CHAOTIC C_0-SEMIGROUPS INDUCED BY SEMIFLOWS IN LEBESGUE AND SOBOLEV SPACES

JAVIER AROZA, THOMAS KALMES, AND ELISABETTA MANGINO

Abstract. We give characterizations of chaos for C_0-semigroups induced by semiflows on $L^p_\rho(\Omega)$ for open $\Omega \subseteq \mathbb{R}$ similar to the characterizations of hypercyclicity and mixing of such C_0-semigroups proved in [18]. Moreover, we characterize hypercyclicity, mixing, and chaos for these classes of C_0-semigroups on $W^{1,p}_*(I)$ for a bounded interval $I \subset \mathbb{R}$ and prove that these C_0-semigroups are never hypercyclic on $W^{1,p}(I)$. We apply our results to concrete first order partial differential equations, such as the von Foerster-Lasota equation.

1. Introduction

The aim of this article is to characterize certain dynamical properties of C_0-semigroups induced by weighted semiflows on different function spaces. Recall that a C_0-semigroup T on a separable Banach space X is called *hypercyclic* if there is $x \in X$ with its orbit $\{T(t)x;\, t \geq 0\}$ under T being dense in X. T is called *chaotic* if T is hypercyclic and the set of periodic points, i.e. $\{x \in X;\, \exists t > 0 : T(t)x = x\}$, is dense in X. It is well-known, that a C_0-semigroup T on a separable Banach space X is hypercyclic if and only if T is *(topologically) transitive*, i.e. for any pair of non-empty, open subsets U, V of X there is $t > 0$ such that $T(t)(U) \cap V \neq \emptyset$. If for any pair of non-empty, open subsets U, V of X there is $t_0 > 0$ such that $T(t)(U) \cap V \neq \emptyset$ whenever $t \geq t_0$ then T is called *mixing*, while T is *weakly mixing* if the direct sum C_0-semigroup $T \oplus T$ is transitive on $X \oplus X$.

The study of chaotic properties for C_0-semigroups has attracted the attention of many researchers. We refer the reader to Chapter 7 of the monograph by Grosse-Erdmann and Peris [15] and the references therein. Some recent papers in the topic are [1, 3, 4, 5, 13, 16, 22, 25]

For $\Omega \subseteq \mathbb{R}$ open and a Borel measure μ over Ω admitting a strictly positive Lebesgue density ρ we consider C_0-semigroups T on $L^p(\Omega, \mu), 1 \leq p < \infty$, of the form

$$T(t)f(x) = h_t(x)f(\varphi(t, x)),$$

where φ is the solution semiflow of an ordinary differential equation

$$\dot{x} = F(x)$$

in Ω and

$$h_t(x) = \exp\Big(\int_0^t h(\varphi(s, x))ds\Big)$$

with $h \in C(\Omega)$. Such C_0-semigroups appear in a natural way when dealing with initial value problems for linear first order partial differential operators. While characterizations of hypercyclicity, (weak) mixing, and chaos of such C_0-semigroups where obtained for open $\Omega \subseteq \mathbb{R}^d$ for arbitrary dimension d in [17], evaluation of these

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conditions in concrete examples is sometimes rather involved. In contrast to general dimension the case $d = 1$ allows for significantly simplified characterizations. In [18] these were given for hypercyclicity and mixing. In section 2, we give a simplified characterization of chaos for such C_0-semigroups. Moreover, we further evaluate and extend the conditions obtained in [18].

In section 3 we investigate the above kind of C_0-semigroups on the Sobolev spaces $W^{1,p}(a,b)$, where $(a,b) \subseteq \mathbb{R}$ is a bounded interval. In case of $F(a) = 0$ we show that hypercyclicity never occurs while we characterize hypercyclicity, (weak) mixing, mixing, and chaos on the closed subspace $W^{1,p}_*(a,b) = \{f \in W^{1,p}(a,b);\ f(a) = 0\}$.

In order to illustrate our results, we consider concrete initial value problems for first order partial differential operators in section 4. Among them, we consider the linear von Foerster- Lasota equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = h(x)u(x) \quad t \geq 0, x \in [0,1] \tag{1}$$

with initial condition

$$u(x,0) = v(x) \quad x \in [0,1],$$

where v belongs to a suitable function space.

(1) is a particular case of the equation

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = f(x,u) \qquad t \geq 0,\ x \in [0,1] \tag{2}$$

that was introduced in [20] to describe the reproduction of a population of red blood cells, mainly in connection with studies about anemia. After the paper [19], this problem has already been studied in different function spaces by several authors either with an ergodic theoretical approach (see [25] and the references quoted therein) or by explicitly constructing hypercyclic and periodic solutions (see [11, 8, 10], or by investigating spectral properties of the differential operator associated to the equation (1) (see [27, 26] and the applications in [9]). In the present paper the results will follow from the characterizations proved in section 3.

2. Characterizations of dynamical properties of C_0-semigroups on $L^p(\Omega,\mu)$ induced by weighted semiflows

Let $\Omega \subseteq \mathbb{R}$ be an open set and $F : \Omega \to \mathbb{R}$ be a C^1-function. Hence, for every $x_0 \in \Omega$ there is a unique solution $\varphi(\cdot, x_0)$ of the initial value problem

$$\dot{x} = F(x),\ x(0) = x_0.$$

Denoting its maximal domain of definition by $J(x_0)$ it is well-known that $J(x_0)$ is an open interval containing 0. Throughout this section we make the general assumption that $[0,\infty) \subset J(x_0)$ for every $x_0 \in \Omega$, i.e. $\varphi : [0,\infty) \to \Omega$. From the uniqueness of the solution it follows that $\varphi(t,\cdot)$ is injective for every $t \geq 0$ and $\varphi(t+s,x) = \varphi(t,\varphi(s,x))$ for all $x \in \Omega$ and $s,t \in J(x)$ with $s+t \in J(x)$. Moreover, for every $t \geq 0$ and $x \in \varphi(t,\Omega)$ we have $[-t,\infty) \subset J(x)$ and for all $s \in [0,t]$ we have $\varphi(-s,x) = \varphi(s,\cdot)^{-1}(x)$. Since F is a C^1-function it is well-known that the same is true for $\varphi(t,\cdot)$ on Ω and $\varphi(-t,\cdot)$ on $\varphi(t,\Omega)$ for every $t \geq 0$. We denote by $\partial_2\varphi(t,\cdot)$ its derivative. Moreover we recall that a subset $M \subseteq \Omega$ is positively invariant under φ if for every $x \in M$ and for every $t \in J(x_0)\cap]0,\infty[$ it holds that $\varphi(t,x) \in M$.

We refer to the monograph of Amann [2] for further results on this topic.

Additionally, fix $h \in C(\Omega)$. We define for $t \geq 0$

$$h_t : \Omega \to \mathbb{C}, x \mapsto \exp\left(\int_0^t h(\varphi(s,x))\,ds\right).$$

It is easily seen, that because of $\varphi(t+s,x) = \varphi(t,\varphi(s,x))$ we have $h_{t+s}(x) = h_t(x)h_s(\varphi(t,x))$ for all $s,t \geq 0$ and $x \in \Omega$.

For a measurable function $\rho : \Omega \to (0, \infty)$ we want to define a C_0-semigroup on $L^p(\Omega, \mu)$ via $(T(t)f)(x) := h_t(x)f(\varphi(t, x))$, where $1 \leq p < \infty$ and μ denotes the Borel measure on Ω with Lebesgue density ρ. The next theorem gives a characterization of when this is possible. For its proof see [17, Theorem 4.7 and Proposition 4.12]. Although in [17] h is assumed to be real valued the proofs of [17, Theorem 4.7 and Proposition 4.12] are valid for complex valued h, too. Observe that for real valued h we have $h_t = |h_t|$ for all $t \geq 0$. From now on we write $L^p_\rho(\Omega)$ instead of $L^p(\Omega, \mu)$ and simply $L^p(\Omega)$ in case of $\rho = 1$.

Theorem 1. *Let $F, h,$ and ρ be as above and $p \in [1, \infty)$. Then the following are equivalent.*

i) *For $t \geq 0$ the operators*
$$T(t) : L^p_\rho(\Omega) \to L^p_\rho(\Omega), (T(t)f)(x) := h_t(x)f(\varphi(t, x))$$
are well-defined, linear and continuous and define a C_0-semigroup $T_{F,h} := (T_{F,h}(t))_{t\geq 0} := (T(t))_{t\geq 0}$ on $L^p_\rho(\Omega)$.

ii) *There are $M \geq 1, \omega \in \mathbb{R}$ such that for every $t \geq 0$*
$$|h_t(x)|^p \rho(x) \leq Me^{\omega t}\rho(\varphi(t, x))|\partial_2\varphi(t, x)| \text{ holds } \lambda\text{-a.e. on } \Omega,$$
where λ denotes Lebesgue measure.

Moreover, if ii) holds the generator of the above C_0-semigroup is an extension of the operator
$$C^1_c(\Omega) \to L^p_\rho(\Omega), f \mapsto Ff' + hf$$
in $L^p_\rho(\Omega)$, where $C^1_c(\Omega)$ denotes the space of compactly supported, continuously differentiable functions on Ω. Additionally, if h is bounded and F is such that for every $x_0 \in \Omega$ the maximal domain of $\varphi(\cdot, x_0)$ equals $\mathbb{R}$, then $C^1_c(\Omega)$ is a core for the generator of the C_0-semigroup $T_{F,h}$.

Definition 2. In what follows, we call a measurable function $\rho : \Omega \to (0, \infty)$ *p-admissible for F and h* ($p \in [1, \infty)$) if there are constants $M \geq 1, \omega \in \mathbb{R}$ such that
$$\forall\, t \geq 0, x \in \Omega : \ |h_t(x)|^p \rho(x) \leq Me^{\omega t}\rho(\varphi(t, x))|\partial_2\varphi(t, x)|.$$
Since $|h_t(x)| = \exp(\int_0^t \operatorname{Re} h(\varphi(s, \cdot))ds)$, p-admissibility of ρ only depends on F and $\operatorname{Re} h$. By the above theorem, we have for a p-admissible ρ for F and h the well-defined C_0-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$.

Our first aim in this section is to give a characterization of chaos for $T_{F,h}$ on $L^p_\rho(\Omega)$ similar to the characterizations of hypercyclicity and mixing for those C_0-semigroups proved in [18]. Before we do so we make the following observation.

Remark 3. (i) Let ρ be p-admissible for F and h. If for $x_0 \in \Omega$ we have $F(x_0) = 0$ it clearly follows that $\varphi(t, x_0) = x_0$ for every $t \geq 0$. Since we have uniqueness for the solutions of the initial value problems
$$\dot{x} = F(x),\ x(0) = x_0\,(x_0 \in \Omega),$$
this immediately implies for $\{F = 0\} := \{x \in \Omega;\, F(x) = 0\}$ that
$$\forall\, t \geq 0 : \ \varphi(t, \Omega\backslash\{F = 0\}) = \varphi(t, \Omega)\backslash\{F = 0\} \subseteq \Omega\backslash\{F = 0\}.$$
Hence we can consider $F_{|\Omega\backslash\{F=0\}}$ and $h_{|\Omega\backslash\{F=0\}}$ on $\Omega\backslash\{F = 0\}$ and $\rho_{|\Omega\backslash\{F=0\}}$ is p-admissible for $F_{|\Omega\backslash\{F=0\}}$ and $h_{|\Omega\backslash\{F=0\}}$. For notational convenience we write again $T_{F,h}$ and $L^p_\rho(\Omega\backslash\{F = 0\})$ instead of $T_{F_{|\Omega\backslash\{F=0\}}, h_{|\Omega\backslash\{F=0\}}}$ and $L^p_{\rho_{|\Omega\backslash\{F=0\}}}(\Omega\backslash\{F = 0\})$. Finally, if $\lambda(\{F = 0\}) = 0$ it follows that $L^p_\rho(\Omega)$ and $L^p_\rho(\Omega\backslash\{F = 0\})$ can be identified in a canonical way.

(ii) $\varphi(t, \cdot) : \Omega \to \varphi(t, \Omega)$ is bijective for $t \geq 0$ with inverse $\varphi(-t, \cdot) : \varphi(t, \Omega) \to \Omega$. For $x_0 \in \Omega$ we define $Z(x_0)$ to be the connected component of $\Omega\backslash\{F = 0\}$

containing x_0 if $F(x_0) \neq 0$ and $Z(x_0) := \{x_0\}$ if $F(x_0) = 0$. It is well-known that $\varphi(t, Z(x_0)) \subseteq Z(x_0)$ for every $t \geq 0$, more precisely

$$\forall\, x_0 \in \Omega : \; \varphi([0,\infty), x_0) = \begin{cases} Z(x_0) \cap [x_0, \infty) & \text{if } F(x_0) \geq 0 \\ Z(x_0) \cap (-\infty, x_0] & \text{if } F(x_0) \leq 0. \end{cases}$$

Herefrom and from the injectivity, it follows easily that $\varphi(t,\cdot) : \Omega \to \Omega$ is strictly increasing for all $t \geq 0$ and thus $\varphi(-t,\cdot) : \varphi(t,\Omega) \to \Omega$ is strictly increasing, too. Since

$$\forall\, t \geq 0, x \in \Omega : \; x = \varphi(-t, \varphi(t,x))$$

we obtain

$$\forall\, t \geq 0, x \in \Omega : \; 1 = \partial_2\varphi(-t, \varphi(t,x))\partial_2\varphi(t,x)$$

so that

$$\forall\, t \geq 0, x \in \Omega : \; \partial_2\varphi(t,x) > 0$$

as well as

$$\forall\, t \geq 0, x \in \varphi(t,\Omega) : \; \partial_2\varphi(-t,x) > 0.$$

In order to formulate our result in a convenient way we introduce the following notions.

Definition 4. *If ρ is p-admissible for F and h we define for $t \geq 0$*

$$\rho_{t,p} : \Omega \to [0,\infty), \rho_{t,p}(x) := \chi_{\varphi(t,\Omega)}(x)|h_t(\varphi(-t,x))|^p\rho(\varphi(-t,x))\partial_2\varphi(-t,x)$$

as well as

$$\rho_{-t,p} : \Omega \to [0,\infty), \rho_{-t,p}(x) := |h_t(x)|^{-p}\rho(\varphi(t,x))\partial_2\varphi(t,x).$$

Obviously, $\rho_{t,p}$ and $\rho_{-t,p}$ depend on F and h but in order to keep notation simple we will not take this into account notationally as there will be no danger of confusion. Observe that $\rho_{0,p} = \rho$. The next lemma will be a crucial tool. For its proof see [18, Lemma 7].

Lemma 5. *Let F be as above and $h \in C(\Omega)$ real valued. Let ρ be p-admissible for F and h and let $[a,b] \subset \Omega\backslash\{F = 0\}$. Setting $\alpha := a, \beta := b$ if $F_{|[a,b]} > 0$, respectively $\alpha := b, \beta := a$ if $F_{|[a,b]} < 0$, there is a constant $C > 0$ such that*

$$\forall\, x \in [a,b] : \; \frac{1}{C} \leq \rho(x) \leq C$$

as well as

$$\forall\, t \geq 0, x \in [a,b] : \; \frac{1}{C}\rho_{t,p}(\alpha) \leq \rho_{t,p}(x) \leq C\rho_{t,p}(\beta)$$

and

$$\forall\, t \geq 0, x \in [a,b] : \; \frac{1}{C}\rho_{-t,p}(\alpha) \leq \rho_{-t,p}(x) \leq C\rho_{-t,p}(\beta).$$

We can now prove a characterization of chaos for the C_0-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$ for real valued h. We denote m-dimensional Lebesgue measure by λ^m and set $\lambda := \lambda^1$.

Theorem 6. *Let $\Omega \subseteq \mathbb{R}$ be open, $F \in C^1(\Omega)$ as above, $h \in C(\Omega)$ real valued, and $\rho : \Omega \to (0,\infty)$ be a measurable function which is p-admissible for F and h. Then the following are equivalent.*

i) *$T_{F,h}$ is chaotic on $L^p_\rho(\Omega)$.*
ii) *$\lambda(\{F = 0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are m different connected components $C_1, \ldots, C_m$ of $\Omega\backslash\{F = 0\}$, for λ^m-almost all choices of $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ there is $t > 0$ such that*

$$\sum_{j=1}^{m}\sum_{l\in\mathbb{Z}} \rho_{lt,p}(x_j) < \infty.$$

Proof. We first show that i) implies ii). Since for all $x \in \{F = 0\}$ and any $t \geq 0$ we have $\varphi(t, x) = x$ it follows

$$\forall f \in L^p_\rho(\Omega) : \big(T(t)f\big)_{|\{F=0\}} = \big(\exp(t\,h)f\big)_{|\{F=0\}}$$

so that $(T(t)f)_{t\geq 0}$ cannot be dense in $L^p_\rho(\Omega)$ for any $f \in L^p_\rho(\Omega)$ if $\lambda(\{F = 0\}) > 0$. Hence, since $T_{F,h}$ is chaotic, we conclude $\lambda(\{F = 0\}) = 0$. As described in Remark 3 i) we are therefore actually dealing with $T_{F,h}$ on $L^p_\rho(\Omega\backslash\{F = 0\})$. Now, if $K \subset \Omega\backslash\{F = 0\}$ is compact there is $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ whenever $t > t_K$. Hence, we can apply [17, Theorem 5.3] saying that, because $T_{F,h}$ is chaotic on $L^p_\rho(\Omega\backslash\{F = 0\})$ for every compact $K \subset \Omega\backslash\{F = 0\}$ there are a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0, \infty)$ and a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K such that $\lim_{n\to\infty} \mu(L_n) = \mu(K)$ and

$$\lim_{n\to\infty} \Big(\sum_{l=1}^{\infty} \int_{L_n} \rho_{lt_n,p}\, d\lambda + \int_{L_n} \rho_{-lt_n,p}\, d\lambda\Big) = 0.$$

(Recall that μ denotes the Borel measure on Ω with Lebesgue density ρ!) We will apply this condition to special compact sets in order to derive ii).

Let $x_1, \ldots, x_m$ be each from different connected components of $\Omega\backslash\{F = 0\}$. As $\Omega\backslash\{F = 0\}$ is open, there is $r < 0$ such that $\varphi(t, x_j)$ is well-defined for all $t \in [r, \infty)$ and every $1 \leq j \leq m$. For each $1 \leq j \leq m$ we set $K_j := \{\varphi(t, x_j);\, t \in [0, 1]\}$ if $F(x_j) > 0$, respectively $K_j := \{\varphi(t, x_j);\, t \in [r, 0]\}$ if $F(x_j) < 0$. It follows that $K_j = [x_j, \varphi(1, x_j)]$ if $F(x_j) > 0$, respectively $K_j = [\varphi(r, x_j), x_j]$ if $F(x_j) < 0$. In particular $\lambda(K_j) > 0$ and thus $\mu(K_j) > 0$ for every j.

$K := \cup_{j=1}^m K_j \subset \Omega\backslash\{F = 0\}$ is compact so that there are $(t_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ as above. Let $L_{n,j} := L_n \cap K_j$ for $1 \leq j \leq m$ and $n \in \mathbb{N}$. Applying Lemma 5 to K_j it follows that there are $C_j > 0\,(1 \leq j \leq m)$ such that for all $n \in \mathbb{N}$

$$\int_{L_{n,j}} \rho_{lt_n,p}(y)\, d\lambda(y) = \int_{L_{n,j}} \frac{\rho_{lt_n,p}(y)}{\rho(y)}\, d\mu(y) \geq C_j \rho_{lt_n,p}(x_j)\mu(L_{n,j})$$

and analogously

$$\int_{L_{n,j}} \rho_{-lt_n,p}(y)\, d\lambda(y) \geq C_j \rho_{-lt_n,p}(x_j)\mu(L_{n,j}).$$

Since for n large enough

$$\begin{aligned}
\infty &> \sum_{l=1}^{\infty} \Big(\int_{L_n} \rho_{lt_n,p}\, d\lambda + \int_{L_n} \rho_{-lt_n,p}\, d\lambda\Big) \\
&= \sum_{j=1}^{m}\sum_{l=1}^{\infty} \Big(\int_{L_{n,j}} \rho_{lt_n,p}\, d\lambda + \int_{L_{n,j}} \rho_{-lt_n,p}\, d\lambda\Big) \\
&\geq \sum_{j=1}^{m} C_j \mu(L_{n,j}) \sum_{l=1}^{\infty} \Big(\rho_{lt_n,p}(x_j) + \rho_{-lt_n,p}(x_j)\Big)
\end{aligned}$$

and $\lim_{n\to\infty} \mu(L_{n,j}) = \mu(K_j) > 0$ we deduce for n large enough

$$\sum_{j=1}^{m}\sum_{l\in\mathbb{Z}} \rho_{lt_n,p}(x_j) < \infty.$$

Since for $t = 0$ we have $\rho_{t,p} = \rho > 0$ the above t_n has to be strictly positive. Thus, ii) is proved.

It remains to show that ii) implies i). Since $\lambda(\{F = 0\}) = 0$ we consider $T_{F,h}$ on $L^p_\rho(\Omega\backslash\{F = 0\})$, as explained in Remark 3 i). If $K \subset \Omega\backslash\{F = 0\}$ is compact there is $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ whenever $t > t_K$. Hence, we can again use [17, Theorem 5.3]. For fixed compact $K \subset \Omega\backslash\{F = 0\}$ there are finitely many intervals

$[a_j,b_j]\subset\Omega\backslash\{F=0\}$ such that each $[a_j,b_j]$ is contained in a different connected component of $\Omega\backslash\{F=0\}$ and $K\subseteq\cup_{j=1}^m[a_j,b_j]$. We define $x_j:=a_j$ if $F_{|[a_j,b_j]}>0$, respectively $x_j:=b_j$ if $F_{|[a_j,b_j]}<0$. Moreover, without loss of generality, we can assume by ii) that there is $t>0$ with

$$\sum_{j=1}^m\sum_{l\in\mathbb{Z}}\rho_{lt,p}(x_j)<\infty.$$

Now it follows from Lemma 5 that for some constants $C_j>0\,(1\leq j\leq m)$ with $t_n:=nt$, $n\in\mathbb{N}$

$$\begin{aligned}\sum_{l=1}^\infty\Big(\int_K\rho_{lt_n,p}\,d\lambda+\int_K\rho_{-lt_n,p}\,d\lambda\Big)&\leq\sum_{j=1}^m\sum_{l=1}^\infty\Big(\int_{[a_j,b_j]}\rho_{lt_n,p}\,d\lambda+\int_{[a_j,b_j]}\rho_{-lt_n,p}\,d\lambda\Big)\\&=\sum_{j=1}^m\sum_{l=1}^\infty\Big(\int_{[a_j,b_j]}\frac{\rho_{lt_n,p}}{\rho}\,d\mu+\int_{[a_j,b_j]}\frac{\rho_{-lt_n,p}}{\rho}\,d\mu\Big)\\&\leq\sum_{j=1}^m C_j\mu([a_j,b_j])\sum_{l=1}^\infty\Big(\rho_{lnt,p}(x_j)+\rho_{-lnt,p}(x_j)\Big)\end{aligned}$$

so that

$$\lim_{n\to\infty}\sum_{l=1}^\infty\Big(\int_K\rho_{lt_n,p}\,d\lambda+\int_K\rho_{-lt_n,p}\,d\lambda\Big)=0.$$

With $L_n:=K$, $n\in\mathbb{N}$ it follows from [17, Theorem 5.3] that $T_{F,h}$ is chaotic on $L^p_\rho(\Omega\backslash\{F=0\})$. This proves the theorem. □

Example 7. (i) Let $\Omega=\mathbb{R},F=1$, and $h=0$ so that $\varphi(t,x)=x+t$ and $h_t(x)=1$. Then a measurable function $\rho:\mathbb{R}\to(0,\infty)$ is p-admissible for F and h for some $p\in[1,\infty)$ if the same holds for every $p\in[1,\infty)$ and the corresponding C_0-semigroup $T_{F,h}$ is the *(bilateral) left translation semigroup* on $L^p_\rho(\mathbb{R})$ and given by $(T(t)f)(x)=f(x+t)$, its generator being an extension of

$$C^1_c(\mathbb{R})\to L^p_\rho(\mathbb{R}),f\mapsto f'.$$

Moreover, we have

$$\rho_{t,p}(x)=\rho(x+t)\text{ and }\rho_{-t,p}=\rho(x-t).$$

Since $\{F=0\}=\emptyset$ we have only a single connected component of $\mathbb{R}\backslash\{F=0\}$ so that by Theorem 6 the left translation semigroup on $L^p_\rho(\mathbb{R})$ is chaotic if and only if for λ-a.e. $x\in\mathbb{R}$ there is $t>0$ such that

$$\sum_{l\in\mathbb{Z}}\rho(x+lt)<\infty.$$

This weight condition is originally due to Matsui et al. [23, Theorem 2] (see also Chapter 7 and related exercises, in [15]). Note that chaos is independent of $p\in[1,\infty)$.

(ii) Let again $\Omega=\mathbb{R}$. Moreover, $F(x):=1-x,h(x)=0\,(x\in\mathbb{R})$ so that $\varphi(t,x)=1+(x-1)e^{-t}$, $h_t(x)=1$, and $\partial_2\varphi(t,x)=e^{-t}$. Then a measurable function $\rho:\mathbb{R}\to(0,\infty)$ is again p-admissible for F and h for some $p\in[1,\infty)$ if the same holds for every $p\in[1,\infty)$ and the corresponding C_0-semigroup $T_{F,h}$ is given by $(T(t)f)(x)=f(1+(x-1)e^{-t})$ with generator being an extension of

$$C^1_c(\mathbb{R})\to L^p_\rho(\mathbb{R}),f\mapsto\big(x\mapsto(1-x)f'(x)\big).$$

Furthermore, $\mathbb{R}\backslash\{F=0\}=(-\infty,1)\cup(1,\infty)$ has two connected components and

$$\forall\,t\in\mathbb{R}:\ \rho_{t,p}(x)=\rho(1+(x-1)e^t)e^t,$$

so that $T_{F,h}$ is chaotic on $L^p_\rho(\mathbb{R})$ if and only if for λ^2-a.e. $(x_1, x_2) \in (-\infty, 1) \times (1, \infty)$ there is $t > 0$ such that

$$\sum_{j=1}^{2} \sum_{l \in \mathbb{Z}} \rho(1 + (x_j - 1)e^{lt})e^{lt} < \infty.$$

Again, the occurrence of chaos is independent of $p \in [1, \infty)$ as $h = 0$.

Analogous characterizations of hypercyclicity and mixing for $T_{F,h}$ on $L^p_\rho(\Omega)$ for some p-admissible ρ for F and real valued h were proved in [18]. Since we will use them in sequel, we include them here for the reader's convenience. For the proofs see [18, Theorem 9 and Remark 12]. It is worth observing that, by applying Theorem 8 to the left translation semigroup as stated in Example 7(i), one recovers the characterizations of hypercyclicity and mixing originally proved in [12] and [6] respectively.

Theorem 8. *Let $\Omega \subseteq \mathbb{R}$ be open, $F \in C^1(\Omega)$ as usual, $h \in C(\Omega)$ real valued, and $\rho : \Omega \to (0, \infty)$ be a measurable function which is p-admissible for F and h.*

a) *For the C_0-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$ the following are equivalent.*
   i) *$T_{F,h}$ is hypercyclic.*
   ii) *$T_{F,h}$ is weakly mixing.*
   iii) *$\lambda(\{F = 0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are m different connected components $C_1, \ldots, C_m$ of $\Omega \backslash \{F = 0\}$, for λ^m-almost all choices of $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ there is a sequence of positive numbers $(t_n)_{n \in \mathbb{N}}$ tending to infinity such that*

$$\forall\, 1 \leq j \leq m : \lim_{n\to\infty} \rho_{t_n,p}(x_j) = \lim_{n\to\infty} \rho_{-t_n,p}(x_j) = 0.$$

b) *For the C_0-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$ the following are equivalent.*
   i) *$T_{F,h}$ is mixing.*
   ii) *$\lambda(\{F = 0\}) = 0$ and for λ-almost every $x \in \Omega$ one has*

$$\lim_{t\to\infty} \rho_{t,p}(x) = \lim_{t\to\infty} \rho_{-t,p}(x) = 0.$$

So far, we have only characterizations of hypercyclicity, mixing, and chaos of $T_{F,h}$ in case of real valued h. In order to obtain some results for at least some complex valued h let us recall the so-called *comparison principle* which is a useful tool in order to identify hypercyclic, (weakly) mixing, or chaotic C_0-semigroups. Let T be a C_0-semigroup on a Banach space Y and S is a C_0-semigroup on a Banach space X which is quasi-conjugate to T, i.e. there is a continuous mapping $\Phi : Y \to X$ with dense range such that $\Phi \circ T(t) = S(t) \circ \Phi$ for every $t \geq 0$. Then, S is hypercyclic, (weakly) mixing, or chaotic respectively, if the same holds for T, see for Example [15, Proposition 7.7]. Recall that T and S are said to be conjugate, if the above Φ is a homeomorphism.

Let $F \in C^1(\Omega)$ be as above and $h, g \in C(\Omega)$ with g real valued. Moreover, let ρ be p-admissible for F and h as well as for F and g. We want to find some measurable function $\psi : \Omega \to \mathbb{C}$ such that $\exp \circ \psi$ induces a continuous, invertible multiplication operator M on $L^p_\rho(\Omega)$ for which the C_0-semigroups $T_{F,h}$ and $T_{F,g}$ are conjugate via M, i.e.

$$\forall\, t \geq 0, f \in L^p_\rho(\Omega) : \exp(\psi)\, T_{F,h}(t)(f) = T_{F,g}(t)(\exp(\psi) f)$$

Being $T_{F,g}(t)(Mf) = \exp(\psi(\varphi(t, \cdot))) \frac{g_t}{h_t} T_{F,h}(f)$, this is satisfied, if for each $x \in \Omega$

$$\forall\, t \geq 0 : \psi(\varphi(t, x)) - \int_0^t (h(\varphi(s, x)) - g(\varphi(s, x)))\, ds = \psi(x). \tag{3}$$

Now, if $x \in \{F=0\}$ it follows from $\varphi(t,x)=x$ that the above expression reduces to

$$\forall\, t \geq 0 : \psi(x) - t(h(x)-g(x)) = \psi(x).$$

Thus, it is necessary that h and g coincide on $\{F=0\}$.

Moreover, for $x \in \{F \neq 0\}$ we also have $F(\varphi(t,x)) \neq 0$ for every $t \geq 0$ so

$$\begin{aligned}\forall\, t \geq 0: \qquad & \psi(\varphi(t,x)) - \int_0^t h(\varphi(s,x)) - g(\varphi(s,x))ds \\ = \quad & \psi(\varphi(t,x)) - \int_0^t \frac{h(\varphi(s,x)) - g(\varphi(s,x))}{F(\varphi(s,x))}\partial_1\varphi(s,x)ds \\ = \quad & \begin{cases}\psi(\varphi(t,x)) - \int_x^{\varphi(t,x)} \frac{h(y)-g(y)}{F(y)}dy & , \text{ if } F(x) > 0 \\ \psi(\varphi(t,x)) + \int_{\varphi(t,x)}^x \frac{h(y)-g(y)}{F(y)}dy & , \text{ if } F(x) < 0.\end{cases}\end{aligned}$$

Hence, if Ω is connected, $\alpha := \inf\Omega, \omega := \sup\Omega$, and the function $\Omega \to \mathbb{R}, y \mapsto \frac{h(y)-g(y)}{F(y)}$ belongs to $L^1((\alpha,\beta))$ for every $\beta \in \Omega$, or to $L^1((\beta,\omega))$ for every $\beta \in \Omega$ we can set

$$\psi : \Omega \to \mathbb{R}, \psi(x) = \int_\alpha^x \frac{h(y)-g(y)}{F(y)}dy,$$

or

$$\psi : \Omega \to \mathbb{R}, \psi(x) = -\int_x^\omega \frac{h(y)-g(y)}{F(y)}dy,$$

respectively, and it follows from the above calculation that (3) holds on $\{F \neq 0\}$ for this ψ.

Proposition 9. *Let $\Omega \subseteq \mathbb{R}$ be an open interval, $F \in C^1(\Omega)$ as above and $h \in C(\Omega)$. Moreover, let ρ be p-admissible for F and h and set $\alpha := \inf\Omega, \omega := \sup\Omega$. Consider the following conditions.*

1) $\forall\, x \in \{F=0\} : h(x) \in \mathbb{R}$.

2a) *The function $\Omega \to \mathbb{R}, y \mapsto \frac{\operatorname{Im} h(y)}{F(y)}$ belongs to $L^1((\alpha,\beta))$ for all $\beta \in \Omega$.*

2b) *The function $\Omega \to \mathbb{R}, y \mapsto -\frac{\operatorname{Im} h(y)}{F(y)}$ belongs to $L^1((\beta,\omega))$ for all $\beta \in \Omega$.*

If 1) and 2 a) are satisfied, let $\psi : \Omega \to \mathbb{R}, \psi(x) = i\int_\alpha^x \frac{\operatorname{Im} h(y)}{F(y)}dy$, if 1) and 2 b) hold, set $\psi : \Omega \to \mathbb{R}, \psi(x) = -i\int_x^\omega \frac{\operatorname{Im} h(y)}{F(y)}dy$. Then $\exp\circ\psi$ defines a continuous, invertible multiplication operator M on $L^p_\rho(\Omega)$ such that the C_0-semigroups $T_{F,h}$ and $T_{F,\operatorname{Re} h}$ on $L^p_\rho(\Omega)$ are conjugate via M.

Proof. By the observation preceding the proposition applied to h and $g = \operatorname{Re} h$ we only have to show that $\exp\circ\psi$ defines a continuous, invertible multiplication operator on $L^p_\rho(\Omega)$. But this is obvious because $|\exp(\psi(x))| = 1$ for all $x \in \Omega$. $\square$

Remark 10. From the above proposition and the comparison principle it follows immediately, that in Theorems 8 and 6 we can replace the hypothesis of h being real valued by the weaker conditions

1) $\forall\, x \in \{F=0\} : h(x) \in \mathbb{R}$.
2) With $\alpha := \inf\Omega$ and $\omega := \sup\Omega$ the function $\Omega \to \mathbb{R}, y \mapsto \frac{\operatorname{Im} h(y)}{F(y)}$ belongs to $L^1((\alpha,\beta))$ for all $\beta \in \Omega$ or to $L^1((\beta,\omega))$ for all $\beta \in \Omega$.

As is seen in Theorems 6 and 8, the dynamical properties of $T_{F,h}$ on $L^p_\rho(\Omega)$ are determined by the asymptotic behavior of the functions

$$\mathbb{R} \mapsto \mathbb{R}, t \mapsto \rho_{t,p}(x),$$

where $x \in \Omega$ is fixed. In the definition of $\rho_{t,p}$ the term $\partial_2\varphi(t,x)$ appears. We have the following representation of $\partial_2\varphi(t,x)$ at our disposal which not only makes it sometimes easier to evaluate the characterizations of the different dynamical properties of $T_{F,h}$ but will also be very useful in section 3.

Proposition 11. *Let $\Omega \subseteq \mathbb{R}$ be open and $F \in C^1(\Omega)$ as above. Then for every $x \in \Omega$ we have*

$$\forall\, t \geq 0: \ \partial_2\varphi(t,x) = \exp\big(\int_0^t F'(\varphi(s,x))\,ds\big).$$

Moreover, for every $x \in \varphi(r,\Omega), r \geq 0$,

$$\forall\, t \in [0,r]: \ \partial_2\varphi(-t,x) = \exp\big(-\int_{-t}^0 F'(\varphi(s,x))\,ds\big).$$

Proof. For $x \in \Omega$ we have $\partial_2\varphi(t,x) > 0$ by hypothesis on F and Remark 3 ii). Since F is C^1 it is well-known that $\partial_1\partial_2\varphi$ exists and is continuous and $\partial_2\varphi(0,x) = 1$ for all $x \in \Omega$. Hence,

$$\begin{aligned}
\forall\, t \geq 0: \int_0^t F'(\varphi(s,x))\,ds &= \int_0^t \frac{F'(\varphi(s,x))\partial_2\varphi(s,x)}{\partial_2\varphi(s,x)}\,ds = \int_0^t \frac{\frac{\partial}{\partial x}(F\circ\varphi(s,\cdot))(x)}{\partial_2\varphi(s,x)}ds \\
&= \int_0^t \frac{\frac{\partial}{\partial x}\frac{\partial}{\partial s}\varphi(s,x)}{\partial_2\varphi(s,x)}\,ds = \int_0^t \frac{\frac{\partial}{\partial s}\partial_2\varphi(s,x)}{\partial_2\varphi(s,x)}\,ds \\
&= \ln\partial_2\varphi(t,x) - \ln\partial_2\varphi(0,x) = \ln\partial_2\varphi(t,x).
\end{aligned}$$

Therefore, $\exp\big(\int_0^t F'(\varphi(s,x))\,ds\big) = \partial_2\varphi(t,x)$ for each $t \geq 0$. Now, if $x \in \varphi(r,\Omega)$ with $r \geq 0$ it follows as above for $t \in [0,r]$

$$-\int_{-t}^0 F'(\varphi(s,x))\,ds = -(\ln\partial_2\varphi(0,x) - \ln\partial_2\varphi(-t,x)) = \ln\partial_2\varphi(-t,x).$$

□

Corollary 12. *Let $\Omega \subseteq \mathbb{R}$ be open, $F \in C^1(\Omega)$ be as usual, $h \in C(\Omega)$, and let ρ be p-admissible for F and h. Then we have for all $t \geq 0$ and $x \in \Omega$*

$$\begin{aligned}
\rho_{t,p}(x) &= \chi_{\varphi(t,\Omega)}(x)\exp\big(p\int_{-t}^0 \operatorname{Re} h(\varphi(s,x)) - \frac{1}{p}F'(\varphi(s,x))ds\big)\rho(\varphi(-t,x)) \\
&= \begin{cases} \exp\big(pt(\operatorname{Re} h(x) - \frac{1}{p}F'(x)\big)\rho(x), & x \in \{F = 0\}, \\ \chi_{\varphi(t,\Omega)}(x)\exp\big(p\int_{\varphi(-t,x)}^x \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big)\rho(\varphi(-t,x)), & x \in \{F \neq 0\} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\rho_{-t,p}(x) &= \exp\big(-p\int_0^t \operatorname{Re} h(\varphi(s,x)) - \frac{1}{p}F'(\varphi(s,x))ds\big)\rho(\varphi(t,x)) \\
&= \begin{cases} \exp\big(-pt(\operatorname{Re} h(x) - \frac{1}{p}F'(x)\big)\rho(x), & x \in \{F = 0\}, \\ \exp\big(-p\int_x^{\varphi(t,x)} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big)\rho(\varphi(t,x)), & x \in \{F \neq 0\} \end{cases}
\end{aligned}$$

Proof. While a straightforward calculation gives

$$\rho_{t,p}(x) = \chi_{\varphi(t,\Omega)}(x)\exp\big(p\int_{-t}^0 \operatorname{Re} h(\varphi(s,x)) - \frac{1}{p}F'(\varphi(s,x))ds\big)\rho(\varphi(-t,x))$$

and

$$\rho_{-t,p}(x) = \exp\big(-p\int_0^t \operatorname{Re} h(\varphi(s,x)) - \frac{1}{p}F'(\varphi(s,x))ds\big)\rho(\varphi(t,x))$$

we observe that for $x \in \{F = 0\}$ we have $\varphi(t,x) = x$ for each t so that in $\{F = 0\}$

$$\rho_{t,p}(x) = \exp\big(pt(\operatorname{Re} h(x) - \frac{1}{p}F'(x)\big)\rho(x)$$

as well as

$$\rho_{-t,p}(x) = \exp\big(-pt(\operatorname{Re} h(x) - \frac{1}{p}F'(x)\big)\rho(x).$$

For $x \in \{F \neq 0\}$ it is well-known that $\varphi(t,x) \in \{F \neq 0\}$ so that in $\{F \neq 0\}$

$$\begin{aligned}\rho_{-t,p}(x) &= \exp\big(-p\int_0^t \frac{\operatorname{Re} h(\varphi(s,x)) - \frac{1}{p}F'(\varphi(s,x))}{F(\varphi(s,x))}\partial_1\varphi(s,x)ds\big)\rho(\varphi(t,x))\\ &= \exp\big(-p\int_x^{\varphi(t,x)} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big)\rho(\varphi(t,x))\end{aligned}$$

and similarly

$$\rho_{t,p}(x) = \chi_{\varphi(t,\Omega)}(x)\exp\big(p\int_{\varphi(-t,x)}^x \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big)\rho(\varphi(-t,x)).$$

□

We finish this section by taking a closer look at the case $\rho = 1$ for some special cases which are of particular interest in section 4.

Theorem 13. *Let $I \subseteq \mathbb{R}$ be an open interval, $F \in C^1(I)$ as usual with $F(x) < 0$ for each $x \in I$ and such that $\varphi(t,I) \neq I$ for some $t > 0$. Moreover, let $h \in C(I)$ be such that with $\alpha := \inf I$ and $\omega := \sup I$ the function*

$$I \to \mathbb{R}, x \mapsto \frac{\operatorname{Im} h(x)}{F(x)}$$

belongs to $L^1((\alpha,\beta))$ for every $\beta \in I$ or to $L^1((\beta,\omega))$ for every $\beta \in I$. Furthermore, let $\rho = 1$ be p-admissible for F and h for some $1 \leq p < \infty$.

a) *For the C_0-semigroup $T_{F,h}$ on $L^p(I)$ the following are equivalent.*
   i) *$T_{F,h}$ is hypercyclic.*
   ii) *$T_{F,h}$ is weakly mixing.*
   iii) *There is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in I converging to α such that for some $x_0 \in I$*

$$\lim_{n\to\infty}\int_{\alpha_n}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = -\infty.$$

b) *For the C_0-semigroup $T_{F,h}$ on $L^p(I)$ the following are equivalent.*
   i) *$T_{F,h}$ is mixing.*
   ii) *For some $x_0 \in I$*

$$\int_{\alpha}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = -\infty.$$

c) *For the C_0-semigroup $T_{F,h}$ on $L^p(I)$ the following are equivalent.*
   i) *$T_{F,h}$ is chaotic.*
   ii) *There is some $x_0 \in I$ such that for every $x \in I$ there is $t > 0$ such that*

$$\sum_{l=1}^{\infty}\exp\big(p\int_{\varphi(lt,x)}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big) < \infty.$$

Proof. From Corollary 12 it follows

$$\forall\, t \geq 0, x \in I:\ \rho_{t,p}(x) = \chi_{\varphi(t,I)}(x)\exp\big(p\int_{\varphi(-t,x)}^x \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big)$$

and

$$\forall\, t \geq 0, x \in I:\ \rho_{-t,p}(x) = \exp\big(-p\int_x^{\varphi(t,x)} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big).$$

As is well-known, for each $x \in I$ the trajectory $\{\varphi(t,x);\, t \geq 0\}$ is either an open subinterval of I or equals $\{x\}$. Since the later occurs if and only if $F(x) = 0$ it follows from $F(x) < 0$ that $\inf\{\varphi(t,x);\, t \geq 0\} = \alpha$ for every $x \in I$. Moreover, the assumption $\varphi(t,I) \neq I$ for some $t > 0$ implies that for every $x \in I$ there is $t_0 > 0$ such that $\chi_{\varphi(t,I)}(x) = 0$ whenever $t > t_0$. In particular, for all $x \in I$ we have $\rho_{t,p}(x) = 0$ for sufficiently large t.

Proof of part a). It follows from Remark 10 and Theorem 8 that i) and ii) in a) are equivalent and by the preceding observation they hold if and only if for λ-a.e. $x \in I$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ tending to infinity such that

$$0 = \lim_{n\to\infty} \rho_{-t_n,p}(x) = \exp\big(p\int_{\varphi(t_n,x)}^{x} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big).$$

Being $\varphi(\cdot,x)$ strictly decreasing and $\alpha = \inf\{\varphi(t,x);\, t \geq 0\}$ for every $x \in I$, the above relation obviously holds if and only if for some sequence $(\alpha_n)_{n\in\mathbb{N}}$ in I converging to α we have

$$\lim_{n\to\infty} \int_{\alpha_n}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = -\infty$$

for some (and then any) $x_0 \in I$. Thus a) is proved.

The proofs of parts b) and c) go along the same lines as the one of part a) by applying Theorem 8 b) and Theorem 6, respectively, instead of Theorem 8 a), so that we omit them. □

Remark 14. i) If under the hypotheses of the above theorem we have $\varphi(t_0, I) = I$ for some $t_0 > 0$ it is easily seen that the same holds for every $t > 0$ so that

$$\rho_{t,p}(x) = \exp\big(p\int_{\varphi(-t,x)}^{x} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big).$$

It follows by the same kind of arguments as in the above proof that $T_{F,h}$ is hypercyclic on $L^p(I)$ if and only if $T_{F,h}$ is weakly mixing if and only if there are sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\omega_n)_{n\in\mathbb{N}}$ in I converging to α and ω respectively, such that for some $x_0 \in I$

$$\lim_{n\to\infty} \int_{\alpha_n}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = -\infty, \lim_{n\to\infty} \int_{x_0}^{\omega_n} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = \infty.$$

Mixing of $T_{F,h}$ then occurs if and only if

$$\int_{\alpha}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = -\infty, \int_{x_0}^{\omega} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = \infty,$$

while $T_{F,h}$ is chaotic if and only if there is some $x_0 \in I$ such that for every $x \in I$ there is $t > 0$ such that

$$\sum_{l=1}^{\infty} \exp\big(p\int_{\varphi(lt,x)}^{x_0} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big) + \sum_{l=1}^{\infty} \exp\big(-p\int_{x_0}^{\varphi(-lt,x)} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy\big) < \infty.$$

ii) Of course it is possible to characterize hypercyclicity, mixing, and chaos of $T_{F,h}$ in case of F being strictly positive on I, too. The other hypotheses of Theorem 12 unchanged it follows for example that $T_{F,h}$ is hypercyclic if and only if there is a sequence $(\omega_n)_{n\in\mathbb{N}}$ in I converging to $\omega = \sup I$ such that for some $x_0 \in I$

$$\lim_{n\to\infty} \int_{x_0}^{\omega_n} \frac{\operatorname{Re} h(y) - \frac{1}{p}F'(y)}{F(y)}dy = \infty.$$

It should be obvious how the conditions characterizing and mixing and chaos change, respectively, so that we do not state them explicitly.

To conclude this section we give a concrete description of the generator of $T_{F,h}$ in case of $\rho = 1$, at least under some mild additional assumptions on F and h.

Theorem 15. *Let F, h be as above, $1 \leq p < \infty$ and assume that $\rho = 1$ is p-admissible for F and h. Assume that additionally $h \in L^\infty(\Omega)$ and that F can be extended as a C^1-function to $\mathbb{R}$ such that*

i) *$F, F' \in L^\infty(\mathbb{R})$,*
ii) *$\overline{\Omega}$ is positively invariant under the flow $\varphi(\cdot,\cdot)$ associated with the problem $\dot{x} = F(x), x(0) = x_0$ (defined on $\mathbb{R} \otimes \mathbb{R}$ by the assumptions on F).*

Then the generator $(A, D(A))$ of the C_0-semigroup $T_{F,h}$ on $L^p(\Omega)$ is given by

$$D(A) = \{f \in L^p(\Omega);\ Ff' \in L^p(\Omega)\}$$

and

$$A : D(A) \to L^p(\Omega), Af = Ff' + hf,$$

where f' denotes the distributional derivative of f.

Proof. Let $D := \{f \in L^p(\Omega);\ Ff' \in L^p(\Omega)\}$ and

$$B : D \to L^p(\Omega), Bf := Ff'.$$

We first observe that the operator (B, D) is a closed operator. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} u_n = f$ and $\lim_{n\to\infty} Bu_n = g$ in $L^p(\Omega)$. Since $F \in C^1(\Omega)$ we obtain for the distributional derivative of Fu_n

$$(Fu_n)' = F'u_n + Fu_n' = F'u_n + Bu_n$$

which converges in $L^p(\Omega)$ to $F'f+g$ because $F' \in L^\infty(\Omega)$. It follows $\lim_{n\to\infty} Fu_n' = g$ in $L^p(\Omega)$. On the other hand $(Fu_n')_{n\in\mathbb{N}}$ converges in the sense of distributions to Ff' and thus $Ff' = g \in L^p(\Omega)$. Hence $f \in D$ and $Bf = g$, so that (B, D) is a closed operator.

Next we show that

$$D_1 := \{f \in C^1(\overline{\Omega}) \cap L^p(\Omega);\ f' \in L^p(\Omega)\}.$$

is a core for (B, D). Being $F \in L^\infty(\Omega)$, we have $D_1 \subseteq D$. Let $\psi \in C_c^\infty(\mathbb{R})$ be such that $\psi \geq 0$ and $\int_R \psi(x)\,dx = 1$ and set $\psi_n(x) := n\psi(nx), n \in \mathbb{N}$. In what follows we extend all functions from $L^p(\Omega)$ by 0 to all $\mathbb{R}$. Fix $f \in L^p(\Omega)$. Then $\psi_n * f \in C^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$, thus its restriction to $\overline{\Omega}$ belongs to $C^1(\Omega)$. Moreover, as is well-known $(\psi_n * f)' = \psi_n' * f \in L^p(\mathbb{R})$ and therefore $(\psi_n * f) \in D_1$ with $\lim_{n\to\infty}(\psi_n * f)_{|\Omega} = f$ in $L^p(\Omega)$. Since we assumed F to be extendable to $\mathbb{R}$ such that $F \in C^1(\mathbb{R})$ with $F, F' \in L^\infty(\mathbb{R})$ it follows from [28, Lemma 1.2.5] (see also [24, pp. 313-315]) that $F(\psi_n * f)' - \psi_n * (Ff') \in L^p(\mathbb{R})$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} F(\psi_n * f)' - \psi_n * (Ff') = 0$ in $L^p(\mathbb{R})$. As $Ff' \in L^p(\mathbb{R})$ by $f \in D$ we have

$$F(\psi_n * f)' - Ff' = \Big(\psi_n * (Ff') - Ff'\Big) + \Big(F(\psi_n * f') - \psi_n * (Ff')\Big),$$

so $\lim_{n\to\infty} F(\psi_n * f)'_{|\Omega} = Ff'$ in $L^p(\Omega)$ implying that $(B(\psi_n * f)_{|\Omega})_{n\in\mathbb{N}}$ converges to Bf in $L^p(\Omega)$. Hence D_1 is dense in D equipped with the graph norm of B, i.e. D_1 is a core of (B, D).

Since we assume $h \in L^\infty(\Omega)$ it follows that $|h_t(x)| \geq e^{-t\|h\|_\infty}$ for all $x \in \Omega$. Since $\rho = 1$ is p-admissible for F and h we conclude that

$$\forall\, x \in \Omega :\ 1 \leq Me^{(\omega+p\|h\|_\infty)t}|\partial_2\varphi(t, x)|$$

so that $\rho = 1$ is p-admissible for F and 0, too. Denote the generator of the C_0-semigroup $T_{F,0} = (T_{F,0}(t))_{t\geq 0}$ on $L^p(\Omega)$ by $(A_0, D(A_0))$. Using Lebesgue's dominated convergence theorem it is straightforward to verify that $D_1 \subseteq D(A_0)$ and $A_0 f = Bf$ for all $f \in D_1$.

Next we show that D_1 is also a core for $(A_0, D(A_0))$. Indeed, as $C_c^1(\Omega) \subseteq D_1$ it follows that D_1 is dense in $L^p(\Omega)$. Moreover, it follows immediately from $T_{F,0}(t)f = f(\varphi(t,\cdot))$ that D_1 is invariant under $T_{F,0}$ because of the additional hypothesis ii). Applying [14, Proposition II.1.7] we conclude that D_1 is a core for $(A_0, D(A_0))$. As both operators (B, D) and $(A_0, D(A_0))$ are closed and coincide on the common core D_1, we obtain $(A_0, D(A_0)) = (B, D)$.

As $h \in L^\infty(\Omega)$, the operator

$$M_h : L^p(\Omega) \to L^p(\Omega), f \mapsto hf$$

is well-defined and continuous. Being a bounded perturbation of $(A_0, D(A_0))$

$$C : D(A_0) \to L^p(\Omega), f \mapsto A_0 f + M_h f = Bf + M_h$$

generates a C_0-semigroup S on $L^p(\Omega)$ (see e.g. [14, Theorem III.1.3]) and D_1 is a core of $(C, D(A_0))$.

Now, let $(A, D(A))$ be the generator of the C_0-semigroup $T_{F,h}$. As above for the special case $h = 0$ one shows that $D_1 \subseteq D(A)$ and that A and C coincide on D_1. Moreover, if $\alpha \in \rho(A) \cap \rho(C)$, it follows from D_1 being a core for $(C, D(A_0))$ that

$$\overline{(\alpha - A)(D_1)} = \overline{(\alpha - C)(D_1)} = L^p(\Omega).$$

From [14, Exercise II.1.15 (2)] we derive that D_1 is a core for $(A, D(A))$. Since $(A, D(A))$ and $(C, D(A_0))$ are both closed operators coinciding on the common core D_1 we finally obtain the assertion.

□

Remark 16. Conditions on F that ensure that ii) holds can be found in [2], Theorem 16.9 and Corollary 16.10. In particular if $\Omega = (a, b)$, then ii) holds if $F(a) \geq 0$ and $F(b) \leq 0$.

3. Dynamical properties of C_0-semigroups on Sobolev spaces generated by first order differential operators

In what follows let $I = (a, b)$ be a bounded open interval in $\mathbb{R}$. For $1 \leq p < \infty$ we denote as usual by $W^{1,p}(I)$ the first order Sobolev space of p-integrable functions on I, i.e.

$$W^{1,p}(I) = \{u \in L^p(I); u' \in L^p(I)\},$$

where u' denotes the distributional derivative of u. We equip $W^{1,p}(I)$ with its usual norm turning it into a Banach space. It is well-known that $W^{1,p}(I) \subseteq C[a, b]$ and that for any $x \in [a, b]$ the point evaluation δ_x in x is a continuous linear form on $W^{1,p}(I)$. We will use the following closed subspace of $W^{1,p}(I)$,

$$W_*^{1,p}(I) := \ker \delta_a.$$

For the boundedness of I we have the topological direct sum

$$W^{1,p}(I) = W_*^{1,p}(I) \oplus \operatorname{span}\{\mathbb{1}\},$$

where $\mathbb{1}$ denotes the constant function with value 1.

Lemma 17. *Let $I = (a, b)$ be a bounded open interval in $\mathbb{R}$ and $1 \leq p < \infty$. Then*

$$\Phi : L^p(I) \to W_*^{1,p}(I), \Phi(f)(x) := \int_a^x f(y)\, dy$$

is a well-defined, linear and continuous bijection with continuous inverse

$$\Phi^{-1} : W_*^{1,p}(I) \to L^p(I), \Phi^{-1}(f) = f'.$$

Proof. A straightforward application of Jensen's inequality gives that

$$\begin{aligned} \int_a^b |\Phi(f)(x)|^p \, dx &\leq \int_a^b (x-a)^{p-1} \, dx \int_a^b |f(y)|^p \, dy \\ &= \frac{1}{p}(b-a)^p \int_a^b |f(y)|^p \, dy \end{aligned} \tag{4}$$

Moreover, as in [7, Lemma 8.2] it follows that the distributional derivative of $\Phi(f)$ equals f, so that Φ is in fact well-defined, obviously linear, and by (4) and $\Phi(f)' = f$ continuous. Injectivity of Φ follows from $\Phi(f)' = f$. Additionally, from $\Phi(u') = u$ for every $u \in W_*^{1,p}(I)$ we obtain the surjectivity of Φ. Obviously, $\Phi^{-1}(f) = f'$ for all $W_*^{1,p}(I)$. $\square$

The comparison principle and Lemma 17 imply the next result.

Proposition 18. *Let $I = (a,b)$ be a bounded open interval, $1 \leq p < \infty$, and let Φ be as in Lemma 17. Moreover, let T be a C_0-semigroup of $L^p(I)$. Then $S := (\Phi \circ T(t) \circ \Phi^{-1})_{t \geq 0}$ is a C_0-semigroup on $W_*^{1,p}(I)$ which is hypercyclic, (weakly) mixing, or chaotic respectively, if and only if the same holds for T.*

Proof. Clearly, S is a C_0-semigroup on $W_*^{1,p}(I)$ by Lemma 17. The rest of the assertion follows immediately from the comparison principle. $\square$

Now, let $F : [a,b] \to \mathbb{R}$ be a C^1-function on the bounded closed interval $[a,b]$. As in section 2 we assume that for every $x_0 \in I = (a,b)$ the unique solution $\varphi(\cdot, x_0)$ of the initial value problem

$$\dot{x} = F(x), \ x(0) = x_0$$

is defined on $[0, \infty)$.

Lemma 19. *Let $I = (a,b)$ be a bounded interval and $F : [a,b] \to \mathbb{R}$ a C^1-function as above. Then the function $\rho = 1$ is p-admissible for F and F' for every $1 \leq p < \infty$, i.e. via*

$$\forall \, t \geq 0, f \in L^p(I) : \, (T(t)f)(x) := \exp\big(\int_0^t F'(\varphi(s,x)) \, ds\big) f(\varphi(t,x))$$

we have a C_0-semigroup $T_{F,F'}$ on $L^p(I)$.

Proof. Since F is C^1 on $[a,b]$ there is $\omega \in \mathbb{R}$ such that $F'(x) \leq \omega$ for all $x \in [a,b]$. Hence, for $t \geq 0$ and $x \in (a,b)$ we have

$$\begin{aligned} 0 &\leq \partial_2 \varphi(t,x) = 1 + \int_0^t \frac{\partial}{\partial t} \partial_2 \varphi(s,x) \, ds = 1 + \int_0^t \frac{\partial}{\partial x} \frac{\partial}{\partial t} \varphi(s,x) \, ds \\ &= 1 + \int_0^t F'(\varphi(s,x)) \partial_2 \varphi(s,x) \, ds \leq 1 + \omega \int_0^t \partial_2 \varphi(s,x) \, ds. \end{aligned}$$

An application of Gronwall's lemma yields

$$\forall \, x \in (a,b), t \geq 0 : \, \partial_2 \varphi(t,x) \leq e^{\omega t}.$$

For $1 \leq p < \infty$ it follows from the above inequality, the hypothesis, and Proposition 11 that for every $x \in (a,b)$ we have

$$\begin{aligned} \forall \, t \geq 0 : \, \Big(\exp\big(\int_0^t F'(\varphi(s,x)) \, ds\big)\Big)^p &= \Big(\partial_2 \varphi(t,x)\Big)^{p-1} |\partial_2 \varphi(t,x)| \\ &\leq e^{(p-1)\omega t} |\partial_2 \varphi(t,x)| \end{aligned}$$

proving the lemma. $\square$

Theorem 20. *Let $I = (a, b)$ be a bounded interval and $F : [a, b] \to \mathbb{R}$ a C^1-function as above with $F(a) = 0$. Then for every $1 \leq p < \infty$ and $\gamma \in \mathbb{R}$*

$$\forall\, t \geq 0, f \in W^{1,p}(I) : \big(S(t)f\big)(x) := e^{\gamma t} f(\varphi(t, x))$$

defines a C_0-semigroup $S_{F,\gamma}$ on $W^{1,p}(I)$. $W^{1,p}_*(I)$ is $S_{F,\gamma}$-invariant. The generator of $S_{F,\gamma}$ in $W^{1,p}(I)$ is given by

$$B : \{f \in W^{1,p}(I);\, Ff'' \in L^p(I)\} \to W^{1,p}(I), Bf = Ff' + \gamma f,$$

while its generator in $W^{1,p}_*(I)$ is

$$B_* : \{f \in W^{1,p}_*(I);\, Ff'' \in L^p(I)\} \to W^{1,p}_*(I), B_* f = Ff' + \gamma f.$$

Proof. By the preceding lemma $T_{F,F'} = (T(t))_{t\geq 0}$ is a well-defined C_0-semigroup on $L^p(I)$ for every $1 \leq p < \infty$. As $F(a) = 0$ we have $\varphi(t, a) = a$ for all $t \geq 0$ so that for all $f \in L^p(I)$ and $t \geq 0$ with Φ as in Lemma 17 and the fact that $\varphi(t, \cdot)$ is increasing

$$\Phi(T(t)f)(x) = \int_a^x f(\varphi(t, y))|\partial_2\varphi(t, y)|\, dy = \int_a^{\varphi(t,x)} f(y)\, dy = \Phi(f)(\varphi(t, x)),$$

where we used Proposition 11. Since Φ is bijective it follows that $S_{F,\gamma} := (S(t))_{t\geq 0} = \Big(\Phi\circ\big(e^{\gamma t}T(t)\big)\circ\Phi^{-1}\Big)_{t\geq 0}$ defines a C_0-semigroup on $W^{1,p}_*(I)$. Clearly, for every $t \geq 0$ the mapping

$$S(t) : \operatorname{span}\{\mathbb{1}\} \to \operatorname{span}\{\mathbb{1}\}, \alpha\mathbb{1} \mapsto e^{\gamma t}\,(\alpha\mathbb{1}) \circ \varphi(t, \cdot)$$

is well-defined, linear and continuous. It follows that $S_{F,\gamma}$ is a well-defined C_0-semigroup on $W^{1,p}(I) = W^{1,p}_*(I) \oplus \operatorname{span}\{\mathbb{1}\}$ such that $W^{1,p}_*(I)$ is $S_{F,\gamma}$-invariant.

The generator of $S_{F,\gamma|W^{1,p}_*(I)}$ is given by $(\Phi\circ A\circ\Phi^{-1}, \Phi(D(A)))$, where $(A, D(A))$ is the generator of $(e^{\gamma t}T(t))_{t\geq 0}$ in $L^p(I)$ which by Theorem 15 is

$$A : \{f \in L^p(I);\, Ff' \in L^p(I)\} \to L^p(I), Af = Ff' + F'f + \gamma f.$$

Therefore

$$\Phi(D(A)) = \{f \in W^{1,p}_*(I);\, Ff'' \in L^p(I)\}$$

and

$$\forall\, f \in \Phi(D(A)) :\ \Phi \circ A \circ \Phi^{-1}(f) = \Phi(Ff'' + F'f' + \gamma f') = Ff' + \gamma f.$$

Obviously, $\operatorname{span}\{\mathbb{1}\}$ is contained in the domain of the generator $(B, D(B))$ of $S_{F,\gamma}$ with $B\mathbb{1} = \gamma\mathbb{1}$. Therefore,

$$D(B) = \Phi(D(A)) \oplus \operatorname{span}\{\mathbb{1}\} = \{f \in W^{1,p}(I);\, Ff'' \in L^p(I)\}$$

and

$$\forall\, f \in D(B) :\ Bf = \Phi \circ A \circ \Phi^{-1}(f - f(a)\mathbb{1}) + \gamma f(a)\mathbb{1} = Ff' + \gamma f.$$

□

Using the above theorem together with the results stated in section 2 we are now able to prove the following theorem.

Theorem 21. *Let $I = (a, b)$ be a bounded interval, $1 \leq p < \infty, \gamma \in \mathbb{R}$, and $F : [a, b] \to \mathbb{R}$ a C^1-function as above with $F(a) = 0$. Then the following holds.*

a) *The C_0-semigroup $S_{F,\gamma}$ is not hypercyclic on $W^{1,p}(I)$.*

b) *On $W^{1,p}_*(I)$ the following are equivalent for the C_0-semigroup $S_{F,\gamma}$.*

  i) *$S_{F,\gamma}$ is weakly mixing on $W^{1,p}_*(I)$,*

  ii) *$S_{F,\gamma}$ is hypercyclic on $W^{1,p}_*(I)$,*

iii) $\lambda(\{F=0\})=0$ *and for every* $m\in\mathbb{N}$ *for which there are* m *different connected components* $C_1,\ldots,C_m$ *of* $I\backslash\{F=0\}$, *for* λ^m*-almost all choices of* $(x_1,\ldots,x_m)\in\Pi_{j=1}^m C_j$ *there is a sequence of positive numbers* $(t_n)_{n\in\mathbb{N}}$ *tending to infinity such that for every* $1\leq j\leq m$

$$\lim_{n\to\infty}\chi_{\varphi(t_n,I)}(x_j)\,e^{p\gamma t_n}\,\partial_2\varphi(-t_n,x_j)^{1-p}=0$$

and

$$\lim_{n\to\infty}e^{-p\gamma t_n}\frac{\partial_2\varphi(t_n,x_j)^{1+p}}{\partial_2\varphi(2t_n,x_j)^p}=0.$$

c) *On* $W_*^{1,p}(I)$ *the following are equivalent for the* C_0*-semigroup* $S_{F,\gamma}$.
  i) $S_{F,\gamma}$ *is mixing on* $W_*^{1,p}(I)$,
  ii) $\lambda(\{F=0\})=0$ *and for* λ*-almost every* $x\in I$

$$\lim_{t\to\infty}\chi_{\varphi(t,I)}(x)\,e^{p\gamma t}\,\partial_2\varphi(-t,x)^{1-p}=\lim_{t\to\infty}e^{-p\gamma t}\frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p}=0.$$

d) *On* $W_*^{1,p}(I)$ *the following are equivalent for the* C_0*-semigroup* $S_{F,\gamma}$.
  i) $S_{F,\gamma}$ *is chaotic on* $W_*^{1,p}(I)$,
  ii) $\lambda(\{F=0\})=0$ *and for every* $m\in\mathbb{N}$ *for which there are* m *different connected components* $C_1,\ldots,C_m$ *of* $I\backslash\{F=0\}$, *for* λ^m*-almost all choices of* $(x_1,\ldots,x_m)\in\Pi_{j=1}^m C_j$ *there is* $t>0$ *such that*

$$\sum_{l=1}^\infty\chi_{\varphi(lt,I)}(x)\,e^{p\gamma lt}\,\partial_2\varphi(-lt,x)^{1-p}+\sum_{l=1}^\infty e^{-p\gamma lt}\frac{\partial_2\varphi(lt,x)^{1+p}}{\partial_2\varphi(2lt,x)^p}<\infty$$

for every $1\leq j\leq m$.

Proof. Let $P_1:W^{1,p}(I)\to\operatorname{span}\{\mathbb{1}\}, P_1(f)=f(a)\mathbb{1}$. As $F(a)=0$ we have $\varphi(t,a)=a$ for all $t\geq 0$ which implies $P_1\circ S(t)=S(t)\circ P_1$. Since there are no hypercyclic C_0-semigroups on finite dimensional spaces (for the single operator analogue of this, see e.g. [15, Corollary 2.59]) part a) follows from the comparison principle.

Now, let Φ be as in Lemma 17. Since $\big(\Phi^{-1}\circ S(t)\circ\Phi\big)_{t\geq 0}=T_{F,F'+\gamma}$ and $\Phi^{-1}\big(W_*^{1,p}(I)\big)=L^p(I)$ it follows from the comparison principle and Theorem 8 a) for $\rho=1$ that i) and ii) in b) are equivalent and that these are equivalent to hypercyclicity of $T_{F,F'+\gamma}$ on $L^p(I)$. For $h(x)=F'(x)+\gamma$ and $\rho(x)=1$ it follows that for $\rho_{t,p}$ and $\rho_{-t,p}$ from definition 4 we have

$$\forall\, t\geq 0, x\in(a,b):\rho_{t,p}(x)=\chi_{\varphi(t,I)}(x)h_t^p(\varphi(-t,x))\partial_2\varphi(-t,x)$$

as well as

$$\forall\, t\geq 0, x\in(a,b):\rho_{-t,p}(x)=h_t^{-p}(x)\partial_2\varphi(t,x).$$

Observe that for $h(x)=F'(x)+\gamma$ we have by Proposition 11

$$\forall\, t\geq 0, x\in(a,b):h_t(x)=\exp(\gamma t+\int_0^t F'(\varphi(s,x))\,ds)=e^{\gamma t}\partial_2\varphi(t,x).$$

Moreover, because $\varphi(s+t,x)=\varphi(s,\varphi(t,x))$ for all $s,t\in\mathbb{R}$ and each $x\in(a,b)$ for which the involved quantities are defined it follows

$$\forall\, t\geq 0, x\in(a,b):\ \partial_2\varphi(2t,x)=\partial_2\varphi(t,\varphi(t,x))\partial_2\varphi(t,x)$$

and thus for every $x\in(a,b)$ we have

$$\forall\, t\geq 0:\ \partial_2\varphi(t,\varphi(t,x))=\frac{\partial_2\varphi(2t,x)}{\partial_2\varphi(t,x)}$$

as well as

$$\forall\, t\geq 0:\ 1=\partial_2\varphi(t,\varphi(-t,x))\partial_2\varphi(-t,x)$$

for every $x\in\varphi(t,I)$. Taking all this into account it follows

$$\begin{aligned}\forall\, t\geq 0, x\in(a,b):\rho_{t,p}(x) &= \chi_{\varphi(t,I)}(x)\, e^{p\gamma t}\left(\partial_2\varphi(t,\varphi(-t,x))\right)^p\partial_2\varphi(-t,x)\\ &= \chi_{\varphi(t,I)}(x)\, e^{p\gamma t}\left(\partial_2\varphi(-t,x)\right)^{1-p}\end{aligned}$$

as well as

$$\begin{aligned}\forall\, t\geq 0, x\in(a,b):\ \rho_{-t,p}(x) &= e^{-p\gamma t}\left(\partial_2\varphi(t,\varphi(t,x))\right)^{-p}\partial_2\varphi(t,x)\\ &= e^{-p\gamma t}\frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p}.\end{aligned}$$

By Theorem 8 a) $T_{F,F'+\gamma}$ is hypercyclic on $L^p(I)$ if and only if $\lambda(\{F=0\})=0$ and for every $m\in\mathbb{N}$ for which there are m different connected components $C_1,\dots,C_m$ of $I\backslash\{F=0\}$, for λ^m-almost all choices of $(x_1,\dots,x_m)\in\Pi_{j=1}^m C_j$ there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that

$$\forall\, 1\leq j\leq m:\ \lim_{n\to\infty}\rho_{t_n,p}(x_j)=\lim_{n\to\infty}\rho_{-t_n,p}(x_j)=0,$$

so that by the above considerations b) follows.

The proofs of part c) and d) of the theorem follow by exactly the same arguments by referring to Theorem 8 b) and Theorem 6, respectively. □

So far, we have only considered γ to be a real constant. If $h\in W^{1,\infty}(I)$ we would like to have similar results to the above for the C_0-semigroup on $W^{1,p}(I)$ generated by

$$f\mapsto Ff'+hf.$$

Since $h\in W^{1,\infty}(I)$ it follows that the corresponding multiplication operator

$$M_h: W^{1,p}(I)\to W^{1,p}(I), M_h(f)=hf$$

is well-defined and continuous. If we denote the generator of $S_{F,0}$ in $W^{1,p}(I)$ by $(A,D(A))$ it follows that $(A+M_h,D(A))$ generates a C_0-semigroup $S_{F,h}$. By a well-known pertubation result (see e.g. [14, Theorem III.1.10]) this semigroup is given by

$$S(t)f=\sum_{n=0}^{\infty}T_n(t)f, \tag{5}$$

where $T_0(t)=S_{F,0}(t)$ and

$$\begin{aligned}T_{n+1}(t)f &= \int_0^t S_{F,0}(t-s)M_hT_n(s)f ds\\ &= \int_0^t h(\varphi(t-s,\cdot))\Big(T_n(s)f\Big)(\varphi(t-s,\cdot))ds,\end{aligned}$$

where the integrals are Riemann integrals in $W^{1,p}(I)$. In order to get an explicit form of $S_{F,h}$ we use the following well-known elementary observation.

Proposition 22. *Let $v:[0,\infty)\to\mathbb{C}$ be continuous. Then*

$$\forall\, n\in\mathbb{N}_0, t\geq 0:\ \int_0^t v(t-s)\Big(\int_0^s v(t-s+r)dr\Big)^n ds=\frac{1}{n+1}\Big(\int_0^t v(s)ds\Big)^{n+1}.$$

Proof. Setting $V(t) := \int_0^t v(s)ds$ it follows that

$$\begin{aligned}
\int_0^t v(t-s)\Big(\int_0^s v(t-s+r)dr\Big)^n ds &= \int_0^t V'(t-s)\Big(V(t)-V(t-s)\Big)^n ds \\
&= \sum_{k=0}^n \binom{n}{k} V^k(t)(-1)^{n-k}\int_0^t V'(u)V^{n-k}(u)du \\
&= \sum_{k=0}^n \binom{n}{k} V^k(t)(-1)^{n-k}\frac{1}{n+1-k}V^{n+1-k}(t) \\
&= \frac{-1}{n+1}V^{n+1}(t)\sum_{k=0}^n \binom{n+1}{k}(-1)^{n+1-k} \\
&= \frac{1}{n+1}\Big(\int_0^t v(s)\,ds\Big)^{n+1}.
\end{aligned}$$

□

Using Proposition 22 for $t \mapsto h(\varphi(t,x))$ with fixed $x \in [a,b]$ and the fact that point evaluations in $W^{1,p}(I)$ are continuous it follows by induction on n that the above operators $T_n(t)$ in equation (5) are given by

$$T_n(t)f(x) = \frac{1}{n!}\Big(\int_0^t h(\varphi(s,x))ds\Big)^n S_{F,0}(t)f(x)$$

which in turn implies the expected expression for the C_0-semigroup $S_{F,h}$

$$\forall\, t \geq 0, f \in W^{1,p}(I) : S_{F,h}(t)f(x) = S(t)f(x) = h_t(x)f(\varphi(t,x)),$$

where again $h_t(x) = \exp(\int_0^t h(\varphi(s,x))ds)$. Thus we obtain the next proposition.

Proposition 23. *Let $I = (a,b)$ be a bounded interval, $1 \leq p < \infty, F : [a,b] \to \mathbb{R}$ a C^1-function as above with $F(a) = 0$ and $h \in W^{1,\infty}(I)$. Then*

$$A : \{f \in W^{1,p}(I);\, Ff'' \in L^p(I)\}, f \mapsto Ff' + hf$$

generates the C_0-semigroup $S_{F,h}$ on $W^{1,p}(I)$ which is given by

$$\forall\, t \geq 0 : S_{F,h}(t)f(x) = S(t)f(x) = h_t(x)f(\varphi(t,x)).$$

Moreover, $W^{1,p}_*(I)$ is $S_{F,h}$-invariant.

Our next aim is to generalize the content of Theorem 21 to the C_0-semigroups $S_{F,h}$, at least under some mild additional hypothesis. As in section 2 we want to find a measurable function $\psi : [a,b] \to \mathbb{R}$ with $\exp\circ\psi$ inducing a continuous, invertible multiplication operator M on $W^{1,p}(I)$ such that for some $\gamma \in \mathbb{R}$ the C_0-semigroups $S_{F,\gamma}$ and $S_{F,h}$ are conjugate via M, i.e.

$$\forall\, t \geq 0, f \in W^{1,p}(I) :\, \exp(\psi)\, S_{F,h}(t)(f) = S_{F,\gamma}(t)(\exp(\psi)f).$$

This is satisfied, if for all $x \in [a,b]$

$$\forall\, t \geq 0 :\, \psi(\varphi(t,x)) - \int_0^t (h(\varphi(s,x)) - \gamma)\, ds = \psi(x). \tag{6}$$

Now, if $x \in \{F = 0\}$ it follows from $\varphi(t,x) = x$ for all $t \geq 0$

$$\forall\, t \geq 0 : \psi(x) - t(h(x) - \gamma) = \psi(x).$$

Thus, since $F(a) = 0$ it is necessary that $h(x) = h(a)$ for every $x \in \{F = 0\}$.

As in section 2, if the function $[a,b] \to \mathbb{R}, y \mapsto \frac{h(y)-h(a)}{F(y)}$ belongs to $L^1(a,b)$ we can set

$$\psi : [a,b] \to \mathbb{R}, \psi(x) = \int_a^x \frac{h(y)-h(a)}{F(y)}dy$$

and it follows that this function ψ satisfies (6) on $\{F \neq 0\}$.

Proposition 24. *Let $I = (a, b)$ be a bounded interval, $1 \leq p < \infty, F : [a, b] \to \mathbb{R}$ a C^1-function as usual with $F(a) = 0$, and $h \in W^{1,\infty}(I)$. Assume that the following conditions are satisfied.*

i) $\forall\, x \in \{F = 0\} : h(x) = h(a)$.
ii) *The function $[a, b] \to \mathbb{R}, y \mapsto \frac{h(y)-h(a)}{F(y)}$ belongs to $L^\infty(I)$.*

Then $\exp \circ \psi$ with $\psi : [a, b] \to \mathbb{R}, \psi(x) = \int_a^x \frac{h(y)-h(a)}{F(y)} dy$ defines a continuous, invertible multiplication operator M such that the C_0-semigroups $S_{F,h(a)}$ and $S_{F,h}$ on $W^{1,p}(I)$ are conjugate via M.

Proof. By the observation preceding the proposition we only have to show that $\exp \circ \psi$ defines a continuous, invertible multiplication operator on $W^{1,p}(I)$. First we note that $\psi \in W^{1,\infty}(I)$ and therefore $\exp \circ \psi \in W^{1,\infty}(I)$, too. Hence,

$$M : W^{1,p}(I) \to W^{1,p}(I), Mf = \exp(\psi) f$$

is a well-defined continuous multiplication operator on $W^{1,p}(I)$ with continuous inverse given by $\exp(-\psi) f$. □

Remark 25. In the discussion preceding the above proposition we only required that $[a, b] \to \mathbb{R}, y \mapsto \frac{h(y)-h(a)}{F(y)}$ belongs to $L^1(a, b)$. Nevertheless, it is not hard to show that this function actually has to belong to $L^\infty(a, b)$ in order for $\exp(\psi)$ to induce a bounded multiplication operator on $W^{1,p}(I)$.

Obviously, the multiplication operator M from the above proposition satisfies $M(W^{1,p}_*(I)) = W^{1,p}_*(I)$ so that the restrictions of the C_0-semigroups $S_{F,h(a)}$ and $S_{F,h}$ to $W^{1,p}_*(I)$ are conjugate via M, too. Combining Theorem 21 with Proposition 24 the next theorem follows directly from the comparison principle.

Theorem 26. *Let $I = (a, b)$ be a bounded interval, $1 \leq p < \infty, F : [a, b] \to \mathbb{R}$ a C^1-function as above with $F(a) = 0$ and $h \in W^{1,\infty}(I)$. Assume that*

1) $\forall\, x \in \{F = 0\} : h(x) = h(a) \in \mathbb{R}$.
2) *The function $[a, b] \to \mathbb{R}, y \mapsto \frac{h(y)-h(a)}{F(y)}$ belongs to $L^\infty(I)$.*

Then the following holds.

a) *The C_0-semigroup $S_{F,h}$ is not hypercyclic on $W^{1,p}(I)$.*
b) *On $W^{1,p}_*(I)$ the following are equivalent for the C_0-semigroup $S_{F,h}$.*
   i) *$S_{F,h}$ is weakly mixing on $W^{1,p}_*(I)$,*
   ii) *$S_{F,h}$ is hypercyclic on $W^{1,p}_*(I)$,*
   iii) *$\lambda(\{F = 0\}) = 0$ and for every $m \in \mathbb{N}$ for which there are m different connected components $C_1, \ldots, C_m$ of $I \backslash \{F = 0\}$, for λ^m-almost all choices of $(x_1, \ldots, x_m) \in \Pi_{j=1}^m C_j$ there is a sequence of positive numbers $(t_n)_{n \in \mathbb{N}}$ tending to infinity such that for every $1 \leq j \leq m$*

$$\lim_{n\to\infty} \chi_{\varphi(t_n,I)}(x_j)\, e^{ph(a)t_n}\, \partial_2\varphi(-t_n, x_j)^{1-p} = 0$$

and

$$\lim_{n\to\infty} e^{-ph(a)t_n} \frac{\partial_2\varphi(t_n, x_j)^{1+p}}{\partial_2\varphi(2t_n, x_j)^p} = 0.$$

c) *On $W^{1,p}_*(I)$ the following are equivalent for the C_0-semigroup $S_{F,h}$.*
   i) *$S_{F,h}$ is mixing on $W^{1,p}_*(I)$,*
   ii) *$\lambda(\{F = 0\}) = 0$ and for λ-almost every $x \in I$*

$$\lim_{t\to\infty} \chi_{\varphi(t,I)}(x)\, e^{ph(a)t}\, \partial_2\varphi(-t, x)^{1-p} = \lim_{t\to\infty} e^{-ph(a)t} \frac{\partial_2\varphi(t, x)^{1+p}}{\partial_2\varphi(2t, x)^p} = 0.$$

d) *On $W^{1,p}_*(I)$ the following are equivalent for the C_0-semigroup $S_{F,h}$.*
   i) *$S_{F,h}$ is chaotic on $W^{1,p}_*(I)$,*
   ii) *$\lambda(\{F=0\})=0$ and for every $m\in\mathbb{N}$ for which there are m different connected components $C_1,\ldots,C_m$ of $I\backslash\{F=0\}$, for λ^m-almost all choices of $(x_1,\ldots,x_m)\in\Pi_{j=1}^m C_j$ there is $t>0$ such that*

$$\sum_{l=1}^{\infty}\chi_{\varphi(lt,I)}(x)\,e^{ph(a)lt}\,\partial_2\varphi(-lt,x)^{1-p}+\sum_{l=1}^{\infty}e^{-ph(a)lt}\frac{\partial_2\varphi(lt,x)^{1+p}}{\partial_2\varphi(2lt,x)^p}<\infty$$

for every $1\leq j\leq m$.

Instead of formulating analogues to Theorem 12 for $W^{1,p}_*(I)$ we apply our results to some concrete examples in the next section.

4. Applications to first order partial differential equations

Example 1. As a first application we consider for $h\in C(0,1)\cap L^\infty(0,1)$ the so-called von Foerster-Lasota equation, that is the first order partial differential equation

$$\frac{\partial}{\partial t}u(t,x)+x\frac{\partial}{\partial x}u(t,x)=h(x)\,u(t,x),\quad t\geq 0,\,0<x<1$$

with the initial condition

$$u(0,x)=v(x),\quad 0<x<1,$$

where v is a given function. Because of Theorem 15 it is natural to consider our results from section 2 for $I=(0,1), F(x)=-x$ and h and the C_0-semigroup $T_{F,h}$ on $L^p(0,1)$. We call this C_0-semigroup the von Foerster-Lasota semigroup on $L^p(0,1)$ associated with h and denote it by T_h.

Moreover, if $h\in W^{1,\infty}(0,1)$, in view of Proposition 23 it is again natural to apply our results from section 3. We call the resulting C_0-semigroup the von Foerster-Lasota semigroup on $W^{1,p}(0,1)$, respectively $W^{1,p}_*(0,1)$, associated with h and denote it by S_h. The dynamical properties of these semigroups are given in the next theorem.

In case of a real valued $h\in C[0,1]$ with $x\mapsto\frac{h(x)-h(0)}{x}$ being in $L^1(0,1)$ it was already shown by Dawidowicz and Poskrobko in [10] that T_h is chaotic on $L^p(0,1)$ if and only if T_h is hypercyclic if and only if $h(0)>-\frac{1}{p}$. In fact, it was even proved in [10] that T_h is strongly stable on $L^p(0,1)$ for $h(0)\leq-\frac{1}{p}$.

Theorem 27. a) *Let h be in $C(0,1)\cap L^\infty(0,1)$. Then the following properties of the associated von Foerster-Lasota semigroup T_h on $L^p(0,1)$ are equivalent.*
   i) *T_h is hypercyclic.*
   ii) *T_h is weakly mixing.*
   iii) *There is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to zero such that for some $x_0\in(0,1)$*

$$\lim_{n\to\infty}\int_{\alpha_n}^{x_0}\frac{\operatorname{Re}h(y)+\frac{1}{p}}{y}dy=\infty.$$

b) *Assume that for $h\in C[0,1]$ the function*

$$[0,1]\to\mathbb{R},x\mapsto\frac{h(x)-\operatorname{Re}h(0)}{x}$$

belongs to $L^1(0,1)$. Then the following properties of the associated von Foerster-Lasota semigroup T_h on $L^p(0,1)$ are equivalent.
   i) *T_h is hypercyclic.*

ii) T_h *is weakly mixing.*
iii) T_h *is mixing.*
iv) T_h *is chaotic.*
v) $\operatorname{Re} h(0) > -\frac{1}{p}$.

c) *Assume that for* $h \in W^{1,\infty}(0,1)$ *the function*

$$[0,1] \to \mathbb{R}, x \mapsto \frac{h(x) - h(0)}{x}$$

belongs to $L^\infty(0,1)$ *and that* $h(0) \in \mathbb{R}$. *Then the von Foerster-Lasota semigroup* S_h *is not hypercyclic on* $W^{1,p}(0,1)$. *For the restriction of* S_h *to* $W^{1,p}_*(0,1)$ *the following are equivalent.*

i) S_h *is hypercyclic on* $W^{1,p}_*(0,1)$.
ii) S_h *is weakly mixing on* $W^{1,p}_*(0,1)$.
iii) S_h *is mixing on* $W^{1,p}_*(0,1)$.
iv) S_h *is chaotic on* $W^{1,p}_*(0,1)$.
v) $h(0) > 1 - \frac{1}{p}$.

Proof. For $F(x) = -x$ we have $\varphi(t,x) = xe^{-t}$ and thus the hypotheses of Theorem 13 are satisfied.

Proof of part a). Since $(0,1) \to \mathbb{R}, x \mapsto \frac{\operatorname{Im} h(x)}{x}$ belongs to $L^1(\beta,1)$ for each $\beta \in (0,1)$ part a) is obviously a direct application of Theorem 13 a).

Proof of part b). Since the hypotheses of a) are satisfied, it follows that i) and ii) are equivalent and that because of $h \in C[0,1]$ i) holds if and only if

$$\lim_{n\to\infty} \int_{\alpha_n}^1 \frac{\operatorname{Re} h(y) + \frac{1}{p}}{y} dy = \infty \tag{7}$$

for some $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to 0. Since

$$\begin{aligned} \int_{\alpha_n}^1 \frac{\operatorname{Re} h(y) + \frac{1}{p}}{y} dy &= \int_{\alpha_n}^1 \frac{\operatorname{Re} h(0) + \frac{1}{p}}{y} dy + \int_{\alpha_n}^1 \frac{\operatorname{Re}(h(y) - h(0))}{y} dy \\ &= -\Big(\operatorname{Re} h(0) + \frac{1}{p}\Big) \ln(\alpha_n) + \int_{\alpha_n}^1 \frac{\operatorname{Re}(h(y) - h(0))}{y} dy \end{aligned}$$

it follows from the hypothesis on h that (7) holds if and only if $\operatorname{Re} h(0) + \frac{1}{p} > 0$. Hence, i), ii), and v) in b) are equivalent. As iii) and iv) imply i), respectively, it remains to prove that v) implies iii) and iv).

Let us denote by c the $L^1(0,1)$-norm of the L^1-function $x \mapsto \frac{h(x) - \operatorname{Re} h(0)}{x}$. If v) holds, it follows that for each $x \in (0,1), t > 0$, and $l \in \mathbb{N}$ we have

$$\begin{aligned} \exp\Big(p \int_x^{xe^{-lt}} \frac{\operatorname{Re} h(y) + \frac{1}{p}}{y} dy\Big) &= \exp\Big(-p \int_{xe^{-lt}}^x \frac{\operatorname{Re} h(0) + \frac{1}{p}}{y} dy\Big) \cdot \\ &\quad \cdot \exp\Big(-p \int_{xe^{-lt}}^x \frac{\operatorname{Re}(h(y) - h(0))}{y} dy\Big) \\ &\leq \Big(\exp(-pt(\operatorname{Re} h(0) + \frac{1}{p}))\Big)^l e^{pc}. \end{aligned} \tag{8}$$

Because of v) it follows that for every $x \in (0,1)$ and $t > 0$

$$\sum_{l=1}^\infty \exp\Big(p \int_x^{xe^{-lt}} \frac{\operatorname{Re} h(y) + \frac{1}{p}}{y} dy\Big) < \infty,$$

hence T_h is chaotic by part c) of Theorem 13.

In order to show that v) implies iii) we observe that as in inequality (8) we obtain for each $x \in (0,1)$ and $t \geq 0$

$$\exp\Big(p\int_x^{xe^{-t}} \frac{\operatorname{Re} h(y) + \frac{1}{p}}{y} dy\Big) \leq \exp(-pt(\operatorname{Re} h(0) + \frac{1}{p}))e^{pc},$$

so that by part b) of Theorem 13 iii) holds. Hence, b) is proved.

Proof of part c). That S_h is not hypercyclic on $W^{1,p}(0,1)$ follows immediately from Theorem 26. As $[0,1]\backslash\{F = 0\}$ has only one connected component, $\chi_{\varphi(t,I)}(x) = 0$ for every $x \in (0,1]$ for sufficiently large $t > 0$, and

$$\forall\, x \in (0,1]:\ e^{-ph(0)t}\frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p} = e^{-t\big(1+(h(0)-1)p\big)}$$

the rest of c) now follows from Theorem 26 b),c), and d). □

Remark 28. i) Of course Theorem 13 also provides characterizations of mixing and chaos for T_h under the general hypotheses of part a) of the above theorem.
ii) It should be noted that the proof of part b) remains valid if we replace the hypothesis $h \in C[0,1]$ by $h \in C(0,1) \cap L^\infty(0,1)$, h continuously extendable into the origin.
iii) The hypothesis in c) is obviously satisfied if $h(0) \in \mathbb{R}$ and h is differentiable at the origin.

Remark 29. In [8, 11] the authors prove that, if h is a continuous real function on $[0,1]$ such that $\dfrac{h(x) - h(0)}{x} \in L^\infty(0,1)$, then the restriction of T_h is a C_0-semigroup on the space

$$V_\alpha = \{h \in h^\alpha([0,1]) \mid h(0) = 0\},$$

where $h^\alpha([0,1])$ is the little Hölder space of order $\alpha \in]0,1[$, that is the closure of $C^1[0,1]$ in the Hölder space $C^\alpha[0,1]$. Observe that h^α is a separable space (see [21]) and since for $p = (1-\alpha)^{-1}$

$$W^{1,p} \hookrightarrow C^\alpha[0,1]$$

continuosly, it is immediate to show that $W^{1,p}_*(0,1)$ is continuously embedded with dense range in V_α.

As a consequence, by Theorem 27, if $h(0) > \alpha = 1 - 1/p$, then the C_0-semigroup T_h is mixing and chaotic on V_α. Thus we obtain the result of [11, Theorem 2.4] and [8, Theorem 3.10 and Corollary 3.11].

Example 2. Let us consider for $h \in C(0,1) \cap L^\infty(0,1)$ and $r > 1$ the first order partial differential equation

$$\frac{\partial}{\partial t}u(t,x) + x^r\frac{\partial}{\partial x}u(t,x) = h(x)\,u(t,x),\ \ t \geq 0,\ 0 < x < 1$$

with the initial condition

$$u(0,x) = v(x),\ \ 0 < x < 1,$$

where v is a given function. As for $F(x) = -x^r$ we have

$$\varphi(t,x) = \frac{x^{\frac{1}{r-1}}}{(1 + (r-1)xt)^{\frac{1}{r-1}}}$$

and therefore $\varphi([0,\infty),(0,1)) \subseteq (0,1)$, it is again natural to consider our results from section 2 for $I = (0,1)$ and h and the C_0-semigroup $T_{F,h}$ on $L^p(0,1)$. We write $T_{r,h}$ instead of $T_{F,h}$.

Again, if $h \in W^{1,\infty}(0,1)$, in view of Proposition 23 it is natural to apply our results from section 3. The corresponding C_0-semigroup on $W^{1,p}(0,1)$, respectively $W^{1,p}_*(0,1)$ is denoted by $S_{r,h}$. The next theorem summarizes the dynamical properties of these semigroups. Observe that contrary to the von Foerster-Lasota semigroup the dynamical properties of $S_{r,h}$ are independent of p!

Theorem 30. a) *Let h belong to $C(0,1)\cap L^\infty(0,1)$. Then the following properties of the C_0-semigroup $T_{r,h}$ on $L^p(0,1)$ are equivalent.*

i) *$T_{r,h}$ is hypercyclic.*
ii) *$T_{r,h}$ is weakly mixing.*
iii) *There is a sequence $(r_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to zero such that for some $x_0 \in (0,1)$*

$$\lim_{n\to\infty}\int_{r_n}^{x_0}\frac{\operatorname{Re} h(y)+\frac{r}{p}y^{r-1}}{y^r}dy=\infty.$$

b) *Assume that for $h \in C[0,1]$ the function*

$$[0,1]\to\mathbb{R}, x\mapsto\frac{h(x)-x^{r-1}\operatorname{Re} h(0)}{x^r}$$

belongs to $L^1(0,1)$. Then the following properties of $T_{r,h}$ on $L^p(0,1)$ are equivalent.

i) *$T_{r,h}$ is hypercyclic.*
ii) *$T_{r,h}$ is weakly mixing.*
iii) *$T_{r,h}$ is mixing.*
iv) *$T_{r,h}$ is chaotic.*
v) $\operatorname{Re} h(0) > \frac{-r}{p}$.

c) *Assume that for $h \in W^{1,\infty}(0,1)$ the function*

$$[0,1]\to\mathbb{R}, x\mapsto\frac{h(x)-h(0)}{x^r}$$

belongs to $L^\infty(0,1)$ and that $h(0)\in\mathbb{R}$. Then the C_0-semigroup $S_{r,h}$ is not hypercyclic on $W^{1,p}(0,1)$. For the restriction of $S_{r,h}$ to $W^{1,p}_*(0,1)$ the following are equivalent.

i) *$S_{r,h}$ is hypercyclic on $W^{1,p}_*(0,1)$.*
ii) *$S_{r,h}$ is weakly mixing on $W^{1,p}_*(0,1)$.*
iii) *$S_{r,h}$ is mixing on $W^{1,p}_*(0,1)$.*
iv) *$S_{r,h}$ is chaotic on $W^{1,p}_*(0,1)$.*
v) $h(0) > 0$.

Proof. For $F(x) = -x^r$ we have $\varphi(t,x) = x^{\frac{1}{r-1}}(1+(r-1)xt)^{\frac{1}{1-r}}$ and thus the hypotheses of Theorem 13 are satisfied. The proofs of parts a) and b) are mutatis mutandis the same as the correspondig parts of Theorem 27. In order to prove part c) we observe

$$\forall\, t\geq 0, x\in(0,1]:\ \partial_2\varphi(t,x)=\frac{1}{r-1}\frac{x^{\frac{1}{r-1}-1}}{(1+(r-1)xt)^{\frac{r}{r-1}}}$$

and thus

$$\begin{aligned}\forall\, t\geq 0, x\in(0,1]: &\qquad \exp(-ph(0)t)\frac{\partial_2\varphi(t,x)^{1+p}}{\partial_2\varphi(2t,x)^p}\\ &= \exp(-ph(0)t)t^{-\frac{r}{r-1}}x^{\frac{1}{r-1}-1}\frac{(\frac{1}{t}+2(r-1)x)^{p\frac{r}{r-1}}}{(\frac{1}{t}+(r-1)x)^{(1+p)\frac{r}{r-1}}}.\end{aligned}$$

Now the proof follows again as the proof of part c) of Theorem 27. □

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Javier Aroza, Institut Universitari de Matemàtica Pura i Aplicada, Edifici 8E, Universitat Politècnica de València, E-46022 València, Spain
E-mail address: jaarben@upvnet.upv.es

Thomas Kalmes, Faculty of Mathematics, Chemnitz Technical University, 09107 Chemnitz, Germany
E-mail address: thomas.kalmes@math.tu-chemnitz.de

Elisabetta M. Mangino, Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, I-73100 Lecce, Italy
E-mail address: elisabetta.mangino@unisalento.it