \rightarrow \Omega \). From the uniqueness of the solution it follows that \( \varphi(t, \cdot) \) is injective for every \( t \geq 0 \) and \( \varphi(t+s, x) = \varphi(t, \varphi(s, x)) \) for all \( x \in \Omega \) and \( s, t \in J(x) \) with \( s + t \in J(x) \). Moreover, for every \( t \geq 0 \) and \( x \in \varphi(t, \Omega) \) we have \([-t, \infty) \subseteq J(x) \) and for all \( s \in [0, t] \) we have \( \varphi(-s, x) = \varphi(s, \cdot)^{-1}(x) \). Since \( F \) is a \( C^1 \)-function it is well-known that the same is true for \( \varphi(t, \cdot) \) on \( \Omega \) and \( \varphi(-t, \cdot) \) on \( \varphi(t, \Omega) \) for every \( t \geq 0 \). We denote by \( \partial_2 \varphi(t, \cdot) \) its derivative. Moreover we recall that a subset \( M \subseteq \Omega \) is positively invariant under \( \varphi \) if for every \( x \in M \) and for every \( t \in J(x_0) \cap [0, \infty) \) it holds that \( \varphi(t, x) \in M \).

We refer to the monograph of Amann [2] for further results on this topic.

Additionally, fix \( h \in C(\Omega) \). We define for \( t \geq 0 \)

\[ h_t : \Omega \rightarrow \mathbb{C}, x \mapsto \exp(\int_0^t h(\varphi(s, x)) \, ds). \]

It is easily seen, that because of \( \varphi(t + s, x) = \varphi(t, \varphi(s, x)) \) we have \( h_{t+s}(x) = h_t(x)h_s(\varphi(t, x)) \) for all \( s, t \geq 0 \) and \( x \in \Omega \).
For a measurable function $\rho : \Omega \to (0, \infty)$ we want to define a $C_0$-semigroup on $L^p(\Omega, \mu)$ via $(T(t)f)(x) := h_t(x)f(\varphi(t, x))$, where $1 \leq p < \infty$ and $\mu$ denotes the Borel measure on $\Omega$ with Lebesgue density $\rho$. The next theorem gives a characterization of when this is possible. For its proof see [17, Theorem 4.7 and Proposition 4.12]. Although in [17] $h$ is assumed to be real valued the proofs of [17, Theorem 4.7 and Proposition 4.12] are valid for complex valued $h$, too. Observe that for real valued $h$ we have $h_t = |h_t|$ for all $t \geq 0$. From now on we write $L^p(\Omega)$ instead of $L^p(\Omega, \mu)$ and simply $L^p(\Omega)$ in case of $\rho = 1$.

**Theorem 1.** Let $F, h$, and $\rho$ be as above and $p \in [1, \infty)$. Then the following are equivalent.

i) For $t \geq 0$ the operators

$$T(t) : L^p_h(\Omega) \to L^p_h(\Omega), (T(t)f)(x) := h_t(x)f(\varphi(t, x))$$

are well-defined, linear and continuous and define a $C_0$-semigroup $T_{F,h} := (T_{F,h}(t))_{t \geq 0} := (T(t))_{t \geq 0}$ on $L^p_h(\Omega)$.

ii) There are $M \geq 1, \omega \in \mathbb{R}$ such that for every $t \geq 0$

$$|h_t(x)|\rho(x) \leq Me^{\omega t}\rho(\varphi(t, x))\partial_x \varphi(t, x)$$

holds $\lambda$-a.e. on $\Omega$, where $\lambda$ denotes Lebesgue measure.

Moreover, if ii) holds the generator of the above $C_0$-semigroup is an extension of the operator

$$C^1_c(\Omega) \to L^p_h(\Omega), f \mapsto Ff' + hf$$

in $L^p_h(\Omega)$, where $C^1_c(\Omega)$ denotes the space of compactly supported, continuously differentiable functions on $\Omega$. Additionally, if $h$ is bounded and $F$ is such that for every $x_0 \in \Omega$ the maximal domain of $\varphi(\cdot, x_0)$ equals $\mathbb{R}$, then $C^1_c(\Omega)$ is a core for the generator of the $C_0$-semigroup $T_{F,h}$.

**Definition 2.** In what follows, we call a measurable function $\rho : \Omega \to (0, \infty)$ $p$-admissible for $F$ and $h$ ($p \in [1, \infty)$) if there are constants $M \geq 1, \omega \in \mathbb{R}$ such that

$$\forall t \geq 0, x \in \Omega : |h_t(x)|\rho(x) \leq Me^{\omega t}\rho(\varphi(t, x))\partial_x \varphi(t, x)|.$$

Since $|h_t(x)| = \exp(\int_0^t \text{Re } h(\varphi(s, \cdot))ds)$, $p$-admissibility of $\rho$ only depends on $F$ and $\text{Re } h$. By the above theorem, we have for a $p$-admissible $\rho$ for $F$ and $h$ the well-defined $C_0$-semigroup $T_{F,h}$ on $L^p_h(\Omega)$.

Our first aim in this section is to give a characterization of chaos for $T_{F,h}$ on $L^p_h(\Omega)$ similar to the characterizations of hypercyclicity and mixing for those $C_0$-semigroups proved in [13]. Before we do so we make the following observation.

**Remark 3.** (i) Let $\rho$ be $p$-admissible for $F$ and $h$. If for $x_0 \in \Omega$ we have $F(x_0) = 0$ it clearly follows that $\varphi(t, x_0) = x_0$ for every $t \geq 0$. Since we have uniqueness for the solutions of the initial value problems

$$\dot{x} = F(x), \ x(0) = x_0 (x_0 \in \Omega),$$

this immediately implies for $\{F = 0\} := \{x \in \Omega; F(x) = 0\}$ that

$$\forall t \geq 0 : \varphi(t, \Omega \setminus \{F = 0\}) = \varphi(t, \Omega \setminus \{F = 0\}) \subseteq \Omega \setminus \{F = 0\}.$$

Hence we can consider $F_{|\Omega \setminus \{F = 0\}}$ and $h_{|\Omega \setminus \{F = 0\}}$ on $\Omega \setminus \{F = 0\}$ and $\rho_{|\Omega \setminus \{F = 0\}}$ is $p$-admissible for $F_{|\Omega \setminus \{F = 0\}}$ and $h_{|\Omega \setminus \{F = 0\}}$. For notational convenience we write again $T_{F,h}$ and $L^p_h(\Omega \setminus \{F = 0\})$ instead of $T_{F_{|\Omega \setminus \{F = 0\}}, h_{|\Omega \setminus \{F = 0\}}}$ and $L^p_{h_{|\Omega \setminus \{F = 0\}}}(\Omega \setminus \{F = 0\})$. Finally, if $\lambda(\{F = 0\}) = 0$ it follows that $L^p_h(\Omega)$ and $L^p_h(\Omega \setminus \{F = 0\})$ can be identified in a canonical way.

(ii) $\varphi(t, \cdot) : \Omega \to \varphi(t, \Omega)$ is bijective for $t \geq 0$ with inverse $\varphi(-t, \cdot) : \varphi(t, \Omega) \to \Omega$. For $x_0 \in \Omega$ we define $Z(x_0)$ to be the connected component of $\Omega \setminus \{F = 0\}$
containing $x_0$ if $F(x_0) \neq 0$ and $Z(x_0) := \{x_0\}$ if $F(x_0) = 0$. It is well-known that
\[ \varphi(t, Z(x_0)) \subseteq Z(x_0) \text{ for every } t \geq 0, \] more precisely
\[ \forall x_0 \in \Omega : \varphi([0, \infty), x_0) = \begin{cases} Z(x_0) \cap [x_0, \infty) & \text{if } F(x_0) > 0 \\ Z(x_0) \cap (-\infty, x_0] & \text{if } F(x_0) \leq 0. \end{cases} \]
Herefrom and from the injectivity, it follows easily that $\varphi(t, \cdot) : \Omega \to \Omega$ is strictly increasing for all $t \geq 0$ and thus $\varphi(-t, \cdot) : \varphi(t, \Omega) \to \Omega$ is strictly increasing, too.

Since \[ \forall t \geq 0, x \in \Omega : x = \varphi(-t, \varphi(t, x)) \]
we obtain \[ \forall t \geq 0, x \in \Omega : 1 = \partial_2 \varphi(-t, \varphi(t, x)) \partial_2 \varphi(t, x) \]
so that \[ \forall t \geq 0, x \in \Omega : \partial_2 \varphi(t, x) > 0 \]
as well as \[ \forall t \geq 0, x \in \varphi(t, \Omega) : \partial_2 \varphi(-t, x) > 0. \]

In order to formulate our result in a convenient way we introduce the following notions.

**Definition 4.** If $\rho$ is $p$-admissible for $F$ and $h$ we define for $t \geq 0$
\[ \rho_{t,p} : \Omega \to [0, \infty), \rho_{t,p}(x) := \chi_{\varphi(t, \Omega)}(x)|h_t(\varphi(t, x))|^p \rho(\varphi(t, x))\partial_2 \varphi(t, x) \]
as well as
\[ \rho_{-t,p} : \Omega \to [0, \infty), \rho_{-t,p}(x) := |h_t(x)|^{-p} \rho(\varphi(t, x))\partial_2 \varphi(t, x). \]

Obviously, $\rho_{t,p}$ and $\rho_{-t,p}$ depend on $F$ and $h$ but in order to keep notation simple we will not take this into account notationally as there will be no danger of confusion. Observe that $\rho_{0,p} = \rho$. The next lemma will be a crucial tool. For its proof see [18, Lemma 7].

**Lemma 5.** Let $F$ be as above and $h \in C(\Omega)$ real valued. Let $\rho$ be $p$-admissible for $F$ and $h$ and let $[a, b] \subset \Omega \setminus \{F = 0\}$. Setting $\alpha := a, \beta := b$ if $F_{[a,b]} > 0$, respectively $\alpha := b, \beta := a$ if $F_{[a,b]} < 0$, there is a constant $C > 0$ such that
\[ \forall x \in [a, b] : \frac{1}{C} \leq \rho(x) \leq C \]
as well as
\[ \forall t \geq 0, x \in [a, b] : \frac{1}{C} \rho_{t,p}(\alpha) \leq \rho_{t,p}(x) \leq C \rho_{t,p}(\beta) \]
and
\[ \forall t \geq 0, x \in [a, b] : \frac{1}{C} \rho_{-t,p}(\alpha) \leq \rho_{-t,p}(x) \leq C \rho_{-t,p}(\beta). \]

We can now prove a characterization of chaos for the $C_0$-semigroup $T_{F,h}$ on $L_p^\infty(\Omega)$ for real valued $h$. We denote $m$-dimensional Lebesgue measure by $\lambda^m$ and set $\lambda := \lambda^1$.

**Theorem 6.** Let $\Omega \subset \mathbb{R}$ be open, $F \in C^1(\Omega)$ as above, $h \in C(\Omega)$ real valued, and $\rho : \Omega \to (0, \infty)$ be a measurable function which is $p$-admissible for $F$ and $h$. Then the following are equivalent.

i) $T_{F,h}$ is chaotic on $L_p^\infty(\Omega)$.

ii) $\lambda(\{F = 0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are $m$ different connected components $C_1, \ldots, C_m$ of $\Omega \setminus \{F = 0\}$, for $\lambda^m$-almost all choices of $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ there is $t > 0$ such that
\[ \sum_{j=1}^m \sum_{t \in \mathbb{Z}} \rho_{t,p}(x_j) < \infty. \]
Proof. We first show that i) implies ii). Since for all $x \in \{F = 0\}$ and any $t \geq 0$ we have $\varphi(t, x) = x$ it follows

$$
\forall f \in L^p(\Omega) : (T(t)f)_{\{|F=0\}} = (\exp(t h)f)_{\{|F=0\}}
$$

so that $(T(t)f)_{t \geq 0}$ cannot be dense in $L^p(\Omega)$ for any $f \in L^p(\Omega)$ if $\lambda(\{F = 0\}) > 0$. Hence, since $T_{F,h}$ is chaotic, we conclude $\lambda(\{F = 0\}) = 0$. As described in Remark [5] we are therefore actually dealing with $T_{F,h}$ on $L^p(\Omega \setminus \{F = 0\})$. Now, if $K \subset \Omega \setminus \{F = 0\}$ is compact there is $\xi_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ whenever $t > t_K$. Hence, we can apply [17] Theorem 5.3 saying that, because $T_{F,h}$ is chaotic on $L^p(\Omega \setminus \{F = 0\})$ for every compact $K \subset \Omega \setminus \{F = 0\}$ there are a sequence $(t_n)_{n \in \mathbb{N}}$ in $[0, \infty)$ and a sequence of measurable subsets $(L_n)_{n \in \mathbb{N}}$ of $K$ such that

$$
\lim_{n \to \infty} \mu(L_n) = \mu(K)
$$

(Recall that $\mu$ denotes the Borel measure on $\Omega$ with Lebesgue density $\rho$!) We will apply this condition to special compact sets in order to derive ii).

Let $x_1, \ldots, x_m$ be each from different connected components of $\Omega \setminus \{F = 0\}$. As $\Omega \setminus \{F = 0\}$ is open, there is $r < 0$ such that $\varphi(t, x_j)$ is well-defined for all $t \in [r, \infty)$ and every $1 \leq j \leq m$. For each $1 \leq j \leq m$ we set $K_j := \{\varphi(t, x_j); t \in [0, 1]\}$ if $F(x_j) > 0$, respectively $K_j := \{\varphi(t, x_j); t \in [r, 0]\}$ if $F(x_j) < 0$. It follows that $K_j = [x_j, \varphi(1, x_j)]$ if $F(x_j) > 0$, respectively $K_j = [\varphi(r, x_j), x_j]$ if $F(x_j) < 0$. In particular $\lambda(K_j) > 0$ and thus $\mu(K_j) > 0$ for every $j$.

Let $\mu(K_j) = \bigcup_{j=1}^m K_j \subset \Omega \setminus \{F = 0\}$ be compact so that there are $(t_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ as above. Let $L_{n,j} := L_n \cap K_j$ for $1 \leq j \leq m$ and $n \in \mathbb{N}$. Applying Lemma [5] to $K_j$ it follows that there are $C_j > 0 (1 \leq j \leq m)$ such that for all $n \in \mathbb{N}$

$$
\int_{L_{n,j}} \rho_{t_{n,p}}(y) \, d\lambda(y) = \int_{L_{n,j}} \frac{\rho_{t_{n,p}}(y)}{\rho(y)} \, d\mu(y) \geq C_j \rho_{t_{n,p}}(x_j) \mu(L_{n,j})
$$

and analogously

$$
\int_{L_{n,j}} \rho_{-t_{n,p}}(y) \, d\lambda(y) \geq C_j \rho_{-t_{n,p}}(x_j) \mu(L_{n,j}).
$$

Since for $n$ large enough

$$
\infty > \sum_{l=1}^\infty \left( \int_{L_n} \rho_{t_{n,p}} \, d\lambda + \int_{L_n} \rho_{-t_{n,p}} \, d\lambda \right)
$$

$$
= \sum_{j=1}^m \sum_{l=1}^\infty \left( \int_{L_{n,j}} \rho_{t_{n,p}} \, d\lambda + \int_{L_{n,j}} \rho_{-t_{n,p}} \, d\lambda \right)
$$

$$
\geq \sum_{j=1}^m C_j \mu(L_{n,j}) \sum_{l=1}^\infty \left( \rho_{t_{n,p}}(x_j) + \rho_{-t_{n,p}}(x_j) \right)
$$

and $\lim_{n \to \infty} \mu(L_{n,j}) = \mu(K_j) > 0$ we deduce for $n$ large enough

$$
\sum_{j=1}^m \sum_{l \in \mathbb{Z}} \rho_{t_{n,p}}(x_j) < \infty.
$$

Since for $t = 0$ we have $\rho_{t,p} = \rho > 0$ the above $t_n$ has to be strictly positive. Thus, ii) is proved.

It remains to show that ii) implies i). Since $\lambda(\{F = 0\}) = 0$ we consider $T_{F,h}$ on $L^p(\Omega \setminus \{F = 0\})$, as explained in Remark [3]. If $K \subset \Omega \setminus \{F = 0\}$ is compact there is $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ whenever $t > t_K$. Hence, we can again use [17] Theorem 5.3. For fixed compact $K \subset \Omega \setminus \{F = 0\}$ there are finitely many intervals
$[a_j, b_j] \subset \Omega \setminus \{F = 0\}$ such that each $[a_j, b_j]$ is contained in a different connected component of $\Omega \setminus \{F = 0\}$ and $\mathcal{K} \subseteq \cup_{m=1}^{\infty} [a_j, b_j]$. We define $x_j := a_j$ if $F[a_j, b_j] > 0$, respectively $x_j := b_j$ if $F[a_j, b_j] < 0$. Moreover, without loss of generality, we can assume by ii) that there is $t > 0$ with

$$\sum_{j=1}^{m} \sum_{l \in \mathbb{Z}} \rho_{t, p}(x_j) < \infty.$$ 

Now it follows from Lemma 5 that for some constants $C_j > 0 (1 \leq j \leq m)$ with $t_n := nt, n \in \mathbb{N}$

$$\sum_{l=1}^{\infty} \left( \int_{K} \rho_{t_n, p} d\lambda + \int_{K} \rho_{-t_n, p} d\lambda \right) \leq \sum_{j=1}^{m} \sum_{l=1}^{\infty} \left( \int_{[a_j, b_j]} \rho_{t_n, p} d\lambda + \int_{[a_j, b_j]} \rho_{-t_n, p} d\lambda \right)$$

$$= \sum_{j=1}^{m} \sum_{l=1}^{\infty} \left( \int_{[a_j, b_j]} \frac{\rho_{t_n, p}}{\rho} d\mu + \int_{[a_j, b_j]} \frac{\rho - \rho_{-t_n, p}}{\rho} d\mu \right)$$

$$\leq \sum_{j=1}^{m} \sum_{l=1}^{\infty} \left( \rho_{t_n, p}(x_j) + \rho_{-t_n, p}(x_j) \right)$$

so that

$$\lim_{n \to \infty} \sum_{l=1}^{\infty} \left( \int_{K} \rho_{t_n, p} d\lambda + \int_{K} \rho_{-t_n, p} d\lambda \right) = 0.$$ 

With $L_n := K, n \in \mathbb{N}$ it follows from [17, Theorem 5.3] that $T_{F,h}$ is chaotic on $L^p_\mu(\Omega \setminus \{F = 0\})$. This proves the theorem. □

**Example 7.** (i) Let $\Omega = \mathbb{R}, F = 1,$ and $h = 0$ so that $\varphi(t, x) = x + t$ and $h_t(x) = 1$. Then a measurable function $p : \mathbb{R} \to (0, \infty)$ is $p$-admissible for $F$ and $h$ for some $p \in [1, \infty)$ if the same holds for every $p \in [1, \infty)$ and the corresponding $C_0$-semigroup $T_{F,h}$ is the (bilateral) left translation semigroup on $L^p_\mu(\mathbb{R})$ and given by $(T(t)f)(x) = f(x + t)$, its generator being an extension of

$$C^1_c(\mathbb{R}) \to L^p_\mu(\mathbb{R}), f \mapsto f'.$$

Moreover, we have

$$\rho_{t, p}(x) = \rho(x + t) \text{ and } \rho_{-t, p} = \rho(x - t).$$

Since $\{F = 0\} = \emptyset$ we have only a single connected component of $\mathbb{R} \setminus \{F = 0\}$ so that by Theorem 6 the left translation semigroup on $L^p_\mu(\mathbb{R})$ is chaotic if and only if for $\lambda$-a.e. $x \in \mathbb{R}$ there is $t > 0$ such that

$$\sum_{l \in \mathbb{Z}} \rho(x + lt) < \infty.$$ 

This weight condition is originally due to Matsui et al. [28, Theorem 2] (see also Chapter 7 and related exercises, in [15]). Note that chaos is independent of $p \in [1, \infty)$.

(ii) Let again $\Omega = \mathbb{R}$. Moreover, $F(x) := 1 - x, h(x) = 0 (x \in \mathbb{R})$ so that $\varphi(t, x) = 1 + (x - 1) e^{-t}, h_t(x) = 1,$ and $\partial_2 \varphi(t, x) = e^{-t}$. Then a measurable function $p : \mathbb{R} \to (0, \infty)$ is again $p$-admissible for $F$ and $h$ for some $p \in [1, \infty)$ if the same holds for every $p \in [1, \infty)$ and the corresponding $C_0$-semigroup $T_{F,h}$ is given by $(T(t)f)(x) = f(1 + (x - 1) e^{-t})$ with generator being an extension of

$$C^1_c(\mathbb{R}) \to L^p_\mu(\mathbb{R}), f \mapsto (x \mapsto (1 - x) f'(x)).$$

Furthermore, $\mathbb{R} \setminus \{F = 0\} = (-\infty, 1) \cup (1, \infty)$ has two connected components and

$$\forall t \in \mathbb{R} : \rho_{t, p}(x) = \rho(1 + (x - 1) e^t) e^t.$$
so that $T_{F,h}$ is chaotic on $L^p_h(\mathbb{R})$ if and only if for $\lambda^2$-a.e. $(x_1, x_2) \in (\infty, 1) \times (1, \infty)$ there is $t > 0$ such that
\[
\sum_{j=1}^2 \sum_{l \in \mathbb{Z}} \rho(1 + (x_j - 1)e^{|l|}e^{lt}) < \infty.
\]

Again, the occurrence of chaos is independent of $p \in [1, \infty)$ as $h = 0$.

Analogous characterizations of hypercyclicity and mixing for $T_{F,h}$ on $L^p_h(\Omega)$ for some $p$-admissible $\rho$ for $F$ and real valued $h$ were proved in [18]. Since we will use them in sequel, we include them here for the reader’s convenience. For the proofs see [18, Theorem 9 and Remark 12]. It is worth observing that, by applying Theorem 8 to the left translation semigroup as stated in Example 7(i), one recovers the characterizations of hypercyclicity and mixing originally proved in [12] and [6] respectively.

**Theorem 8.** Let $\Omega \subseteq \mathbb{R}$ be open, $F \in C^1(\Omega)$ as usual, $h \in C(\Omega)$ real valued, and $\rho : \Omega \to (0, \infty)$ be a measurable function which is $p$-admissible for $F$ and $h$.

a) For the $C_0$-semigroup $T_{F,h}$ on $L^p_h(\Omega)$ the following are equivalent.

i) $T_{F,h}$ is hypercyclic.

ii) $T_{F,h}$ is weakly mixing.

iii) $\lambda(\{F = 0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are $m$ different connected components $C_1, \ldots, C_m$ of $\Omega \setminus \{F = 0\}$, for $\lambda^m$-almost all choices of $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ there is a sequence of positive numbers $(t_n)_{n \in \mathbb{N}}$ tending to infinity such that
\[
\forall 1 \leq j \leq m : \lim_{n \to \infty} \rho_{t_n,p}(x_j) = \lim_{n \to \infty} \rho_{-t_n,p}(x_j) = 0.
\]

b) For the $C_0$-semigroup $T_{F,h}$ on $L^p_h(\Omega)$ the following are equivalent.

i) $T_{F,h}$ is mixing.

ii) $\lambda(\{F = 0\}) = 0$ and for $\lambda$-almost every $x \in \Omega$ one has
\[
\lim_{t \to \infty} \rho_{t,p}(x) = \lim_{t \to \infty} \rho_{-t,p}(x) = 0.
\]

So far, we have only characterizations of hypercyclicity, mixing, and chaos of $T_{F,h}$ in case of real valued $h$. In order to obtain some results for at least some complex valued $h$ let us recall the so-called *comparison principle* which is a useful tool in order to identify hypercyclic, (weakly) mixing, or chaotic $C_0$-semigroups. Let $T$ be a $C_0$-semigroup on a Banach space $Y$ and $S$ is a $C_0$-semigroup on a Banach space $X$ which is quasi-conjugate to $T$, i.e. there is a continuous mapping $\Phi : Y \to X$ with dense range such that $\Phi \circ T(t) = S(t) \circ \Phi$ for every $t \geq 0$. Then, $S$ is hypercyclic, (weakly) mixing, or chaotic respectively, if the same holds for $T$, see for Example 7.7. Recall that $T$ and $S$ are said to be conjugate, if the above $\Phi$ is a homeomorphism.

Let $F \in C^1(\Omega)$ be as above and $h, g \in C(\Omega)$ with $g$ real valued. Moreover, let $\rho$ be $p$-admissible for $F$ and $h$ as well as for $F$ and $g$. We want to find some measurable function $\psi : \Omega \to \mathbb{C}$ such that $\exp(\psi)$ induces a continuous, invertible multiplication operator $M$ on $L^p_h(\Omega)$ for which the $C_0$-semigroups $T_{F,h}$ and $T_{F,g}$ are conjugate via $M$, i.e.
\[
\forall t \geq 0, f \in L^p_h(\Omega) : \exp(\psi) T_{F,h}(t)(f) = T_{F,g}(t)(\exp(\psi)f)
\]

Being $T_{F,g}(t)(Mf) = \exp(\psi(\varphi(t, \cdot)))_{\Omega_h} T_{F,h}(f)$, this is satisfied, if for each $x \in \Omega$
\[
\forall t \geq 0 : \psi(\varphi(t, x)) - \int_0^t (h(\varphi(s, x)) - g(\varphi(s, x))) ds = \psi(x).
\]
Now, if \( x \in \{ F = 0 \} \) it follows from \( \varphi(t, x) = x \) that the above expression reduces to
\[
\forall t \geq 0 : \psi(x) - t(h(x) - g(x)) = \psi(x).
\]
Thus, it is necessary that \( h \) and \( g \) coincide on \( \{ F = 0 \} \).

Moreover, for \( x \in \{ F \neq 0 \} \) we also have \( F(\varphi(t, x)) \neq 0 \) for every \( t \geq 0 \) so
\[
\forall t \geq 0 : \psi(\varphi(t, x)) = \psi(t) = 0.
\]

Hence, if \( \Omega \) is connected, \( \alpha := \inf \Omega, \omega := \sup \Omega \), and the function \( \Omega \to \mathbb{R}, y \mapsto \frac{h(y) - g(y)}{F(y)} \) belongs to \( L^1((\alpha, \beta)) \) for every \( \beta \in \Omega \), or to \( L^1((\beta, \omega)) \) for every \( \beta \in \Omega \) we can set
\[
\psi : \Omega \to \mathbb{R}, \psi(x) = \int_{\alpha}^{x} \frac{h(y) - g(y)}{F(y)} dy,
\]
or
\[
\psi : \Omega \to \mathbb{R}, \psi(x) = -\int_{x}^{\omega} \frac{h(y) - g(y)}{F(y)} dy,
\]
respectively, and it follows from the above calculation that (\ref{eq:3}) holds on \( \{ F \neq 0 \} \) for this \( \psi \).

**Proposition 9.** Let \( \Omega \subseteq \mathbb{R} \) be an open interval, \( F \in C^1(\Omega) \) as above and \( h \in C(\Omega) \).
Moreover, let \( \rho \) be \( p \)-admissible for \( F \) and \( h \) and set \( \alpha := \inf \Omega, \omega := \sup \Omega \). Consider the following conditions.

1) \( \forall x \in \{ F = 0 \} : h(x) \in \mathbb{R} \).

2a) The function \( \Omega \to \mathbb{R}, y \mapsto \frac{\Im h(y)}{F(y)} \) belongs to \( L^1((\alpha, \beta)) \) for all \( \beta \in \Omega \).

2b) The function \( \Omega \to \mathbb{R}, y \mapsto -\frac{\Im h(y)}{F(y)} \) belongs to \( L^1((\beta, \omega)) \) for all \( \beta \in \Omega \).

If 1) and 2a) are satisfied, let \( \psi : \Omega \to \mathbb{R}, \psi(x) = i \int_{\alpha}^{x} \frac{\Im h(y)}{F(y)} dy \), if 1) and 2b) hold, set \( \psi : \Omega \to \mathbb{R}, \psi(x) = -i \int_{x}^{\omega} \frac{\Im h(y)}{F(y)} dy \). Then \( \exp \psi \) defines a continuous, invertible multiplication operator \( M \) on \( \mathbb{L}^p(\Omega) \) such that the \( C_0 \)-semigroups \( T_{F,h} \) and \( T_{F,Re,h} \) on \( \mathbb{L}^p(\Omega) \) are conjugate via \( M \).

**Proof.** By the observation preceding the proposition applied to \( h \) and \( g = \Re h \) we only have to show that \( \exp \psi \) defines a continuous, invertible multiplication operator on \( \mathbb{L}^p(\Omega) \). But this is obvious because \( |\exp(\psi(x))| = 1 \) for all \( x \in \Omega \). \( \square \)

**Remark 10.** From the above proposition and the comparison principle it follows immediately, that in Theorems 6 and 8 we can replace the hypothesis of \( h \) being real valued by the weaker conditions

1) \( \forall x \in \{ F = 0 \} : h(x) \in \mathbb{R} \).

2) With \( \alpha := \inf \Omega \) and \( \omega := \sup \Omega \) the function \( \Omega \to \mathbb{R}, y \mapsto \frac{\Im h(y)}{F(y)} \) belongs to \( L^1((\alpha, \beta)) \) for all \( \beta \in \Omega \) or to \( L^1((\beta, \omega)) \) for all \( \beta \in \Omega \).

As is seen in Theorems 6 and 8 the dynamical properties of \( T_{F,\rho} \) on \( \mathbb{L}^p(\Omega) \) are determined by the asymptotic behavior of the functions
\[
\mathbb{R} \to \mathbb{R}, t \mapsto \rho_{t,p}(x),
\]
where \( x \in \Omega \) is fixed. In the definition of \( \rho_{t,p} \) the term \( \partial_2 \varphi(t, x) \) appears. We have the following representation of \( \partial_2 \varphi(t, x) \) at our disposal which not only makes
it sometimes easier to evaluate the characterizations of the different dynamical properties of \( T_{F,h} \) but will also be very useful in section 3.

**Proposition 11.** Let \( \Omega \subseteq \mathbb{R} \) be open and \( F \in C^1(\Omega) \) as above. Then for every \( x \in \Omega \) we have

\[
\forall t \geq 0 : \partial_x \varphi(t, x) = \exp(t \int_0^t F'(\varphi(s, x)) \, ds).
\]

Moreover, for every \( x \in \varphi(r, \Omega), r \geq 0, \)

\[
\forall t \in [0, r] : \partial_x \varphi(-t, x) = \exp(-t \int_{-t}^0 F'(\varphi(s, x)) \, ds).
\]

**Proof.** For \( x \in \Omega \) we have \( \partial_x \varphi(t, x) > 0 \) by hypothesis on \( F \) and Remark 3 ii). Since \( F \) is \( C^1 \) it is well-known that \( \partial_x \varphi \) exists and is continuous and \( \partial_x \varphi(0, x) = 1 \) for all \( x \in \Omega \). Hence,

\[
\forall t \geq 0 : \int_0^t F'(\varphi(s, x)) \, ds = \int_0^t \frac{\partial}{\partial x}(F \circ \varphi(s, x))(x) \, ds = \int_0^t \frac{\partial}{\partial x} \varphi(s, x) \, ds.
\]

Therefore, \( \exp(\int_0^t F'(\varphi(s, x)) \, ds) = \partial_x \varphi(t, x) \) for each \( t \geq 0 \). Now, if \( x \in \varphi(r, \Omega) \) with \( r \geq 0 \) it follows as above for \( t \in [0, r] \)

\[
- \int_{-t}^0 F'(\varphi(s, x)) \, ds = \ln \partial_x \varphi(0, x) \quad \text{for each } t \in [0, r]
\]

which is the desired result.

**Corollary 12.** Let \( \Omega \subseteq \mathbb{R} \) be open, \( F \in C^1(\Omega) \) be as usual, \( h \in C(\Omega) \), and let \( \rho \) be \( p \)-admissible for \( F \) and \( h \). Then we have for all \( t \geq 0 \) and \( x \in \Omega \)

\[
\rho_{t,p}(x) = \chi_{\varphi(t, \Omega)}(x) \exp\left(p \int_0^t \text{Re} h(\varphi(s, x)) - \frac{1}{p} F'(\varphi(s, x)) ds\right) \rho(\varphi(-t, x))
\]

\[
= \begin{cases} 
\exp\left(p \int_0^t \text{Re} h(\varphi(s, x)) - \frac{1}{p} F'(\varphi(s, x)) ds\right) \rho(\varphi(-t, x)), & x \in \{F = 0\}, \\
\chi_{\varphi(t, \Omega)}(x) \exp\left(p \int_{\varphi(-t, x)}^{\varphi(t, x)} \frac{\text{Re} h(y) - \frac{1}{p} F'(y)}{F'(y)} \, dy\right) \rho(\varphi(-t, x)), & x \in \{F \neq 0\}
\end{cases}
\]

and

\[
\rho_{-t,p}(x) = \begin{cases} 
\exp\left(-p \int_0^t \text{Re} h(\varphi(s, x)) - \frac{1}{p} F'(\varphi(s, x)) ds\right) \rho(\varphi(t, x)) \\
\exp\left(-p \int_{\varphi(-t, x)}^{\varphi(t, x)} \frac{\text{Re} h(y) - \frac{1}{p} F'(y)}{F'(y)} \, dy\right) \rho(\varphi(t, x)), & x \in \{F \neq 0\}
\end{cases}
\]

**Proof.** While a straightforward calculation gives

\[
\rho_{t,p}(x) = \chi_{\varphi(t, \Omega)}(x) \exp\left(p \int_0^t \text{Re} h(\varphi(s, x)) - \frac{1}{p} F'(\varphi(s, x)) ds\right) \rho(\varphi(-t, x))
\]

and

\[
\rho_{-t,p}(x) = \exp\left(-p \int_0^t \frac{\text{Re} h(\varphi(s, x)) - \frac{1}{p} F'(\varphi(s, x)) ds}{F'(\varphi(s, x))}\rho(\varphi(t, x))
\]

we observe that for \( x \in \{F = 0\} \) we have \( \varphi(t, x) = x \) for each \( t \) so that in \( \{F = 0\} \)

\[
\rho_{t,p}(x) = \exp\left(p \text{Re} h(x) - \frac{1}{p} F'(x)\right) \rho(x)
\]

and

\[
\rho_{-t,p}(x) = \exp\left(-p \text{Re} h(x) + \frac{1}{p} F'(x)\right) \rho(x)
\]
as well as
\[ \rho_{-t,p}(x) = \exp \left( -pt(\Re h(x) - \frac{1}{p}\Phi'(x))\rho(x) \right). \]

For \( x \in \{ F \neq 0 \} \) it is well-known that \( \varphi(t, x) \in \{ F \neq 0 \} \) so that in \( \{ F \neq 0 \} \)
\[ \rho_{-t,p}(x) = \exp \left( -p \int_0^t \frac{\Re h(\varphi(s, x)) - \frac{1}{p}\Phi'(\varphi(s, x))}{F(\varphi(s, x))} \partial_t \varphi(s, x) ds \rho(\varphi(t, x)) \right) \]
and similarly
\[ \rho_{t,p}(x) = \chi_{\varphi(t, \Omega)}(x) \exp \left( p \int_{\varphi(-t, x)}^x \frac{\Re h(y) - \frac{1}{p}\Phi'(y)}{F(y)} dy \right) \rho(\varphi(-t, x)). \]

\[ \square \]

We finish this section by taking a closer look at the case \( \rho = 1 \) for some special cases which are of particular interest in section 4.

**Theorem 13.** Let \( I \subseteq \mathbb{R} \) be an open interval, \( F \in C^1(I) \) as usual with \( F(x) < 0 \) for each \( x \in I \) and such that \( \varphi(t, I) \neq I \) for some \( t > 0 \). Moreover, let \( h \in C(I) \) be such that with \( \alpha := \inf I \) and \( \omega := \sup I \) the function
\[ I \to \mathbb{R}, x \mapsto \frac{\Im h(x)}{F(x)} \]
belongs to \( L^1(\alpha, \beta) \) for every \( \beta \in I \) or to \( L^1(\beta, \omega) \) for every \( \beta \in I \). Furthermore, let \( \rho = 1 \) be \( p \)-admissible for \( F \) and \( h \) for some \( 1 \leq p < \infty \).

a) For the \( C_0 \)-semigroup \( T_{F,h} \) on \( L^p(I) \) the following are equivalent.
   i) \( T_{F,h} \) is hypercyclic.
   ii) \( T_{F,h} \) is weakly mixing.
   iii) There is a sequence \( (\alpha_n)_{n \in \mathbb{N}} \) in \( I \) converging to \( \alpha \) such that for some \( x_0 \in I \)
   \[ \lim_{n \to \infty} \int_{\alpha_n}^{x_0} \frac{\Re h(y) - \frac{1}{p}\Phi'(y)}{F(y)} dy = -\infty. \]

b) For the \( C_0 \)-semigroup \( T_{F,h} \) on \( L^p(I) \) the following are equivalent.
   i) \( T_{F,h} \) is mixing.
   ii) For some \( x_0 \in I \)
   \[ \int_{\alpha}^{x_0} \frac{\Re h(y) - \frac{1}{p}\Phi'(y)}{F(y)} dy = -\infty. \]

c) For the \( C_0 \)-semigroup \( T_{F,h} \) on \( L^p(I) \) the following are equivalent.
   i) \( T_{F,h} \) is chaotic.
   ii) There is some \( x_0 \in I \) such that for every \( x \in I \) there is \( t > 0 \) such that
   \[ \sum_{l=1}^{\infty} \exp \left( p \int_{\varphi(t, x)}^{\varphi(t, x)} \frac{\Re h(y) - \frac{1}{p}\Phi'(y)}{F(y)} dy \right) < 1. \]

**Proof.** From Corollary 12 it follows
\[ \forall t \geq 0, x \in I : \rho_{t,p}(x) = \chi_{\varphi(t, I)}(x) \exp \left( p \int_{\varphi(-t, x)}^x \frac{\Re h(y) - \frac{1}{p}\Phi'(y)}{F(y)} dy \right) \]
and
\[ \forall t \geq 0, x \in I : \rho_{-t,p}(x) = \exp \left( -p \int_x^{\varphi(t, x)} \frac{\Re h(y) - \frac{1}{p}\Phi'(y)}{F(y)} dy \right). \]
As is well-known, for each \( x \in I \) the trajectory \( \{ \varphi(t, x); t \geq 0 \} \) is either an open subinterval of \( I \) or equals \( \{ x \} \). Since the later occurs if and only if \( F(x) = 0 \) it follows from \( F(x) < 0 \) that \( \inf \{ \varphi(t, x); t \geq 0 \} = \alpha \) for every \( x \in I \). Moreover, the assumption \( \varphi(t, I) \neq I \) for some \( t \geq 0 \) implies that for every \( x \in I \) there is \( t_0 > 0 \) such that \( \chi_{\varphi(t, I)}(x) = 0 \) whenever \( t > t_0 \). In particular, for all \( x \in I \) we have \( \rho_{t, p}(x) = 0 \) for sufficiently large \( t \).

**Proof of part a.** It follows from Remark 10 and Theorem 8 that i) and ii) in a) are equivalent and by the preceding observation they hold if and only if for \( \lambda \)-a.e. \( x \in I \) there is a sequence \( (t_n)_{n \in \mathbb{N}} \) in \( (0, \infty) \) tending to infinity such that
\[
0 = \lim_{n \to \infty} \rho_{-t_n, p}(x) = \exp \left( p \int_{\varphi(t_n, x)}^{x} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy \right).
\]
Being \( \varphi(\cdot, x) \) strictly decreasing and \( \alpha = \inf \{ \varphi(t, x); t \geq 0 \} \) for every \( x \in I \), the above relation obviously holds if and only if for some sequence \( (\alpha_n)_{n \in \mathbb{N}} \) in \( I \) converging to \( \alpha \) we have
\[
\lim_{n \to \infty} \int_{\alpha_n}^{\alpha_0} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy = -\infty
\]
for some (and then any) \( x_0 \in I \). Thus a) is proved.

The proofs of parts b) and c) go along the same lines as the one of part a) by applying Theorem 8 b) and Theorem 6 respectively, instead of Theorem 8 a), so that we omit them.

**Remark 14.** i) If under the hypotheses of the above theorem we have \( \varphi(t_0, I) = I \) for some \( t_0 > 0 \) it is easily seen that the same holds for every \( t > 0 \) so that
\[
\rho_{t, p}(x) = \exp \left( p \int_{\varphi(-t, x)}^{x} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy \right).
\]
It follows by the same kind of arguments as in the above proof that \( T_{F, h} \) is hypercyclic on \( L^p(I) \) if and only if \( T_{F, h} \) is weakly mixing if and only if there are sequences \( (\alpha_n)_{n \in \mathbb{N}} \) and \( (\omega_n)_{n \in \mathbb{N}} \) in \( I \) converging to \( \alpha \) and \( \omega \) respectively, such that for some \( x_0 \in I \)
\[
\lim_{n \to \infty} \int_{\alpha_n}^{\alpha_0} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy = -\infty, \quad \lim_{n \to \infty} \int_{x_0}^{\omega_n} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy = \infty.
\]
Mixing of \( T_{F, h} \) then occurs if and only if
\[
\int_{\alpha}^{\omega} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy = -\infty, \quad \int_{x_0}^{\omega} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy = \infty,
\]
while \( T_{F, h} \) is chaotic if and only if there is some \( x_0 \in I \) such that for every \( x \in I \) there is \( t > 0 \) such that
\[
\sum_{l=1}^{\infty} \exp \left( p \int_{\varphi(l t, x)}^{\varphi((l+1) t, x)} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy \right)
+ \sum_{l=1}^{\infty} \exp \left( - p \int_{x_0}^{\varphi(l t, x)} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy \right) < \infty.
\]
ii) Of course it is possible to characterize hypercyclicity, mixing, and chaos of \( T_{F, h} \) in case of \( F \) being strictly positive on \( I \), too. The other hypotheses of Theorem 12 unchanged it follows for example that \( T_{F, h} \) is hypercyclic if and only if there is a sequence \( (\omega_n)_{n \in \mathbb{N}} \) in \( I \) converging to \( \omega = \sup I \) such that for some \( x_0 \in I \)
\[
\lim_{n \to \infty} \int_{x_0}^{\omega_n} \frac{\Re h(y) - \frac{1}{p} F'(y)}{F(y)} dy = \infty.
\]
It should be obvious how the conditions characterizing and mixing and chaos change, respectively, so that we do not state them explicitly.

To conclude this section we give a concrete description of the generator of $T_{F,h}$ in case of $\rho = 1$, at least under some mild additional assumptions on $F$ and $h$.

**Theorem 15.** Let $F, h$ be as above, $1 \leq p < \infty$ and assume that $\rho = 1$ is $p$-admissible for $F$ and $h$. Assume that additionally $h \in L^\infty(\Omega)$ and that $F$ can be extended as a $C^1$-function to $\mathbb{R}$ such that

i) $F, F' \in L^\infty(\mathbb{R})$,

ii) $\Omega$ is positively invariant under the flow $\phi(\cdot, t)$ associated with the problem $\dot{x} = F(x), x(0) = x_0$ (defined on $\mathbb{R} \times \mathbb{R}$ by the assumptions on $F$).

Then the generator $(A, D(A))$ of the $C_0$-semigroup $T_{F,h}$ on $L^p(\Omega)$ is given by

$$D(A) = \{f \in L^p(\Omega); Ff' \in L^p(\Omega)\}$$

and

$$A : D(A) \to L^p(\Omega), Af = Ff' + hf,$$

where $f'$ denotes the distributional derivative of $f$.

**Proof.** Let $D := \{f \in L^p(\Omega); Ff' \in L^p(\Omega)\}$ and

$$B : D \to L^p(\Omega), Bf := Ff'.$$

We first observe that the operator $(B, D)$ is a closed operator. Indeed, let $(u_n)_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \to \infty} u_n = f$ and $\lim_{n \to \infty} Bu_n = g$ in $L^p(\Omega)$. Since $F \in C^1(\Omega)$ we obtain for the distributional derivative of $Fu_n$

$$(Fu_n)' = F'u_n + F'u_n = F'u_n + Bu_n$$

which converges in $L^p(\Omega)$ to $F'f + g$ because $F' \in L^\infty(\Omega)$. It follows $\lim_{n \to \infty} Fu_n = g$ in $L^p(\Omega)$. On the other hand $(Fu_n)'_{n \in \mathbb{N}}$ converges in the sense of distributions to $Ff'$ and thus $Ff = g \in L^p(\Omega)$. Hence $f \in D$ and $Bf = g$, so that $(B, D)$ is a closed operator.

Next we show that

$$D_1 := \{f \in C^1(\Omega) \cap L^p(\Omega); f' \in L^p(\Omega)\},$$

is a core for $(B, D)$. Being $F \in L^\infty(\Omega)$, we have $D_1 \subseteq D$. Let $\psi \in C^\infty_c(\mathbb{R})$ be such that $\psi \geq 0$ and $\int_{\mathbb{R}} \psi(x) dx = 1$ and $\psi_n(x) := n\psi(nx), n \in \mathbb{N}$. In what follows we extend all functions from $L^p(\Omega)$ by 0 to all $\mathbb{R}$. Fix $f \in L^p(\Omega)$. Then $\psi_n * f \in C^\infty_c(\mathbb{R}) \cap L^p(\mathbb{R})$, thus its restriction to $\Omega$ belongs to $C^1(\Omega)$. Moreover, as is well-known $(\psi_n * f)' = \psi_n' * f \in L^p(\mathbb{R})$ and therefore $(\psi_n * f) \in D_1$ with $\lim_{n \to \infty} (\psi_n * f) = f$ in $L^p(\Omega)$. Since we assumed $F$ to be extendable to $\mathbb{R}$ such that $F \in C^1(\mathbb{R})$ with $F, F' \in L^\infty(\mathbb{R})$ it follows from [38, Lemma 1.2.5] (see also [24, pp. 313-315]) that $F(\psi_n * f)' - \psi_n * (Ff') \in L^p(\mathbb{R})$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} F(\psi_n * f)' = Ff'$ in $L^p(\mathbb{R})$ but $\lim_{n \to \infty} F(\psi_n * f) = Ff$ does not hold in $L^p(\mathbb{R})$.

As $Ff' \in L^p(\mathbb{R})$ by $f \in D$ we have

$$F(\psi_n * f)' - Ff' = \left(\psi_n * (Ff') - Ff'\right) + \left(F(\psi_n * f)' - \psi_n * (Ff')\right),$$

so $\lim_{n \to \infty} F(\psi_n * f)' = Ff'$ implying that $(B(\psi_n * f)_{n \in \mathbb{N}})$ converges to $Bf$ in $L^p(\Omega)$. Hence $D_1$ is dense in $D$ equipped with the graph norm of $B$, i.e., $D_1$ is a core of $(B, D)$.

Since we assume $h \in L^\infty(\Omega)$ it follows that $|h_t(x)| \geq e^{-t\|h\|_\infty}$ for all $x \in \Omega$. Since $\rho = 1$ is $p$-admissible for $F$ and $h$ we conclude that

$$\forall x \in \Omega : 1 \leq M e^{(\omega + Ef \|h\|_\infty) t} |\partial_2 \varphi(t, x)|$$

so that $\rho = 1$ is $p$-admissible for $F$ and 0, too. Denote the generator of the $C_0$-semigroup $T_{F,0} = (T_{F,0}(t))_{t \geq 0}$ on $L^p(\Omega)$ by $(A_0, D(A_0))$. Using Lebesgue's
dominated convergence theorem it is straightforward to verify that $D_1 \subseteq D(A_0)$ and $A_0f = Bf$ for all $f \in D_1$.

Next we show that $D_1$ is also a core for $(A_0, D(A_0))$. Indeed, as $C^\infty_0(\Omega) \subseteq D_1$ it follows that $D_1$ is dense in $L^p(\Omega)$. Moreover, it follows immediately from $T_{F,0}(t)f = f(\varphi(t, \cdot))$ that $D_1$ is invariant under $T_{F,0}$ because of the additional hypothesis ii).

Applying [14, Proposition II.1.7] we conclude that $D_1$ is a core for $(A_0, D(A_0))$. As both operators $(B, D)$ and $(A_0, D(A_0))$ are closed and coincide on the common core $D_1$, we obtain $(A_0, D(A_0)) = (B, D)$.

As $h \in L^\infty(\Omega)$, the operator

$$M_h : L^p(\Omega) \to L^p(\Omega), f \mapsto hf$$

is well-defined and continuous. Being a bounded perturbation of $(A_0, D(A_0))$

$$C : D(A_0) \to L^p(\Omega), f \mapsto A_0f + M_hf = Bf + M_h$$

generates a $C_0$-semigroup $S$ on $L^p(\Omega)$ (see e.g. [14, Theorem III.1.3]) and $D_1$ is a core of $(C, D(A_0))$.

Now, let $(A, D(A))$ be the generator of the $C_0$-semigroup $T_{F,h}$. As above for the special case $h = 0$ one shows that $D_1 \subseteq D(A)$ and that $A$ and $C$ coincide on $D_1$.

Moreover, if $\alpha \in \rho(A) \cap \rho(C)$, it follows from $D_1$ being a core for $(C, D(A_0))$ that

$$\bar{(\alpha - A)(D_1)} = (\alpha - C)(D_1) = L^p(\Omega).$$

From [14, Exercise II.1.15 (2)] we derive that $D_1$ is a core for $(A, D(A))$. Since $(A, D(A))$ and $(C, D(A_0))$ are both closed operators coinciding on the common core $D_1$ we finally obtain the assertion.

\[\square\]

**Remark 16.** Conditions on $F$ that ensure that ii) holds can be found in [2], Theorem 16.9 and Corollary 16.10. In particular if $\Omega = (a, b)$, then ii) holds if $F(a) \geq 0$ and $F(b) \leq 0$.

3. **Dynamical properties of $C_0$-semigroups on Sobolev spaces generated by first order differential operators**

In what follows let $I = (a, b)$ be a bounded open interval in $\mathbb{R}$. For $1 \leq p < \infty$ we denote as usual by $W^{1,p}(I)$ the first order Sobolev space of $p$-integrable functions on $I$, i.e.

$$W^{1,p}(I) = \{u \in L^p(I); u' \in L^p(I)\},$$

where $u'$ denotes the distributional derivative of $u$. We equip $W^{1,p}(I)$ with its usual norm turning it into a Banach space. It is well-known that $W^{1,p}(I) \subseteq C[a, b]$ and that for any $x \in [a, b]$ the point evaluation $\delta_x$ in $x$ is a continuous linear form on $W^{1,p}(I)$. We will use the following closed subspace of $W^{1,p}(I)$,

$$W^{1,p}_*(I) := \ker \delta_a.$$

For the boundedness of $I$ we have the topological direct sum

$$W^{1,p}(I) = W^{1,p}_*(I) \oplus \text{span} \{\mathbb{1}\},$$

where $\mathbb{1}$ denotes the constant function with value 1.

**Lemma 17.** Let $I = (a, b)$ be a bounded open interval in $\mathbb{R}$ and $1 \leq p < \infty$. Then

$$\Phi : L^p(I) \to W^{1,p}_*(I), \Phi(f)(x) := \int_a^x f(y) \, dy$$

is a well-defined, linear and continuous bijection with continuous inverse

$$\Phi^{-1} : W^{1,p}_*(I) \to L^p(I), \Phi^{-1}(f) = f'.$$
Proof. A straightforward application of Jensen’s inequality gives that
\[
\int_a^b |\Phi(f)(x)|^p \, dx \leq \int_a^b (x-a)^{p-1} \, dx \int_a^b |f(y)|^p \, dy
\]
(4)
Moreover, as in [7, Lemma 8.2] it follows that the distributional derivative of \( \Phi(f) \) equals \( f \), so that \( \Phi \) is in fact well-defined, obviously linear, and by [7] and \( \Phi(f)' = f \) continuous. Injectivity of \( \Phi \) follows from \( \Phi(f)' = f \). Additionally, from \( \Phi(u) = u \) for every \( u \in W_{1,p}^*(I) \) we obtain the surjectivity of \( \Phi \). Obviously, \( \Phi^{-1}(f) = f' \) for all \( W_{1,p}^*(I) \).

The comparison principle and Lemma [17] imply the next result.

Proposition 18. Let \( I = (a, b) \) be a bounded open interval, \( 1 \leq p < \infty \), and let \( \Phi \) be as in Lemma [17]. Moreover, let \( T \) be a \( C_0 \)-semigroup of \( L^p(I) \). Then \( S := (\Phi \circ T)(t) \circ \Phi^{-1} \) is a \( C_0 \)-semigroup on \( W_{1,p}^*(I) \) which is hypercyclic, (weakly) mixing, or chaotic respectively, if and only if the same holds for \( T \).

Proof. Clearly, \( S \) is a \( C_0 \)-semigroup on \( W_{1,p}^*(I) \) by Lemma [17]. The rest of the assertion follows immediately from the comparison principle.

Now, let \( F : [a, b] \to \mathbb{R} \) be a \( C^1 \)-function on the bounded closed interval \( [a, b] \).
As in section 2 we assume that for every \( x_0 \in I = (a, b) \) the unique solution \( \varphi(\cdot, x_0) \) of the initial value problem
\[
\dot{x} = F(x), \quad x(0) = x_0
\]
is defined on \([0, \infty)\).

Lemma 19. Let \( I = (a, b) \) be a bounded interval and \( F : [a, b] \to \mathbb{R} \) a \( C^1 \)-function as above. Then the function \( \rho = 1 \) is \( p \)-admissible for \( F \) and \( F' \) for every \( 1 \leq p < \infty \), i.e. via
\[
\forall t \geq 0, f \in L^p(I) : (T(t)f)(x) := \exp \left( \int_0^t F'(\varphi(s, x)) \, ds \right) f(\varphi(t, x))
\]
we have a \( C_0 \)-semigroup \( T_{F,F'} \) on \( L^p(I) \).

Proof. Since \( F \) is \( C^1 \) on \([a, b] \) there is \( \omega \in \mathbb{R} \) such that \( F'(x) \leq \omega \) for all \( x \in [a, b] \). Hence, for \( t \geq 0 \) and \( x \in (a, b) \) we have
\[
0 \leq \partial_2 \varphi(t, x) = 1 + \int_0^t \frac{\partial}{\partial t} \partial_2 \varphi(s, x) \, ds = 1 + \int_0^t \frac{\partial}{\partial x} \frac{\partial}{\partial t} \varphi(s, x) \, ds
\]
\[
= 1 + \int_0^t F'(\varphi(s, x)) \partial_2 \varphi(s, x) \, ds \leq 1 + \omega \int_0^t \partial_2 \varphi(s, x) \, ds.
\]
An application of Gronwall’s lemma yields
\[
\forall x \in (a, b), t \geq 0 : \partial_2 \varphi(t, x) \leq e^{\omega t}.
\]
For \( 1 \leq p < \infty \) it follows from the above inequality, the hypothesis, and Proposition [17] that for every \( x \in (a, b) \) we have
\[
\forall t \geq 0 : \left( \exp \left( \int_0^t F'(\varphi(s, x)) \, ds \right) \right)^p = \left( \partial_2 \varphi(t, x) \right)^{p-1} |\partial_2 \varphi(t, x)|
\]
\[
\leq e^{(p-1)\omega t} |\partial_2 \varphi(t, x)|
\]
proving the lemma.
Theorem 20. Let \( I = (a, b) \) be a bounded interval and \( F : [a, b] \to \mathbb{R} \) a \( C^1 \)-function as above with \( F(a) = 0 \). Then for every \( 1 \leq p < \infty \) and \( \gamma \in \mathbb{R} \)
\[
\forall t \geq 0, f \in W^{1,p}(I) : (S(t)f)(x) := e^{\gamma t}\Phi(\varphi(t, x))
\]
defines a \( C_0 \)-semigroup \( S_{F,\gamma} \) on \( W^{1,p}(I) \). \( W^{1,p}(I) \) is \( S_{F,\gamma} \)-invariant. The generator of \( S_{F,\gamma} \) in \( W^{1,p}(I) \) is given by
\[
B : \{ f \in W^{1,p}(I); Ff'' \in L^p(I) \} \to W^{1,p}(I), Bf = Ff' + \gamma f,
\]
while its generator in \( W^{1,p}(I) \) is
\[
B_* : \{ f \in W^{1,p}_*(I); Ff'' \in L^p(I) \} \to W^{1,p}_*(I), B_*f = Ff' + \gamma f.
\]

Proof. By the preceding lemma \( T_{F,F} = (T(t))_{t \geq 0} \) is a well-defined \( C_0 \)-semigroup on \( L^p(I) \) for every \( 1 \leq p < \infty \). As \( F(a) = 0 \) we have \( \varphi(t, a) = a \) for all \( t \geq 0 \) so that for all \( f \in L^p(I) \) and \( t \geq 0 \) with \( \Phi \) as in Lemma 17 and the fact that \( \varphi(t, \cdot) \) is increasing
\[
\Phi(T(t)f)(x) = \int_0^2 f(\varphi(t, y))\varphi(t, y)dy = \int_0^{\varphi(t,x)} f(y)dy = \Phi(f)(\varphi(t, x)),
\]
where we used Proposition 11. Since \( \Phi \) is bijective it follows that \( S_{F,\gamma} := (S(t))_{t \geq 0} = \left(\Phi \circ (e^{\gamma t}T(t)) \circ \Phi^{-1}\right)_{t \geq 0} \) defines a \( C_0 \)-semigroup on \( W^{1,p}_*(I) \). Clearly, for every \( t \geq 0 \) the mapping
\[
S(t) : \text{span} \{ 1 \} \to \text{span} \{ 1 \}, \alpha \mapsto e^{\gamma t}(\alpha 1) \circ \varphi(t, \cdot)
\]
is well-defined, linear and continuous. It follows that \( S_{F,\gamma} \) is a well-defined \( C_0 \)-semigroup on \( W^{1,p}(I) = W^{1,p}_*(I) \oplus \text{span} \{ 1 \} \) such that \( W^{1,p}_*(I) \) is \( S_{F,\gamma} \)-invariant.

The generator of \( S_{F,\gamma}|W^{1,p}_*(I) \) is given by \((\Phi \circ A \circ \Phi^{-1}, \Phi(D(A)))\), where \((A, D(A))\) is the generator of \((e^{\gamma t}T(t))_{t \geq 0} \) in \( L^p(I) \) which by Theorem 15 is
\[
A : \{ f \in L^p(I); Ff' \in L^p(I) \} \to L^p(I), Af = Ff' + F'f + \gamma f.
\]
Therefore
\[
\Phi(D(A)) = \{ f \in W^{1,p}_*(I); Ff'' \in L^p(I) \}
\]
and
\[
\forall f \in \Phi(D(A)) : \Phi \circ A \circ \Phi^{-1}(f) = \Phi(Ff'' + F'f' + \gamma f) = Ff' + \gamma f.
\]
Obviously, \( \text{span} \{ 1 \} \) is contained in the domain of the generator \((B, D(B))\) of \( S_{F,\gamma} \) with \( B1 = \gamma 1 \).

Therefore,
\[
D(B) = \Phi(D(A)) \oplus \text{span} \{ 1 \} = \{ f \in W^{1,p}(I); Ff'' \in L^p(I) \}
\]
and
\[
\forall f \in D(B) : Bf = \Phi \circ A \circ \Phi^{-1}(f - f(a)1) + \gamma f(a)1 = Ff' + \gamma f.
\]

Theorem 21. Let \( I = (a, b) \) be a bounded interval, \( 1 \leq p < \infty \), \( \gamma \in \mathbb{R} \), and \( F : [a, b] \to \mathbb{R} \) a \( C^1 \)-function as above with \( F(a) = 0 \). Then the following holds.

a) The \( C_0 \)-semigroup \( S_{F,\gamma} \) is not hypercyclic on \( W^{1,p}(I) \).

b) On \( W^{1,p} \) the following are equivalent for the \( C_0 \)-semigroup \( S_{F,\gamma} \).

i) \( S_{F,\gamma} \) is weakly mixing on \( W^{1,p}(I) \),

ii) \( S_{F,\gamma} \) is hypercyclic on \( W^{1,p}(I) \),

\( \square \)
iii) \( \lambda(\{F = 0\}) = 0 \) and for every \( m \in \mathbb{N} \) for which there are \( m \) different connected components \( C_1, \ldots, C_m \) of \( \varGamma \setminus \{F = 0\} \), for \( \lambda^m \)-almost all choices of \( (x_1, \ldots, x_m) \in \prod_{j=1}^m C_j \) there is a sequence of positive numbers \( (t_n)_{n \in \mathbb{N}} \) tending to infinity such that for every \( 1 \leq j \leq m \)

\[
\lim_{n \to \infty} \chi_{\varphi(t_n, I)}(x_j) e^{\varphi_{t_n}} \partial_2 \varphi(-t_n, x_j)^{1-p} = 0
\]

and

\[
\lim_{n \to \infty} e^{-\varphi_{t_n}} \partial_2 \varphi(t_n, x_j)^{1+p} \frac{\partial_2 \varphi(t_n, x_j)^p}{\partial_2 \varphi(2t_n, x_j)^p} = 0.
\]

c) On \( W^1_p(\varGamma) \) the following are equivalent for the \( C_0 \)-semigroup \( S_{F, \gamma} \).

i) \( S_{F, \gamma} \) is mixing on \( W^1_p(\varGamma) \),

ii) \( \lambda(\{F = 0\}) = 0 \) and for \( \lambda \)-almost every \( x \in \varGamma \)

\[
\lim_{t \to \infty} \chi_{\varphi(t, I)}(x) e^{\varphi t} \partial_2 \varphi(-t, x)^{1-p} = \lim_{t \to \infty} e^{-\varphi_t} \partial_2 \varphi(t, x)^{1+p} = 0.
\]

d) On \( W^1_p(\varGamma) \) the following are equivalent for the \( C_0 \)-semigroup \( S_{F, \gamma} \).

i) \( S_{F, \gamma} \) is chaotic on \( W^1_p(\varGamma) \),

ii) \( \lambda(\{F = 0\}) = 0 \) and for every \( m \in \mathbb{N} \) for which there are \( m \) different connected components \( C_1, \ldots, C_m \) of \( \varGamma \setminus \{F = 0\} \), for \( \lambda^m \)-almost all choices of \( (x_1, \ldots, x_m) \in \prod_{j=1}^m C_j \) there is \( t > 0 \) such that

\[
\sum_{l=1}^\infty \chi_{\varphi(lt, I)}(x) e^{\varphi_{lt}} \partial_2 \varphi(-lt, x)^{1-p} + \sum_{l=1}^\infty e^{-\varphi_{lt}} \partial_2 \varphi(lt, x)^{1+p} < \infty
\]

for every \( 1 \leq j \leq m \).

**Proof.** Let \( P_1 : W^1_p(\varGamma) \to \text{span} \{\mathbb{I}\} \), \( P_1(f) = f(a) \mathbb{I} \). As \( F(a) = 0 \) we have \( \varphi(t, a) = a \) for all \( t \geq 0 \) which implies \( P_1 \circ S(t) = S(t) \circ P_1 \). Since there are no hypercyclic \( C_0 \)-semigroups on finite dimensional spaces (for the single operator analogue of this, see e.g. [15 Corollary 2.59]) part \( a) \) follows from the comparison principle.

Now, let \( \Phi \) be as in Lemma [17]. Since \( (\Phi^{-1} \circ S(t) \circ \Phi)_{t \geq 0} = T_{F, F^r, \gamma} \) and \( \Phi^{-1}(W^1_p(\varGamma)) = L^p(\varGamma) \) it follows from the comparison principle and Theorem [8] \( a) \) for \( p = 1 \) that i) and ii) in b) are equivalent and that these are equivalent to hypercyclicity of \( T_{F, F^r, \gamma} \) on \( L^p(\varGamma) \). For \( h(x) = F'(x) + \gamma \) and \( \rho(x) = 1 \) it follows that for \( \rho_{t,p} \) and \( \rho_{-t,p} \) from definition \( 2 \) we have

\[
\forall t \geq 0, x \in (a, b) : \rho_{t,p}(x) = \chi_{\varphi(t, I)}(x) h^p_t(\varphi(-t, x)) \partial_2 \varphi(-t, x)
\]

as well as

\[
\forall t \geq 0, x \in (a, b) : \rho_{-t,p}(x) = h^p_t(x) \partial_2 \varphi(t, x).
\]

Observe that for \( h(x) = F'(x) + \gamma \) we have by Proposition [11]

\[
\forall t \geq 0, x \in (a, b) : h_t(x) = \exp(\gamma t + \int_0^t F'(\varphi(s, x)) \, ds) = e^{\gamma t} \partial_2 \varphi(t, x).
\]

Moreover, because \( \varphi(s + t, x) = \varphi(s, \varphi(t, x)) \) for all \( s, t \in \mathbb{R} \) and each \( x \in (a, b) \) for which the involved quantities are defined it follows

\[
\forall t \geq 0, x \in (a, b) : \partial_2 \varphi(2t, x) = \partial_2 \varphi(t, \varphi(t, x)) \partial_2 \varphi(t, x)
\]

and thus for every \( x \in (a, b) \) we have

\[
\forall t \geq 0 : \partial_2 \varphi(t, \varphi(t, x)) = \frac{\partial_2 \varphi(2t, x)}{\partial_2 \varphi(t, x)}
\]

as well as

\[
\forall t \geq 0 : 1 = \partial_2 \varphi(t, \varphi(-t, x)) \partial_2 \varphi(-t, x)
\]
for every $x \in \varphi(t, I)$. Taking all this into account it follows

$$\forall t \geq 0, x \in (a, b) : \rho_{t,p}(x) = \chi_{\varphi(t, I)}(x) e^{\gamma t} (\partial_2 \varphi(t, \varphi(-t, x)))^p \partial_2 \varphi(-t, x)$$

as well as

$$\forall t \geq 0, x \in (a, b) : \rho_{-t,p}(x) = e^{-\gamma t} (\partial_2 \varphi(t, \varphi(t, x)))^{-p} \partial_2 \varphi(t, x)$$

By Theorem 8 (a) $T_{F,F'+\gamma}$ is hypercyclic on $L^p(I)$ if and only if $\lambda(\{F = 0\}) = 0$ and

for every $m \in \mathbb{N}$ for which there are $m$ different connected components $C_1, \ldots, C_m$

of $I \setminus \{F = 0\}$, for $\lambda^m$-almost all choices of $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ there is a sequence

of positive numbers $(t_n)_{n \in \mathbb{N}}$ tending to infinity such that

$$\forall 1 \leq j \leq m : \lim_{n \to \infty} \rho_{t_n,p}(x_j) = \lim_{n \to \infty} \rho_{-t_n,p}(x_j) = 0,$$

so that by the above considerations b) follows.

The proofs of part c) and d) of the theorem follow by exactly the same arguments by referring to Theorem 8 (b) and Theorem 5 respectively. \(\square\)

So far, we have only considered $\gamma$ to be a real constant. If $h \in W^{1,\infty}(I)$ we would like to have similar results to the above for the $C_0$-semigroup on $W^{1,p}(I)$ generated by

$$f \mapsto Ff' + hf.$$ 

Since $h \in W^{1,\infty}(I)$ it follows that the corresponding multiplication operator

$$M_h : W^{1,p}(I) \to W^{1,p}(I), M_h(f) = hf$$

is well-defined and continuous. If we denote the generator of $S_{F,0}$ in $W^{1,p}(I)$ by $(A, D(A))$ it follows that $(A + M_h, D(A))$ generates a $C_0$-semigroup $S_{F,h}$. By a well-known perturbation result (see e.g. [13] Theorem III.1.10]) this semigroup is given by

$$S(t)f = \sum_{n=0}^{\infty} T_n(t)f,$$

where $T_0(t) = S_{F,0}(t)$ and

$$T_{n+1}(t)f = \int_0^t S_{F,0}(t-s)M_h T_n(s)f ds$$

$$= \int_0^t h(\varphi(t-s, \cdot))(T_n(s)f)(\varphi(t-s, \cdot)) ds,$$

where the integrals are Riemann integrals in $W^{1,p}(I)$. In order to get an explicit form of $S_{F,h}$ we use the following well-known elementary observation.

**Proposition 22.** Let $v : [0, \infty) \to \mathbb{C}$ be continuous. Then

$$\forall n \in \mathbb{N}_0, t \geq 0 : \int_0^t v(t-s) \left( \int_0^s v(t-s + r) dr \right)^n ds = \frac{1}{n+1} \left( \int_0^t v(s) ds \right)^{n+1}.$$
Proposition 23. Let \( h \) be a measurable function on a bounded interval, at least under some mild additional hypothesis. As in section 2 we want to find \( A : \{ f \in W^{1,p}(I) : Ff'' \in L^p(I) \}, f \mapsto Ff' + hf \) generates the \( C_0 \)-semigroup \( S_{F,h} \) on \( W^{1,p}(I) \) which is given by

\[
S_{F,h}(t) = \exp(t Fh) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left( \int_0^t h(s) ds \right)^n.
\]

Using Proposition 22 for \( t \mapsto h(\varphi(t,x)) \) with fixed \( x \in [a, b] \) and the fact that point evaluations in \( W^{1,p}(I) \) are continuous it follows by induction on \( n \) that the above operators \( T_n(t) \) in equation (9) are given by

\[
T_n(t)f(x) = \frac{1}{n!} \left( \int_0^t h(\varphi(s,x)) ds \right)^n S_{F,0}(t)f(x)
\]

which in turn implies the expected expression for the \( C_0 \)-semigroup \( S_{F,h} \)

\[
\forall t \geq 0, f \in W^{1,p}(I) : S_{F,h}(t)f(x) = S(t)f(x) = h_t(x)f(\varphi(t,x)),
\]

where again \( h_t(x) = \exp(\int_0^t h(\varphi(s,x)) ds) \). Thus we obtain the next proposition.

Proposition 24. Let \( I = (a, b) \) be a bounded interval, \( 1 \leq p < \infty, F : [a, b] \to \mathbb{R} \) a \( C^1 \)-function as above with \( F(a) = 0 \) and \( h \in W^{1,\infty}(I) \). Then

\[
A : \{ f \in W^{1,p}(I) : Ff'' \in L^p(I) \}, f \mapsto Ff' + hf
\]
generates the \( C_0 \)-semigroup \( S_{F,h} \) on \( W^{1,p}(I) \) which is given by

\[
\forall t \geq 0 : S_{F,h}(t)f(x) = S(t)f(x) = h_t(x)f(\varphi(t,x)).
\]

Moreover, \( W^{1,p}(I) \) is \( S_{F,h} \)-invariant.

Our next aim is to generalize the content of Theorem 21 to the \( C_0 \)-semigroups \( S_{F,h} \), at least under some mild additional hypothesis. As in section 2 we want to find a measurable function \( \psi : [a, b] \to \mathbb{R} \) with \( \exp \circ \psi \) inducing a continuous, invertible multiplication operator \( M \) on \( W^{1,p}(I) \) such that for some \( \gamma \in \mathbb{R} \) the \( C_0 \)-semigroups \( S_{F,\gamma} \) and \( S_{F,h} \) are conjugate via \( M \), i.e.

\[
\forall t \geq 0, f \in W^{1,p}(I) : \exp(\psi) S_{F,h}(t)(f) = S_{F,\gamma}(t)(\exp(\psi)f).
\]

This is satisfied, if for all \( x \in [a, b] \)

\[
\forall t \geq 0 : \psi(\varphi(t,x)) - \int_0^t (h(\varphi(s,x)) - \gamma) ds = \psi(x).
\]

Now, if \( x \in \{ F = 0 \} \) it follows from \( \varphi(t,x) = x \) for all \( t \geq 0 \)

\[
\forall t \geq 0 : \psi(x) - t(h(x) - \gamma) = \psi(x).
\]

Thus, since \( F(a) = 0 \) it is necessary that \( h(x) = h(a) \) for every \( x \in \{ F = 0 \} \).

As in section 2, if the function \( [a, b] \to \mathbb{R}, y \mapsto \frac{h(y) - h(a)}{F(y)} \) belongs to \( L^1(a, b) \) we can set

\[
\psi : [a, b] \to \mathbb{R}, \psi(x) = \int_a^x \frac{h(y) - h(a)}{F(y)} dy.
\]
Proof. Induce a bounded multiplication operator on \( W^L\) and show that this function actually has to belong to \( \psi \). Let \( S \) be a well-defined continuous multiplication operator on \( W^L \) invertible multiplication operator \( \exp(\psi) \) so that the restrictions of the \( \psi \)-functions as above with \( F, h \) are conjugate via \( S \). The function \( \exp(\psi) \) is weakly mixing on \( W^1_p(I) \) and therefore \( \exp(\psi) \in W^1_\infty(I) \), too. Hence, \( M : W^1_p(I) \to W^1_p(I), Mf = \exp(\psi)f \) is a well-defined continuous multiplication operator on \( W^1_p(I) \) with continuous inverse given by \( \exp(\psi)^{-1}f \).

Remark 25. In the discussion preceding the above proposition we only required that \( [a, b] \to \mathbb{R}, y \mapsto \frac{h(y) - h(a)}{F(y)} \) belongs to \( L^1(a, b) \). Nevertheless, it is not hard to show that this function actually has to belong to \( L^\infty(a, b) \) in order for \( \exp(\psi) \) to induce a bounded multiplication operator on \( W^1_p(I) \).

Obviously, the multiplication operator \( M \) from the above proposition satisfies \( M(W^1_p(I)) = W^1_p(I) \) so that the restrictions of the \( \psi \)-semigroups \( S_{F,h}(a) \) and \( S_{F,h} \) to \( W^1_p(I) \) are conjugate via \( M \). Combining Theorem 21 with Proposition 24 the next theorem follows directly from the comparison principle.

Theorem 26. Let \( I = (a, b) \) be a bounded interval, \( 1 \leq p < \infty, F : [a, b] \to \mathbb{R} \) a \( C^1 \)-function as above with \( F(a) = 0 \) and \( h \in W^1_\infty(I) \). Assume that

1) \( \forall x \in \{ F = 0 \} : h(x) = h(a) \in \mathbb{R} \).

2) The function \( [a, b] \to \mathbb{R}, y \mapsto \frac{h(y) - h(a)}{F(y)} \) belongs to \( L^\infty(I) \).

Then the following holds.

a) The \( \psi \)-semigroup \( S_{F,h} \) is not hypercyclic on \( W^1_p(I) \).

b) On \( W^1_p(I) \) the following are equivalent for the \( \psi \)-semigroup \( S_{F,h} \).

i) \( S_{F,h} \) is weakly mixing on \( W^1_p(I) \),

ii) \( S_{F,h} \) is hypercyclic on \( W^1_p(I) \),

iii) \( \lambda(\{ F = 0 \}) = 0 \) and for every \( m \in \mathbb{N} \) for which there are \( m \) different connected components \( C_1, \ldots, C_m \) of \( I \setminus \{ F = 0 \} \), for \( m \)-almost all choices of \( (x_1, \ldots, x_m) \in \prod_{j=1}^m C_j \) there is a sequence of positive numbers \( (t_n)_{n \in \mathbb{N}} \) tending to infinity such that for every \( 1 \leq j \leq m \)

\[
\lim_{n \to \infty} \chi_{\varphi(t_n, x_j)}(x) e^{ph(a)t_n} \frac{\partial^p \varphi(-t_n, x_j)}{\partial^p \varphi(2t_n, x_j)} = 0
\]

and

\[
\lim_{n \to \infty} e^{-ph(a)t_n} \frac{\partial^p \varphi(t_n, x_j)}{\partial^p \varphi(2t_n, x_j)} = 0.
\]

c) On \( W^1_p(I) \) the following are equivalent for the \( \psi \)-semigroup \( S_{F,h} \).

i) \( S_{F,h} \) is mixing on \( W^*_p(I) \),

ii) \( \lambda(\{ F = 0 \}) = 0 \) and for \( \lambda \)-almost every \( x \in I \)

\[
\lim_{t \to \infty} \chi_{\varphi(t, I)}(x) e^{ph(a)t} \frac{\partial^p \varphi(-t, x)}{\partial^p \varphi(2t, x)} = 0.
\]
Theorem 27.

a) Let \( h \) be in \( C(0,1) \cap L^\infty(0,1) \). Then the following properties of the associated von Foerster-Lasota semigroup \( T_h \) on \( L^p(0,1) \) are equivalent.

i) \( T_h \) is hypercyclic.

ii) \( T_h \) is weakly mixing.

iii) There is a sequence \( (\alpha_n)_{n \in \mathbb{N}} \) in \( (0,1) \) converging to zero such that for some \( x_0 \in (0,1) \)

\[
\lim_{n \to \infty} \int_{\alpha_n}^{x_0} \frac{\mathrm{Re} h(y) + \frac{1}{2}}{y^2} \, dy = \infty.
\]

b) Assume that for \( h \in C[0,1] \) the function

\[
[0,1] \to \mathbb{R}, \, x \mapsto \frac{h(x) - \text{Re} \, h(0)}{x}
\]

belongs to \( L^1(0,1) \). Then the following properties of the associated von Foerster-Lasota semigroup \( T_h \) on \( L^p(0,1) \) are equivalent.

i) \( T_h \) is hypercyclic.
ii) $T_h$ is weakly mixing.
iii) $T_h$ is mixing.
iv) $T_h$ is chaotic.
v) $\Re h(0) > -\frac{1}{2}$.

c) Assume that for $h \in W^{1,\infty}(0,1)$ the function

$$[0,1] \to \mathbb{R}, x \mapsto \frac{h(x) - h(0)}{x}$$

belongs to $L^\infty(0,1)$ and that $h(0) \in \mathbb{R}$. Then the von Foerster-Lasota semigroup $S_h$ is not hypercyclic on $W^{1,p}(0,1)$. For the restriction of $S_h$ to $W^{1,p}(0,1)$ the following are equivalent.
i) $S_h$ is hypercyclic on $W^{1,p}(0,1)$.
ii) $S_h$ is weakly mixing on $W^{1,p}(0,1)$.
iii) $S_h$ is mixing on $W^{1,p}(0,1)$.
iv) $S_h$ is chaotic on $W^{1,p}(0,1)$.
v) $h(0) > 1 - \frac{1}{p}$.

Proof. For $F(x) = -x$ we have $\varphi(t, x) = xe^{-t}$ and thus the hypotheses of Theorem 13 are satisfied.

Proof of part a). Since $(0,1) \to \mathbb{R}, x \mapsto \frac{\Im h(x)}{x}$ belongs to $L^1(\beta,1)$ for each $\beta \in (0,1)$ part a) is obviously a direct application of Theorem 13 a).

Proof of part b). Since the hypotheses of a) are satisfied, it follows that i) and ii) are equivalent and that because of $h \in C[0,1]$ i) holds if and only if

$$\lim_{n \to \infty} \int_{\alpha_n}^{1} \frac{\Re h(y) + \frac{1}{p}}{y} dy = \infty$$

for some $(\alpha_n)_{n \in \mathbb{N}}$ in $(0,1)$ converging to 0. Since

$$\int_{\alpha_n}^{1} \frac{\Re h(y) + \frac{1}{p}}{y} dy = \int_{\alpha_n}^{1} \frac{\Re h(0) + \frac{1}{p}}{y} dy + \int_{\alpha_n}^{1} \frac{\Re (h(y) - h(0))}{y} dy = -\left(\Re h(0) + \frac{1}{p}\right) \ln(\alpha_n) + \int_{\alpha_n}^{1} \frac{\Re (h(y) - h(0))}{y} dy$$

it follows from the hypothesis on $h$ that (7) holds if and only if $\Re h(0) + \frac{1}{p} > 0$. Hence, i), ii), and v) in b) are equivalent. As iii) and iv) imply i), respectively, it remains to prove that v) implies iii) and iv).

Let us denote by $c$ the $L^1(0,1)$-norm of the $L^1$-function $x \mapsto \frac{h(x) - \Re h(0)}{x}$. If v) holds, it follows that for each $x \in (0,1), t > 0$, and $l \in \mathbb{N}$ we have

$$\exp \left( p \int_{x}^{xe^{-it}} \frac{\Re h(y) + \frac{1}{p}}{y} dy \right) = \exp \left( -p \int_{xe^{-it}}^{x} \frac{\Re h(0) + \frac{1}{p}}{y} dy \right) \cdot \exp \left( -p \int_{xe^{-it}}^{x} \frac{\Re (h(y) - h(0))}{y} dy \right) \leq \left( \exp(-p(\Re h(0) + \frac{1}{p})) \right)^{l} e^{pc}.$$  

Because of v) it follows that for every $x \in (0,1)$ and $t > 0$

$$\sum_{l=1}^{\infty} \exp \left( p \int_{x}^{xe^{-it}} \frac{\Re h(y) + \frac{1}{p}}{y} dy \right) < \infty,$$

hence $T_h$ is chaotic by part c) of Theorem 13.
In order to show that v) implies iii) we observe that as in inequality (8) we obtain for each \( x \in (0,1) \) and \( t \geq 0 \)
\[
\exp \left( \int_0^x \frac{\Re h(y) + \frac{1}{y}}{2} dy \right) \leq \exp(-pt(\Re h(0) + \frac{1}{p}))e^{pc},
\]
so that by part b) of Theorem 13 iii) holds. Hence, b) is proved.

Proof of part c). That \( S_h \) is not hypercyclic on \( W^{1,p}(0,1) \) follows immediately from Theorem 26. As \([0,1]\setminus \{F = 0\}\) has only one connected component, \( \chi_{\varphi(t,1)}(x) = 0 \) for every \( x \in (0,1) \) for sufficiently large \( t > 0 \), and
\[
\forall x \in (0,1) : e^{-ph(0)t} \frac{\partial^2 \varphi(t,x)}{\partial^2 t} = e^{-t(1+(h(0)-1)p)}
\]
the rest of c) now follows from Theorem 26(b),c), and d). \( \square \)

Remark 28. i) Of course Theorem 13 also provides characterizations of mixing and chaos for \( T_h \) under the general hypotheses of part a) of the above theorem.
ii) It should be noted that the proof of part b) remains valid if we replace the hypothesis \( h \in C[0,1] \) by \( h \in C(0,1) \cap L^{\infty}(0,1) \), \( h \) continuously extendable into the origin.
iii) The hypothesis in c) is obviously satisfied if \( h(0) \in \mathbb{R} \) and \( h \) is differentiable at the origin.

Remark 29. In [8] the authors prove that, if \( h \) is a continuous real function on \([0,1] \) such that \( h(x) - h(0) \in L^{\infty}(0,1) \), then the restriction of \( T_h \) is a \( C_0 \)-semigroup on the space
\[
V_\alpha = \{ h \in h^\alpha([0,1]) \mid h(0) = 0 \},
\]
where \( h^\alpha([0,1]) \) is the little Hölder space of order \( \alpha \in (0,1] \), that is the closure of \( C^1([0,1]) \) in the Hölder space \( C^{\alpha}[0,1] \). Observe that \( h^\alpha \) is a separable space (see [21]) and since for \( p = (1-\alpha)^{-1} \)
\[
W^{1,p} \hookrightarrow C^\alpha[0,1]
\]
continuously, it is immediate to show that \( W^{1,p}_r(0,1) \) is continuously embedded with dense range in \( V_\alpha \).

As a consequence, by Theorem 27 if \( h(0) > \alpha = 1-1/p \), then the \( C_0 \)-semigroup \( T_h \) is mixing and chaotic on \( V_\alpha \). Thus we obtain the result of [11] Theorem 2.4 and [8] Theorem 3.10 and Corollary 3.11.

Example 2. Let us consider for \( h \in C(0,1) \cap L^{\infty}(0,1) \) and \( r > 1 \) the first order partial differential equation
\[
\frac{\partial}{\partial t} u(t,x) + x^r \frac{\partial}{\partial x} u(t,x) = h(x) u(t,x), \quad t \geq 0, \quad 0 < x < 1
\]
with the initial condition
\[
u(0,x) = v(x), \quad 0 < x < 1,
\]
where \( v \) is a given function. As for \( F(x) = -x^r \) we have
\[
\varphi(t,x) = \frac{x^{\frac{1}{1-r}}}{(1 + (r - 1)xt)^{\frac{1}{1-r}}}
\]
and therefore \( \varphi(0,\infty);(0,1) \subset (0,1) \), it is again natural to consider our results from section 2 for \( I = (0,1) \) and \( h \) and the \( C_0 \)-semigroup \( T_{F,h} \) on \( L^p(0,1) \). We write \( T_{r,h} \) instead of \( T_{F,h} \).
Again, if \( h \in W^{1,\infty}(0,1) \), in view of Proposition 23 it is natural to apply our results from section 3. The corresponding \( C_0 \)-semigroup on \( W^{1,p}(0,1) \), respectively \( W_s^{1,p}(0,1) \) is denoted by \( S_{r,h} \). The next theorem summarizes the dynamical properties of these semigroups. Observe that contrary to the von Foerster-Lasota dynamical properties of \( S_{r,h} \) are independent of \( p! \)

**Theorem 30.**

a) Let \( h \) belong to \( C(0,1) \cap L^\infty(0,1) \). Then the following properties of the \( C_0 \)-semigroup \( T_{r,h} \) on \( L^p(0,1) \) are equivalent.
   
   i) \( T_{r,h} \) is hypercyclic.
   
   ii) \( T_{r,h} \) is weakly mixing.
   
   iii) There is a sequence \((r_n)_{n \in \mathbb{N}} \) in \((0,1)\) converging to zero such that for some \( x_0 \in (0,1) \)
   
   \[ \lim_{n \to \infty} \int_{r_n}^{x_0} \frac{\Re h(y) + \frac{r}{y} y^{r-1}}{y^r} \, dy = \infty. \]

b) Assume that for \( h \in C[0,1] \) the function

\[ [0,1] \to \mathbb{R}, x \mapsto \frac{h(x) - x^{r-1}\Re h(0)}{x^r} \]

belongs to \( L^1(0,1) \). Then the following properties of \( T_{r,h} \) on \( L^p(0,1) \) are equivalent.

i) \( T_{r,h} \) is hypercyclic.

ii) \( T_{r,h} \) is weakly mixing.

iii) \( T_{r,h} \) is mixing.

iv) \( T_{r,h} \) is chaotic.

v) \( \Re h(0) > \frac{r}{r-1} \).

c) Assume that for \( h \in W^{1,\infty}(0,1) \) the function

\[ [0,1] \to \mathbb{R}, x \mapsto \frac{h(x) - h(0)}{x^r} \]

belongs to \( L^\infty(0,1) \) and that \( h(0) \in \mathbb{R} \). Then the \( C_0 \)-semigroup \( S_{r,h} \) is not hypercyclic on \( W^{1,p}(0,1) \). For the restriction of \( S_{r,h} \) to \( W_s^{1,p}(0,1) \) the following are equivalent.

i) \( S_{r,h} \) is hypercyclic on \( W_s^{1,p}(0,1) \).

ii) \( S_{r,h} \) is weakly mixing on \( W_s^{1,p}(0,1) \).

iii) \( S_{r,h} \) is mixing on \( W_s^{1,p}(0,1) \).

iv) \( S_{r,h} \) is chaotic on \( W_s^{1,p}(0,1) \).

v) \( h(0) > 0 \).

**Proof.** For \( F(x) = -x^r \) we have \( \varphi(t,x) = x^{\frac{1}{1+r}} (1 + (r-1)tx)^{\frac{1}{r-1}} \) and thus the hypotheses of Theorem 13 are satisfied. The proofs of parts a) and b) are mutatis mutandis the same as the correspondig parts of Theorem 27. In order to prove part c) we observe

\[ \forall t \geq 0, x \in (0,1]: \partial_2 \varphi(t,x) = \frac{1}{r-1} \left( \frac{x^{\frac{1}{r}}}{1 + (r-1)tx} \right)^{\frac{1}{r-1}} \]

and thus

\[ \forall t \geq 0, x \in (0,1]: \exp(-ph(0)t) \frac{\partial_2 \varphi(t,x)^{1+p}}{\partial_2 \varphi(2t,x)^p} = \exp(-ph(0)t) t^{-\frac{r}{1+r}} x^{\frac{1}{1+r}-1} \left( \frac{1}{r} + 2(r-1)tx^{\frac{1}{r-1}} \right)^{\frac{1}{r-1}} \frac{1}{(1 + (r-1)tx^{\frac{1}{r-1}})^{(1+p)}^{\frac{1}{r}}} \]

Now the proof follows again as the proof of part c) of Theorem 27. \( \square \)
References


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