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Composite Media and Asymptotic Dirichlet Forms

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We introduce a metric topology on the space of Dirichlet forms and study compactness and closure properties of families of local and non-local forms.

INTRODUCTION

Loosely speaking, a *composite medium* is a body which exhibits a fragmented and physically heterogeneous structure at a "microscopic" scale—as in the case of a fine mixture of two materials of different physical characteristics—and whose behavior is observed at a larger "macroscopic" scale.

An adequate mathematical description, for example of the spectral properties of the body, can be based on a variational principle of a suitable *asymptotic* character. The general pattern can be outlined as follows. A sequence of *approximate* energy forms is first written down, which matches the specific fine structure of the body. A limit form is shown to exist, which is interpreted as the *energy form* of the underlying composite. An explicit integral expression is then given to the energy, builded on *effective characteristics* for the composite.

For mixtures that are periodically spaced, this method is known as *homogenization*. The approximate energies are then simply obtained by scaling-down an initial energy form possessing a suitable periodic structure and, in the limit, *constant* effective characteristics are found that describe the homogenized body.

This variational approach has been extended also to *non-periodic* composites, with various degree of generality and detail. Moreover, some useful tools such as appropriate *variational convergences* have been developed,

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which have greatly widened the range of application of the method. As general references to the subject we mention, for example, [A, BLP, HM, KONZ, RT].

The available general theory, however, seems to be inadequate to describe highly non-homogeneous and non-isotropic structures, like the *thin-layer* models described in Example 6.4.1 and Example 6.5.1 of Section 6, which involve measure-valued singular conductivities or measure-valued gradients. Falling outside the existing theory, confined to local functionals, are also the *non-local* examples of Section 6, namely, Example 6.1.1 to Example 6.3.1, in which asymptotic non-local potentials arise from the limit of pointwise singular conductivities.

The aim of this paper is to show that the beautiful theory of the *Dirichlet forms*, initiated by A. Beurling and J. Deny [BD1, BD2], provides an appropriate functional framework to the variational description of such irregular media and to the asymptotic variational principles of the kind mentioned before. The main new tools are suitable *topologies* on the space of Markovian forms and related *compactness* and *closure* theorems.

Before proceeding to describe the content of the paper, let us remark that the role of Dirichlet forms in dealing with selfadjoint operators not explicitely realizable as classic boundary value problems is well known. As a recent application of this kind we refer, for example, to the stochastic models of the euclidean quantum field theories, in the approach of R. Høegh-Krohn, S. Albeverio, M. Röckner, and others; see, for instance, [AR] and references therein.

We should also mention that Dirichlet forms are strictly related to quite general classes of Markov processes, as shown by M. L. Silverstein [S1, S2], M. Fukushima [F], and others. Despite this relation provides a rich probabilistic background, very apt to the description of the "irregularities" that are the object of our study, we shall confine ourselves in the present paper to the "analytic" theory and methods, having in mind a PDE approach. However, we shall occasionally adopt the suggestive probabilistic terminology.

The basic definitions and properties of Markovian and Dirichlet forms needed in the following are summarized in Section 1 and Section 3. The topological notions on the space of forms are discussed in Section 2 and Section 4. They may have an independent interest in the general theory of Dirichlet forms. The application to composite media is given in Section 5 and Section 6.

More precisely, in Section 2 we consider the space of all Markovian, symmetric forms defined on a Hilbert space $H = L^2(X, m)$, where X is a separable measure space and m a given σ-finite positive measure on X. On this space of forms we then define two *variational convergences*.

The first convergence, which is of a metrizable topological nature, is a special case of a variational convergence for convex sets and functionals introduced in [M1, M3], in connection with a perturbation theory for variational inequalities and related boundary value problems. Its main feature is that it can be caracterized in terms of convergence of the *resolvent operators*, *semigroups*, and *spectral families* associated with the forms, as described by Theorems 2.4.1 and Corollaries 2.6.1 and 2.7.1 of Section 2. A further property of this convergence, not exploited however in the present paper, is its stability under Legendre duality [M4, J].

A second, weaker convergence in the space of forms is the one related to the so-called *Γ-convergence* of functionals. This convergence was introduced by E. De Giorgi and T. Franzoni [DGF], in the general context of relaxation and functional convergence in the calculus of variations; see [A] for further references. In the present setting, the main interest of this convergence relies on the *compactness* properties it gives to quite general families of forms, as described by Theorem 2.8.1 of Section 2.

In Section 4, we further describe the general variational theory of Section 2 in the more explicit framework of *regular* Dirichlet forms in $H = L^2(X, m)$, X being now a separable locally compact space and m a positive Radon measure on X.

We first apply the compactness results of Section 2 under an *asymptotic regularity* assumption on the sequences of forms. This allows us to make use of the Beurling and Deny representation theory of regular Dirichlet forms and leads to useful integral expression for the asymptotic energies, Theorem 4.1.2. Furthermore, under an additional *asymptotic compactness* assumption on the sequence of forms, additional compactness properties for the sequence of the resolvent operators associated with the forms, as well as for their semigroups and spectral families, are also established, Theorem 4.2.1.

Forms of diffusion type may converge to forms with non-trivial killing or jumping parts, as shown by Examples 6.1.1 to 6.3.1 of Section 6. Therefore, a deeper analysis of the convergence properties is carried on, in order to prove *closure* theorems for families of local forms, Theorem 4.3.2, and strongly local forms, Theorem 4.4.1.

In Section 5, as already mentioned, we interpret the results of Section 2 and Section 4 from the point of view of composite media. In particular, we introduce suitable *irregular* Dirichlet forms, with arbitrary *Borel measures* in the role of killing and jumping measures, and show that they provide an appropriate tool for the variational definition of the effective characteristics of quite general classes of composite media, as outlined at the beginning of this Introduction.

This approach goes very much along the lines of [DMM1, DMM2, BDMM, DMGM]. In [DMM1, DMM2, BDMM], Borel killing measures were introduced in connection with so-called *relaxed Dirichlet problems*, in order to give a unifying framework both to Dirichlet problems with homogeneous boundary conditions on possibly very irregular sets and to Schrödinger equations with possibly highly singular potentials. In [DMGM], Borel jumping measures were used to describe energy forms on domains and manifolds with non-trivial and possibly wildly changing *topological type*.

In the final Section 6 we describe some examples that illustrate in particular the results of Section 5.

The main results of this paper were presented in [M5].

1. Preliminaries on Dirichlet Forms

We summarize the main definitions and properties of Dirichlet forms in a separable Hilbert space.

(a) *The Setting*

We consider the Hilbert space $H = L^2(X, m)$, where X is a given separable measurable space and m a σ-finite positive measure on X.

By $(u, v) = \int_X uvm(dx)$ we denote the inner product of H, and by $\|\cdot\|$ the related norm.

(b) *Forms*

A *form* a in H will be any non-negative definite symmetric bilinear form $a(u, v)$ defined on a linear subspace $D[a]$ of H, the *domain* of a. We point out that every form is intended here to be non-negative definite and symmetric. Moreover, the forms are not assumed *a priori* to be densely defined in H.

If a is a form in H, we extend the quadratic functional $a(u, u)$, $u \in D[a]$, on the whole space H, by defining $a(u, u) = +\infty$ for every $u \in H - D[a]$. Rather improperly, we shall denote this functional by $a(u, u)$ leaving to the context to distinguish when the same notation is used to denote the value of the functional at a given u of H. Also, we put $D[a(u, u)] := \{u \in H : a(u, u) < +\infty\}$, therefore $D[a] = D[a(u, u)]$.

The form and its quadratic functional are related by the polarization identity,

$$a(u, v) = \tfrac{1}{2}\{a(u+v, u+v) - a(u, u) - a(v, v)\}$$
$$\text{for every} \quad u, v \in D[a] = D[a(u, u)],$$

which uniquely defines one in term of the other.

We shall occasionally use the notation $F := D[a]$ and $F_b := D[a]_b$, where $D[a]_b := D[a] \cap L^\infty(X, m)$.

(c) *Closed Forms*

A form a is *closed* in H if its domain $D[a]$ is complete under the inner product $a(u, v)+(u, v)$. This inner product will be referred to as the *intrinsic inner product* (or, *metric*) of $D[a]$.

The closedness of a given form in H can be read on its quadratic functional only. In fact, the following useful characterization holds:

> *A form a is closed in H if and only if the quadratic functional $a(u, u)$ is lower semicontinuous on H.*

Remark. This property plays a basic role in the following. Let us recall that the implication "a *closed* $\Rightarrow a(u, u)$ l.s.c." is a consequence of the reflexivity of $D[a]$ under the intrinsic norm. In fact, if $u_n \to u$ in H and $\liminf a(u_n, u_n) < +\infty$, then $u_n \in D[a]$ and a subsequence exists that converges weakly in $D[a]$, hence also in H, to a vector that necessarily coincides with u; therefore $u \in D[a]$ and $a(u, u) \leqslant \liminf a(u_n, u_n)$. We note, incidentally, that the analogous implication is violated if, for instance, we replace $D[a]$ with the Sobolev space $W^{1,1}$, which is indeed complete for the norm $\|u\| + \|Du\|$, $\|\cdot\|$ being here the L^1 norm, whereas the domain of the relaxation in L^1 of the functional $\|Du\|$ is known to be the space BV and $W^{1,1} \subsetneqq BV \subsetneqq L^1$. For completeness, we sketch also the simple proof of the opposite implication "$a(u, u)$ l.s.c $\Rightarrow a$ *closed*." Let $u_n \in D[E]$, $(u_n - u_m, u_n - u_m) + a(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$. Then, there exists $u \in H$ such that $u_m \to u$ in H as $m \to \infty$. By denoting the quadratic functional $a(u, u)$ by F, we then have $0 \leqslant F(u_m - u_n) = a(u_m - u_n, u_m - u_n) \leqslant (u_n - u_m, u_n - u_m) + a(u_n - u_m, u_n - u_m) \to 0$ as $m, n \to \infty$. Since F is l.s.c. on H, for each fixed n we have $F(u - u_n) \leqslant \liminf F(u_m - u_n)$ as $m \to \infty$, hence $0 \leqslant \limsup_{n \to \infty} F(u - u_n) \leqslant \limsup_{n \to \infty} \liminf_{m \to \infty} F(u_m - u_n) = 0$. This implies, in particular, $F(u - u_n) < +\infty$, hence also $F(u) \leqslant 2[F(u - u_n) + F(u_n)] < +\infty$, for n large enough, hence $u \in D[a]$; moreover, $u - u_n \in D[a]$ and $a(u - u_n, u - u_n) = F(u - u_n) \to 0$ as $n \to \infty$. Therefore, $(u - u_n, u - u_n) + a(u - u_n, u - u_n) \to 0$.

A form a in H is closed if and only if there exists a non-negative self-adjoint operator $-A$ in the closure $\overline{D[a]}$ of $D[a]$ in H, with domain $D[-A] \subset D[\sqrt{-A}] = D[a]$, such that $a(u, v) = (\sqrt{-A}\, u, \sqrt{-A}\, v)$ for every $u, v \in D[a]$. Moreover, $a(u, v) = (-Au, v)$ for every $u \in D[A]$, $v \in D[a]$, see, e.g., [K]. The operator A, densely defined in $\overline{D[a]}$, is the *generator* of a.

(d) *Closure*

We will often be interested in closed forms that are initially known, or represented, only on some linear subspace of their final domain. In this respect two standard procedures exist, both depending on the space H, by which a *closed* form in H can be uniquely associated with a given non-closed form in H. The first procedure is the *closure* of a form in H and only applies to forms which are *closable* in H. The second procedure, described in (e) below, is the *relaxation* of a form in H and applies to arbitrary forms. Both procedures lead however to the same (closed) form, in case the initial form is closable.

A form a is *closable* in H if $\{u_n\} \subset D[a]$, $a(u_n - u_m, u_n - u_m) \to 0$, $(u_n, u_n) \to 0$ as $n, m \to \infty$ imply $a(u_n, u_n) \to 0$ as $n \to \infty$.

A form a is closable in H if and only if there exists a closed *extension* $\tilde{a}$ of a in H, i.e., a closed form $\tilde{a}$ with domain $D[\tilde{a}] \supset D[a]$, such that $\tilde{a}(u, v) = a(u, v)$ for every $u, v \in D[a]$.

Moreover, a form a is closable in H if and only if the (abstract) completion of $D[a]$ for the inner product $a(u, v) + (u, v)$ is *injected* in the space H.

Any closable form a in H possesses a *smallest* closed extension in H, that is an extension with smallest domain, which is necessarily unique and is denoted by $\bar{a}$ and called the *closure* of a in H. The domain $D[\bar{a}]$ of $\bar{a}$ coincides with the completion of $D[a]$ for the intrinsic inner product of a, identified with a subspace of H. Therefore, $D[\bar{a}] = \{u \in H : \exists u_n \in D[a],\ u_n \to u$ in H, $a(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty\}$ and for every $u, v \in D[\bar{a}]$, we have then unambiguously $\bar{a}(u, v) = \lim a(u_n, v_n)$ for arbitrary $u_n \to u$, $v_n \to v$ in H as $n \to \infty$, with $a(u_n - u_m, u_n - u_m) \to 0$, $a(v_n - v_m, v_n - v_m) \to 0$ as $n, m \to \infty$.

In particular, with any closable form a in H we can associate a non-negative self-adjoint operator $-A$, with (dense) domain in the closure of $D[a]$ in H, by choosing A to be the generator of the closure $\bar{a}$ of a in H. We recall that if $a(u, v) := (-Su, v)$, $D[a] = D[S]$, where $-S$ is a given non-negative definite, symmetric densely defined operator in H, then $-A$ is the *Friedrichs extension* of $-S$; see [K, VI, Sect. 3; F, Sect. 2.3].

(e) *Relaxation*

Given a form a in H, not necessarily closable, there exists a *greatest* lower semicontinuous functional on H which is a minorant of the quadratic functional $a(u, u)$ associated with a on H. This uniquely determined l.s.c. functional on H, with extended real values in $[0, +\infty]$, is also quadratic and will be denoted by $\underline{a}(u, u)$. A closed form $\underline{a}(u, v)$ is then defined by polarization on the domain $D[\underline{a}] = \{u \in H : \underline{a}(u, u) < +\infty\}$. This form, uniquely determined by the initial a, is the *relaxation* of a in H. We have $\underline{a}(u, u) \leqslant a(u, u)$ for every $u \in H$, hence $D[\underline{a}] \supset D[a]$, and for every $u \in D[\underline{a}]$, $\underline{a}(u, u) = \min\{\liminf a(u_n, u_n) : u_n \to u \text{ in } H \text{ as } n \to +\infty\}$.

If we apply the relaxation procedure to a closable form, then the relaxed form will coincide with the closure of the given form in H. In fact, if a is closable in H, then it is easy to see that $\underline{a}$ itself is an *extension* of a, indeed, it is the *smallest* closed extension of a in H. In fact, since $\bar{a}(u, u)$ is a l.s.c. extension of $a(u, u)$, it is in particular a l.s.c. minorant of $a(u, u)$ on H, hence we have $\bar{a}(u, u) \leqslant \underline{a}(u, u) \leqslant a(u, u)$ for every $u \in H$, therefore $\underline{a}(u, u) = a(u, u)$ for every $u \in D[a]$. Since $\underline{a}(u, u) \leqslant a(u, u)$ for every $u \in H$, $\underline{a}$ is an extension of a, and since $D[\underline{a}] \subset D[\bar{a}]$, $\underline{a}$ is the smallest closed extension of a in H. Therefore, if a is closable in H, then $\underline{a} = \bar{a}$.

Remark. As already noted in (c), in non-reflexive spaces closure and relaxation are distinct procedures even for closable functionals.

(f) *Markovian Forms*

A form a in H is *Markovian* if the following condition is satisfied: for every $\varepsilon > 0$ there exists $\eta_\varepsilon : \mathbb{R} \to [-\varepsilon, 1+\varepsilon]$, with $\eta_\varepsilon(t) \equiv t$ for $t \in [0, 1]$ and $0 \leqslant \eta_\varepsilon(t') - \eta_\varepsilon(t) \leqslant t' - t$ for every $t' < t$, such that $\eta_\varepsilon \circ u \in D[a]$ and $a(\eta_\varepsilon \circ u, \eta_\varepsilon \circ u) \leqslant a(u, u)$ whenever $u \in D[a]$.

From the characterization of closed forms in terms of the lower semicontinuity of their quadratic functionals, it follows easily that a closed form a in H is Markovian if and only if the following condition is satisfied:

$$u \in D[a], \qquad v := (0 \vee u) \wedge 1 \Rightarrow v \in D[a] \qquad \text{and} \qquad a(v, v) \leqslant a(u, u),$$

where $(0 \vee u) \wedge 1 = \inf\{\sup\{u, 0\}, 1\}$.

(g) *Dirichlet Forms*

A *Dirichlet form* in H is a closed Markovian form in H, or, equivalently, a closed form that satisfies last condition in (f).

If a is a Dirichlet form in H, with domain $F = D[a]$, then $F_b = D[a] \cap L^\infty(X, m)$ is an *algebra* and

$$a(uv, uv) \leqslant \|u\|_\infty^2 \, a(v, v) + \|v\|_\infty^2 \, a(u, u) \qquad \text{for every} \quad u, v \in F_b,$$

where $\|u\|_\infty$ denotes the essential sup norm of $L^\infty(X, m)$ [F, Thm. 1.4.2(ii)].

(h) *Resolvents*

Given a closed form a in H, the *resolvent* $\{G_\beta : \beta > 0\}$ is uniquely defined for each $\beta > 0$ by the identity

$$a_\beta(G_\beta u, v) = (u, v) \qquad \text{for every} \quad v \in D[a],$$

where $\alpha_\beta(u, v) := a(u, v) + \beta(u, v)$ for every $u, v \in D[a]$. The existence, given $u \in H$, of a unique $G_\beta u$ satisfying the identity above follows from the Riesz representation theorem.

This resolvent is not necessarily strongly continuous in H as $\beta \to \infty$, being such, that is, $(\beta G_\beta u - u, \beta G_\beta u - u) \to 0$ as $\beta \to \infty$, if and only if $D[a]$ is dense in H.

(i) *Deny–Yosida Approximations*

The forms $a^{(\beta)}$, defined for every $\beta > 0$ on the whole of H by

$$a^{(\beta)}(u, v) := \beta(u - \beta G_\beta u, v), \qquad u, v \in H,$$

are called the *Deny–Yosida approximations* of the form a. These forms have the property

$$D[a] = \{u \in H : \lim a^{(\beta)}(u, u) < +\infty \text{ as } \beta \to \infty\},$$
$$a(u, v) = \lim a^{(\beta)}(u, v) \text{ as } \beta \to \infty,$$

with $a^{(\beta)}(u, u)$ non-decreasing as $\beta \to \infty$ for every $u \in H$; see [F, Thm. 1.3.2].

Moreover, it is easy to check that for every $\beta > 0$ and every $u \in H$ we have

$$a^{(\beta)}(u, u) = \min_{v \in H} \{a(v, v) + \beta \|u - v\|^2\} = a(\beta G_\beta u, \beta G_\beta u) + \beta \|u - \beta G_\beta u\|^2.$$

2. Topologies on Forms

In this section we keep the general setting of Section 1(a) for the space H. We introduce two notions of convergence in the space of forms and describe related compactness and comparison properties.

2.1. *Convergence of Forms*

We first introduce a convergence in the space of forms, according to [M3], as follows:

Definition 2.1.1. A sequence of forms $\{a_h\}$ *converges* to a form a in H if:

(a) For every v_h converging *weakly* to u in H, $\liminf a_h(v_h, v_h) \geqslant a(u, u)$ as $h \to \infty$,

(b) For every $u \in H$, there exists u_h converging *strongly* to u in H, such that $\limsup a_h(u_h, u_h) \leqslant a(u, u)$ as $h \to \infty$.

According to our general notation of Section 1(b), the quadratic functionals occurring in the previous definition are defined on the whole of H and take the value $+\infty$ outside the domain of the form.

Remark. If $\{a_h\}$ converges to a in H, then the quadratic functional $a(u, u)$ is easily seen to be l.s.c. on H, therefore, as observed in Section 1(c), the form a is closed in H. Moreover, $\{a_h\}$ converges to a in H if and only if the sequence of the relaxed forms in H, $\{\underline{a}_h\}$, converges to a in H.

It can be shown that a *topology* can be introduced in the space of all closed forms on H, that makes it a *Polish space* (that is, a metrizable separable topological space, complete for a metric inducing the topology) and reduces on sequences to the convergence just defined. We shall not use general topological notions in the present paper and for more details on this point we refer to [J; A, Thm. 3.36] and to [B1, B2].

2.2. *Γ-Convergence of Forms*

We now introduce a *weaker* convergence in the space of forms, following [DGF]:

DEFINITION 2.2.1. A sequence of forms $\{a_h\}$ *Γ-converges* to a form a in H if:

(a) For every $v_h \to u$ in H, $\liminf a_h(v_h, v_h) \geq a(u, u)$ as $h \to \infty$,

(b) For every $u \in H$, there exists $u_h \to u$ in H, such that $\limsup a_h(u_h, u_h) \leq a(u, u)$ as $h \to \infty$.

Again, the quadratic forms involved in this definition are extended to be $+\infty$ on H outside their domain.

Remark. As before, if $\{a_h\}$ Γ-converges to a in H, then a is closed in H. Moreover, $\{a_h\}$ Γ-converges to a in H if and only if $\{\underline{a}_h\}$ Γ-converges to a in H.

Remark. A topology can also be defined on the space of all (closed) forms on H, which reduces to Γ-convergence on sequences, however, such a topology will not be in general a Hausdorff topology; see [DM, Thm. 9.16, 9.17; A, Sect. 2.8].

2.3. *Asymptotic Compactness of Forms*

On special subsets of forms the two convergences just defined coincide.

DEFINITION 2.3.1. We say that a sequence of forms $\{a_h\}$ is *asymptotically compact* in H if every sequence $\{u_h\}$ in H with $\liminf\{a_h(u_h, u_h) + (u_h, u_h)\} < +\infty$ has a subsequence that converges strongly in H.

If $\{a_h\}$ is asymptotically compact, then any subsequence $\{a_{h_k}\}$ of $\{a_h\}$ is also asymptotically compact. In fact, if $\{v_k\}$ is such that $\liminf[a_{h_k}(v_k, v_k) + (v_k, v_k)] < +\infty$ as $k \to \infty$, by defining $u_1 = \cdots = u_{h_1} = v_1$, $u_{h_1+1} = \cdots = u_{h_2} = v_2, \ldots$, we obtain a sequence $\{u_h\}$ such that

$\liminf\{a_h(u_h, u_h) + (u_h, u_h)\} < +\infty$, therefore a subsequence of $\{u_h\}$ exists that converges strongly in H, hence also a subsequence of $\{v_k\}$ exists, converging strongly in H.

LEMMA 2.3.2. *Let the sequence of forms $\{a_h\}$ be asymptotically compact in H. Then, $\{a_h\}$ converges to a form a in H if and only if $\{a_h\}$ Γ-converges to a in H.*

Proof. It suffices only to prove that (a) of Definition 2.2.1 implies (a) of Definition 2.1.1. Suppose that, for some sequence $\{v_h\}$ converging weakly to u in H, we have $\liminf a_h(v_h, v_h) < a(u, u)$ as $h \to \infty$. Then, by possibly extracting a subsequence, we have $\lim a_h(v_h, v_h) < a(u, u)$, as well as $\liminf[a_h(v_h, v_h) + (v_h, v_h)] < +\infty$, as $h \to \infty$. Since, as remarked above, the asymptotic compactness of the initial sequence $\{a_h\}$ is inherited by its subsequences, a subsequence $\{v_{h'}\}$ of $\{v_h\}$ exists, that converges strongly to some vector $\tilde{u}$ in H as $h' \to \infty$, and necessarily $\tilde{u} = u$. On the other hand, the subsequence $\{a_{h'}\}$ of $\{a_h\}$ Γ-converges to a. By condition (a) of Definition 2.2.1, this implies that $\liminf a_{h'}(v_{h'}, v_{h'}) \geqslant a(u, u)$ as $h' \to \infty$. Since $\{v_{h'}\}$ is a subsequence of $\{v_h\}$, for which we have $\lim a_h(v_h, v_h) < a(u, u)$ as $h \to \infty$, we have reached a contradiction. ∎

Remark. A special case of a sequence asysmptotically compact in H is that of a sequence $\{a_h\}$ which is *inf-compact* in H, in the following sense:

> There exists a compact subset K of H, independent of h, such that $\{u \in H : (u, u) + a_h(u, u) \leqslant 1\} \subset K$ for every h.

2.4. *Convergence of Resolvents*

The convergence of forms according to Definition 2.1.1 can be characterized in terms of convergence of the resolvent operators of the relaxed forms. Namely,

THEOREM 2.4.1. *A sequence of forms $\{a_h\}$ converges to a form a in H, according to Definition 2.1.1, if and only if, for every $\beta > 0$, the sequence $\{G_{h,\beta}\}$ of the resolvent operators associated with the relaxed forms $\underline{a}_h$ in H converges to the resolvent operator G_β of the form a in the strong operator topology of H.*

Proof. As remarked in Section 2.1, it is not restrictive to assume that the forms a_h are themselves closed. We first prove:

(i) If a_h converges to a, then for each $\beta > 0$ and for every $z \in H$, $u_h := G_{h,\beta} z$ converges to $u := G_\beta z$ in H, as $h \to \infty$.

The vector u is characterized as the unique minimizer of $a(v, v) + \beta(v, v) - 2(z, v)$ over H, and a similar characterization holds for each u_h.

Since the norm of $G_{h,\beta}$ as an operator of H into itself is bounded by β^{-1}, there exists a subsequence of $\{u_h\}$, still denoted $\{u_h\}$ in the following, that converges weakly to a vector $\tilde{u}$ of H. For an arbitrary given $v \in H$, by condition (b) of Definition 2.1.1 we can find a sequence $v_h \to v$ in H, such that $\limsup a_h(v_h, v_h) \leqslant a(v, v)$ as $h \to \infty$. Since for every h,

$$a_h(u_h, u_h) + \beta(u_h, u_h) - 2(z, u_h) \leqslant a_h(v_h, v_h) + \beta(v_h, v_h) - 2(z, v_h),$$

by taking condition (a) of Definition 2.1.1 into account, we find in the limit as $h \to \infty$

$$a(\tilde{u}, \tilde{u}) + \beta(\tilde{u}, \tilde{u}) - 2(z, \tilde{u}) \leqslant a(v, v) + \beta(v, v) - 2(z, v)$$

therefore $\tilde{u} = G_\beta z$. By the uniqueness of such a $\tilde{u}$, this proves that u_h converges to u weakly in H as $h \to \infty$. We now prove that $(u_h, u_h) \to (u, u)$. In fact, by condition (b), we can choose $v_h \to u$ in H, such that $\lim a_h(v_h, v_h) = a(u, u)$ as $h \to \infty$, therefore, by rewriting the first inequality above as

$$a_h(u_h, u_h) + \beta \, \|u_h - z/\beta\|^2 \leqslant a_h(v_h, v_h) + \beta \, \|v_h - z/\beta\|^2,$$

we get in the limit, again by condition (a),

$$\beta \limsup \|u_h - z/\beta\|^2 \leqslant -a(u, u) + a(u, u) + \beta \, \|u - z/\beta\|^2,$$

hence $\|u_h - z/\beta\|^2 \to \|u - z/\beta\|^2$, and this concludes the proof of (i).

The proof of the opposite implication will be split in two steps. We first prove:

(j) If, for each $\beta > 0$, $G_{h,\beta}$ converges strongly to G_β as $h \to \infty$, then condition (a) of Definition 2.1.1 is satisfied, namely, for every v_h converging weakly to u in H, $\liminf a_h(v_h, v_h) \geqslant a(u, u)$ as $h \to \infty$.

By the definition of the approximate forms $a_h^{(\beta)}$ in Section 1(i), for every $\beta > 0$ and every $u \in H$, we have $a_h^{(\beta)}(u, u) \to a^{(\beta)}(u, u)$ as $h \to \infty$. Moreover, for every h and every $\beta > 0$,

$$a_h(v_h, v_h) \geqslant a_h^{(\beta)}(v_h, v_h) \geqslant a_h^{(\beta)}(u, u) + 2\beta(u - \beta G_{h,\beta} u, v_h - u).$$

Therefore, $\liminf a_h(v_h, v_h) \geqslant a^{(\beta)}(u, u)$ as $h \to \infty$ for every $\beta > 0$, hence $\liminf a_h(v_h, v_h) \geqslant a(u, u)$ as $h \to \infty$ and this proves (j).

We conclude the proof of the theorem by proving

(jj) Under the assumption in (j), condition (b) of Definition 2.1.1 is satisfied, namely, for every $u \in H$, there exists u_h converging strongly to u in H, such that $\limsup a_h(u_h, u_h) \leqslant a(u, u)$ as $h \to \infty$.

By a diagonal argument, we can choose an increasing sequence $\beta_h \to \infty$ as $h \to \infty$, such that

$$a(u, u) \geqslant \lim_{\beta \to \infty} \lim_{h \to \infty} a_h^{(\beta)}(u, u) \geqslant \lim_{h \to \infty} a_h^{(\beta_h)}(u, u).$$

By choosing $u_h := \beta_h G_{h, \beta_h} u$ for every h, we have, as seen in Section 1(i),

$$a_h^{(\beta_h)}(u, u) = a_h(u_h, u_h) + \beta_h \|u - u_h\|^2.$$

It clearly suffices to prove (b) for a given $u \in D[a]$. Then, $u_h \to u$ in H and

$$a(u, u) \geqslant \limsup a_h(u_h, u_h) \qquad \text{as} \quad h \to \infty.$$

This concludes the proof of the theorem. ▮

Remark. Variational convergences of convex sets and convex functionals leading to the convergence of minimizers and of solutions of more general variational inequalities were first considered in [M1, M3, J]. The characterization of the convergence of forms provided by Theorem 2.4.1 is a special case of a general result for convex functionals and their resolvents, due to H. Attouch [A, Thm. 3.26]. In the quadratic Hilbert case considered above the proof simplifies considerably and has been given for the reader's convenience. A related result from [M2] will be described in Proposition 2.7.3 below.

2.5. *Convergence of Yosida–Deny Approximations*

As a by-product of the previous proof, we get the following further characterization of the convergence according to Definition 2.1.1:

PROPOSITION 2.5.1. *A sequence of forms $\{a_h\}$ converges to a form a in H if and only if, for every $\beta > 0$, the sequence of approximate forms $\{a_h^{(\beta)}\}$ converges pointwise to the approximate form $a^{(\beta)}$ as $h \to \infty$.*

2.6. *Convergence of Semigroups*

As a consequence of a well known Trotter–Kato characterization of convergence of resolvents in terms of convergence of the related semigroups, see, e.g., [K, Thm. IX 2.16; P, Chap. III, Thm. 4.2], we get from Theorem 2.4.1 the following:

COROLLARY 2.6.1. *A sequence of densely defined forms $\{a_h\}$ converges to a densely defined form a in H, according to Definition* 2.1.1, *if and only if for every $t > 0$ the sequence of the semigroup operators $\{T_h(t)\}$ associated with the relaxed forms $\underline{a}_h$ in H converges to the semigroup operator $T(t)$ associated with a in the strong operator topology of H, uniformly on every interval $0 < t \leqslant t_1$.*

2.7. *Spectral Convergence*

A further consequence of Theorem 2.4.1 is the following result about the convergence of spectral families and spectral subspaces:

COROLLARY 2.7.1. *If the sequence of densely defined forms* $\{a_h\}$ *converges to a densely defined form a in H, then, for every points of continuity* $\lambda > \mu$ *of the spectral family* $P(\lambda)$ *of the generator of a, the sequence of spectral operators* $\{P_h(\lambda) - P_h(\mu)\}$ *of the generators of the relaxed forms* $\{\underline{a}_h\}$ *converges strongly to the spectral operator* $P(\lambda) - P(\mu)$ *of a in H. Furthermore, the sequence of spectral subspaces* $\{P_h(\lambda)\, H - P_h(\mu) H\}$ *converges in H to the spectral subspace* $P(\lambda)\, H - P(\mu) H$.

The convergence of the spectral operators is a well known consequence of the strong convergence of the resolvents; see, e.g., [K, VIII, Thm. 1.15; SK, XI, Thm. 11.4]. As to the convergence of the spectral subspaces, as stated in the second part of the corollary, we first recall the following definition from [M1]:

DEFINITION 2.7.2. A sequence of subsets $\{K_h\}$ of H *converges* to a subset K of H if:

(a) For every subsequence $\{K_{h'}\}$ of $\{K_h\}$ and every $v_{h'} \in K_{h'}$ converging *weakly* to u in H as $h' \to \infty$, we have $u \in K$.

(b) For every $u \in K$, there exists $u_h \in K_h$ converging *strongly* to u in H as $h \to \infty$.

To complete the proof of the corollary it suffices then to apply the following result from [M2]:

PROPOSITION 2.7.3. *A sequence of closed linear subspaces* $\{M_h\}$ *of H converges to a closed linear subspace M of H, according to Definition* 2.7.2, *if and only if the sequence of orthogonal projections* $\{P_{M_h}\}$ *converges strongly in H to the orthogonal projection* P_M.

Remark. The preceding proposition is of the same nature than the characterization of the convergence of resolvents in Theorem 2.4.1. In fact, if M is a closed subspace of a Hilbert space H, then the orthogonal projection P_M of H on M coincides with the resolvent operator of the convex functional Φ_M, $\Phi_M(u) := 0$ if $u \in M$, $\Phi_M(u) := +\infty$ if $u \in H - M$, as defined, for instance, in [B, A].

Remark. If the resolvents converge in the uniform operator topology, as for example in the case that $D[a]$ is compactly injected in H, then the spectra converge in the Hausdorff metric of closed subsets of $\mathbb{R}$, [N1, N2], [M].

2.8. *Γ-Compactness of families of Markovian forms*

The main interest of Definition 2.2.1 relies on the following general compactness theorem:

THEOREM 2.8.1. *Let $\{a_h\}$ be a sequence of Markovian forms in H. Then, there exists a Dirichlet form a in H and a subsequence $\{a_{h'}\}$ of $\{a_h\}$, such that $a_{h'}$ Γ-converges to a in H as $h' \to \infty$.*

Proof. By a classical theorem by Kuratowski [KU], there exists a l.s.c. functional $F: H \to [-\infty, +\infty]$ such that the sequence of quadratic functionals $\{a_h(u, u)\}$, up to a subsequence, Γ-converges to F in H, that is, (a) and (b) of Definition 2.2.1 are satisfied, with $F(u)$ in place of $a(u, u)$.

By standard Γ-convergence arguments it is shown that the following properties of the quadratic functionals $a_h(u, u)$ are inherited by the limit F:

(i) $F(u) \geqslant 0$, $F(0) = 0$, and $F(tu) = t^2 F(u)$ for every $u \in H$ and every $t \in \mathbb{R}$,

(ii) $F(u+v) + F(u-v) = 2[F(u) + F(v)]$ for every $u, v \in H$, such that $F(u) < +\infty$, $F(v) < +\infty$.

By polarization, we then define a form a in H with domain $D[a] := \{u \in H: F(u) < +\infty\}$, $a(u, v) := 1/2\{F(u+v) - F(u) - F(v)\}$. Then, $D[a]$ is a linear subspace of H and a is a non-negative definite, symmetric bilinear form on $D[a]$.

Since $a(u, u) = F(u)$ for every $u \in H$ and F is l.s.c. on H, it follows, as remarked in Section 1(c), that the form a is closed in H.

We now prove that a is Markovian. Since a is closed, this amounts to prove that if $u \in D[a]$ and $v := \inf\{\sup\{u, 0\}, 1\}$, then $v \in D[a]$ and $a(v, v) \leqslant a(u, u)$. It suffices to prove that $F(v) \leqslant F(u)$, since this implies $F(v) < +\infty$, hence $a(v, v) = F(v) \leqslant F(u) = a(u, u)$. We have $F(u) = \lim a_h(u_h, u_h)$ for a suitable sequence $u_h \to u$ in H, hence, in particular, $u_h \in D[a_h]$ for all h large enough. Let $\eta_{h,\varepsilon} = \eta_{h,\varepsilon}(r)$, $\varepsilon > 0$, $r \in \mathbb{R}$, be the function occurring in the Markovianity condition satisfied by a_h. Therefore, for every h and every ε, $\eta_{h,\varepsilon}(u_h) \in D[a_h]$ and $a_h(\eta_{h,\varepsilon}(u_h), \eta_{h,\varepsilon}(u_h)) \leqslant a_h(u_h, u_h)$. We choose $\varepsilon = 1/h$ and define $v_h := \eta_{h,\varepsilon}(u_h)$. Then, v_h converges in $L^2_{\text{loc}}(X, m)$ to $v = \inf\{\sup\{u, 0\}, 1\} \in L^2(X, m)$, hence v_h converges to v in $H = L^2(X, m)$, therefore $F(v) \leqslant \liminf a_h(v_h, v_h) \leqslant \lim a_h(u_h, u_h) = F(u)$. ∎

If a is an arbitrary given Markovian form in H, the necessarily unique Dirichlet form in H, whose existence is trivially assured by the preceding theorem when $a_h = a$ for every h, clearly coincides with the *relaxation* of a in H, already introduced in Section 1(f) and denoted by $\underline{a}$. Therefore, we have the

COROLLARY 2.8.2. *The relaxation $\underline{a}$ of a Markovian form a in H is a Dirichlet form in H.*

In view of Section 1(e), we find, in particular, that the closure $\bar{a}$ of a closable Markovian form a is a Dirichlet form, in agreement with Theorem 2.1.1 of [F].

2.9. *Spectral Compactness*

By taking Lemma 2.3.2 and Theorem 2.4.1 and its Corollaries 2.6.1 and 2.7.1 into account, we derive from Theorem 2.8.1 the following stronger compactness result:

THEOREM 2.9.1. *Let $\{a_h\}$ be a sequence of Markovian forms, which is asymptotically compact in H. Then, there exist a Dirichlet form a in H and a subsequence $\{a_{h'}\}$ of $\{a_h\}$, such that $a_{h'}$ converges to a in H as $h' \to \infty$, according to Definition* 2.1.1. *Moreover, for every $\beta > 0$, the resolvent operator $G_{h',\beta}$ of the relaxed form $\underline{a}_{h'}$ converges strongly in H to the resolvent operator G_β of a, as $h' \to \infty$. Furthermore, if $D[a_h]$ and $D[a]$ are dense in H, then the sequences of the semigroup operators $\{T_{h'}(t)\}$, associated with the relaxed forms $\underline{a}_{h'}$, as well as the spectral operators $\{P_{h'}(\lambda) - P_{h'}(\mu)\}$ and spectral subspaces $\{P_{h'}(\lambda)H - P_{h'}(\mu)H\}$, all converge to the corresponding operators and subspaces associated with the limit form a, as described in Corollaries* 2.6.1 *and* 2.7.1.

2.10. *Comparison Criteria*

We conclude this section with some *comparison criteria* for Γ-converging forms. These criteria can be applied, for example, in order to get additional information on the forms whose existence is assured by the compactness theorems of Sections 2.8 and 2.9. They show indeed how to get lower and upper bounds for a limit form, hence also lower and upper inclusion bounds for its domain. These criteria are more conveniently stated by introducing the following notion of *lim inf* and *lim sup* of sequences of forms:

DEFINITION 2.10.1. Given an arbitrary sequence of forms $\{a_h\}$ in H, we define the functionals lim inf$\{a_h\}$ and lim sup$\{a_h\}$ on H, with extended real values, by setting for every $u \in H$,

$$\liminf\{a_h\}(u) := \min\{\liminf a_h(u_h, u_h) : u_h \in D[a_h],\, u_h \to u \text{ in } H \text{ as } h \to \infty\},$$

$$\limsup\{a_h\}(u) := \min\{\limsup a_h(u_h, u_h) : u_h \in D[a_h],\, u_h \to u \text{ in } H \text{ as } h \to \infty\},$$

with the convention $\min \varnothing = +\infty$.

Both the above functionals are l.s.c. on H. Moreover, a sequence $\{a_h\}$ Γ-converges to a form a in H according to Definition 2.2.1 if and only if $\limsup\{a_h\} \leqslant \liminf\{a_h\}$ on H, and then $\liminf\{a_h\} = \limsup\{a_h\} = a(u, u)$, where $a(u, u)$ denotes the quadratic functional of a; see [DGF, Prop. 1.8].

In the special case that a is a given form in H and $a_h = a$ for every h, then $\liminf\{a_h\} = \limsup\{a_h\} = \underline{a}(u, u)$ on H, where $\underline{a}(u, u)$ is the quadratic functional of the relaxation of a in H. In particular, if a is closable in H, then $\liminf\{a_h\} = \limsup\{a_h\} = \bar{a}(u, u)$, where $\bar{a}(u, u)$ is the quadratic functional of the closure of a in H.

PROPOSITION 2.10.2. *Let $\{a_h\}$, $\{\alpha_h\}$, $\{\beta_h\}$ be sequences of forms in H and $0 < \lambda \leqslant \Lambda$ two constants, such that*

$$\lambda \alpha_h(u, u) \leqslant a_h(u, u) \leqslant \Lambda \beta_h(u, u) \qquad \textit{for every } h \textit{ and for every } u \in H.$$

Let $\{a_h\}$ Γ-converge to a form a in H, as $h \to \infty$. Then,

$$\lambda \limsup\{\alpha_h\}(u) \leqslant a(u, u) \leqslant \Lambda \liminf\{\beta_h\}(u) \qquad \textit{for every} \quad u \in H,$$

in particular,

$$D[\liminf \beta_h] \subset D[a] \subset D[\limsup \alpha_h].$$

The proposition shows that inclusion *lower* bounds for $D[a]$ can be obtained from suitable *equi-continuity* properties of the forms a_h, while inclusion *upper* bounds for $D[a]$ require suitable *equi-coerciveness* properties of the forms a_h. Useful special cases are in fact the following ones:

COROLLARY 2.10.3. *Let $\{a_h\}$ be a sequence of forms in H and let α, β be two forms in H, such that the condition*

$$\lambda \alpha(u, u) \leqslant a_h(u, u) \leqslant \Lambda \beta(u, u) \qquad \textit{for every} \quad u \in H,$$

is satisfied for every h, with constants $0 < \lambda \leqslant \Lambda$ independent of h. Let a_h Γ-converge to a form a in H as $h \to \infty$. Then, $D[\underline{\beta}] \subset D[a] \subset D[\underline{\alpha}]$, and

$$\lambda \underline{\alpha}(u, u) \leqslant a(u, u) \leqslant \Lambda \underline{\beta}(u, u) \qquad \textit{for every} \quad u \in H.$$

If, in addition, α is inf-compact in H then a_h converges to a in H according to Definition 2.1.1 as $h \to \infty$, and if α is also closable in H, then $D[\underline{\beta}] \subset D[a] \subset D[\bar{a}]$ and

$$\lambda \bar{\alpha}(u, u) \leqslant a(u, u) \leqslant \Lambda \underline{\beta}(u, u) \qquad \textit{for every } u \in H,$$

where $\bar{\alpha}$ is the closure of α in H.

We recall from Section 2.3 that α is inf-compact in H provided the set $\{u \in H \colon (u, u) + \alpha(u, u) \leqslant 1\}$ is relatively compact in H.

3. Preliminaries on Regular Dirichlet Forms

In this section we summarize the main properties of *regular* Dirichlet forms on a locally compact space, needed in the following sections. For these forms a rich representation theory is available, based on the fundamental formulae of A. Beurling and J. Deny [BD1, BD2]. We refer also to [F, LJ, S1, S2] for the general theory and for the proofs of the results summarized below.

(a) *The Topological Setting*

We now take X to be an arbitrary locally compact separable Hausdorff space and m a given positive Radon measure supported on the whole of X. By H we denote again the Hilbert space $H = L^2(X, m)$, with inner product $(u, v) = \int_X uvm(dx)$ and norm $\|\cdot\| = (u, u)^{1/2}$. The support of an arbitrary $u \in L^2(X, m)$, supp u, is defined to be the (compact) support of the measure $u \cdot m$ in X.

(b) *Normal Contractions*

We say that *the normal contractions* operate on a form a in H if $T \circ u \in D[a]$ and $a(T \circ u, T \circ u) \leqslant a(u, v)$, whenever $u \in D[a]$ and $T\colon \mathbb{R} \to \mathbb{R}$, $T(0) = 0$, $|T(x) - T(y)| \leqslant |x - y|$ for every $x, y \in \mathbb{R}$. The normal contractions operate on every densely defined closed Markovian form, that is, on every densely defined Dirichlet form [F, Thm. 1.4.1].

(c) *Regularity*

A form a in H is *regular* if it possesses a core, a *core* being any subset C of $D[a] \cap C_0(X)$, which is dense both in $C_0(X)$ with the uniform norm and in $D[a]$ with the intrinsic norm $(a(u, u) + (u, u))^{1/2}$. In particular, a regular Dirichlet form is densely defined.

If a is a closable Markovian form defined on a dense subset C of $C_0(X)$, then its closure $\bar{a}$ in H is a regular Dirichlet form which admits C as a core.

(d) *Capacity*

Associated with any Dirichlet form a in H there is a *Choquet capacity* defined on the subsets of X, with related notions of null sets, i.e., sets of capacity zero, quasi-continuity, q.e. properties, and q.e. equivalence; see [F, Thm. 3.1.1].

If a is regular in H, then every $u \in D[a]$ admits a *quasi-continuous modification* $\tilde{u}$, that is, there exists a quasi-continuous function $\tilde{u}$, unique up to q.e. equivalence, such that $\tilde{u} = u$ m-a.e. on X [F, Thm. 3.1.3].

We recall that two quasi-continuous functions which are equal (or, $\leqslant$) m-a.e. on an open subset of X are also equal (or, $\leqslant$) q.e. on that set [F, Lemma 3.1.4].

(e) *Beurling–Deny Formulae*

According to the fundamental theory of Beurling–Deny, [BD1, BD2], and its extensions due to M. L. Silverstein [S1, S2], M. Fukushima [F], Y. Le Jean [LJ], any regular Dirichlet form a on the space $H=L^2(X, m)$ can be expressed on its domain $D[a]$ as follows; see [F, Thm. 2.2.1 and Theorem 4.5.2]:

$$a(u, v)=a^{(c)}(u, v)+\iint_{X\times X-d}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))\, j(dx, dy) + \int_X \tilde{u}\tilde{v}k(dx). \tag{3.1}$$

The form $a^{(c)}$, the measures $j(dx, dy)$ and $k(dx)$ occurring in this representation formula are uniquely determined by a and are called the *diffusion part*, the *jumping measure*, and the *killing measure of a.*

More precisely:

(f) *The Jumping Measure*

The jumping measure of a is the unique positive Radon measure $j(dx, dy)$ on $X\times X$ off the diagonal d, that satisfies the identity

$$\iint_{X\times X-d} u(x)\, v(y)\, j(dx, dy)= -\frac{1}{2}\, a(u, v) \tag{3.2}$$

for every u, $v\in D[a]\cap C_0(X)$ with disjoint supports.

The measure $j(dx, dy)$ does not charge any subset of $X\times X-d$ whose projection on the factor X has capacity zero.

(g) *The Killing Measure*

The killing measure of a is the unique positive Radon measure $k(dx)$ on X satisfying the identity

$$\int_X u(x)\, v(x)\, k(dx)=a(u, v)-\iint_{X\times X-d}(u(x)-u(y))(v(x)-v(y))\, j(dx, dy) \tag{3.3}$$

for every u, $v\in D[a]\cap C_0(X)$, v constant on a neighborhood of supp u. For every $u\in D[a]$, $\tilde{u}$ belongs to $L^2(X, k(dx))$ and the following identity holds:

$$\int_X \tilde{u}^2k(dx)=\lim \int_X \beta(1-\beta G_\beta 1)\, \tilde{u}^2m(dx) \qquad \text{as} \quad \beta\to\infty.$$

Here G_β is the resolvent of a, extended to non-negative bounded function by monotonicity [LJ, Prop. 1.3.3].

The measure $k(dx)$ does not charge any subset of X of capacity zero.

(h) *The Diffusion Part*

The diffusion part $a^{(c)}$ is the form, with domain $D[a^{(c)}]=D[a]$, uniquely defined by the identity (3.1). It has the property that every normal contraction operates on it, in particular, $a^{(c)}$ is a Markovian form. In addition, $a^{(c)}$ has the property

$$a^{(c)}(u, v)=0 \qquad \text{for every } u, v \in D[a^{(c)}],$$
$$v \text{ constant on a neighborhood of supp } u$$

[F, Thm. 2.2.1].

(i) *Local and Strongly Local Forms, Diffusions*

We say that a form a on H is *local*, if the following condition holds:

$$a(u, v)=0 \qquad \text{for every } u, v \in D[a] \text{ with disjoint (compact) supports.}$$

If a has the stronger property

$$a(u, v)=0 \qquad \text{for every } u, v \in D[a],\ v \text{ constant on a neighborhood of supp } u,$$

then we say that a is *strongly local.*

The diffusion part $a^{(c)}$ of a regular Dirichlet form is strongly local.

A regular Dirichlet form is local if and only if its jumping measure vanishes.

DEFINITION. We say that a subset C of $C_0(X)$ is *separating* if the following separation property holds: For every compact set K in X and every relatively compact open set $A \supset K$, there exists a non-negative function $\alpha \in C$, such that $\alpha(x)=1$ for $x \in K$ and $\alpha(x)=0$ for $x \in X-A$.

If a closable Markovian form a defined on a dense separating subalgebra of $C_0(X)$ is local, then its closure $\bar{a}$ in H is also local [F, Thm. 2.1.2]. Hence $\bar{a}$ is a local regular Dirichlet form, whose jumping measure vanishes.

A regular Dirichlet form is strongly local if and only if it is local and its killing measure vanishes [F, Thm. 4.5.3].

A strongly local regular Dirichlet form in H will be also called a *Dirichlet form of diffusion type*, or simply a *diffusion* (we note that we are using here the term "diffusion" to denote what is also named a "diffusion without killing inside X"). Thus, a regular Dirichlet form is a diffusion if and only if both its killing and jumping measures vanish.

If a closable Markovian form a defined on a dense separating subalgebra C of $C_0(X)$ is strongly local, then its closure $\bar{a}$ in H is also strongly local, hence it is a diffusion. In fact, as we have seen before, the jumping measure of $\bar{a}$ vanishes, therefore, the conclusion follows from (3.3), by taking $v=\alpha$, α as in the definition above with $K=\text{supp } u$.

(j) *The Energy Measure*

The form $a^{(c)}$ occurring in (3.1) admits an integral expression, namely

$$a^{(c)}(u, v) = \int_X \mu(u, v)(dx), \tag{3.4}$$

which involves the *energy measure* μ of a, whose definition will be given below.

For every $u \in F_b$, a positive Radon measure $\tilde{\mu}(u, u)$ is uniquely defined on X by the identity

$$\int_X \phi(x)\, \tilde{\mu}(u, u)(dx) = a(u\phi, u) - \frac{1}{2} a(u^2, \phi) - \frac{1}{2} \int_X \phi \tilde{u}^2 k(dx), \tag{3.5}$$

for every $\phi \in D[a] \cap C_0(X)$ [LJ, Prop. 1.4.1]. The measure $\tilde{\mu}(u, u)$ is defined for every $u \in F$ as the increasing limit of the measures $\tilde{\mu}(u_n, u_n)$ as $n \to \infty$, where $u_n := \max\{-n, \min\{u, n\}\}$.

The measure $\mu(u, u)$ is then defined for every $u \in F$ by

$$\mu(u, u) := \tilde{\mu}(u, u) - \int_{x \neq y} (\tilde{u}(x) - \tilde{u}(\cdot))^2\, j(dx, \cdot), \tag{3.6}$$

[LJ, Prop. 1.5.1(b)]. The (signed) Radon measure $\mu(u, v)$, u, $v \in F$, is defined by polarization:

$$\mu(u, v) = \tfrac{1}{2}\{\mu(u + v, u + v) - \mu(u, u) - \mu(v, v)\}.$$

These are Radon measures on X, uniquely associated with every u, $v \in D[a]$, and they do not charge sets of capacity zero. We shall occasionally use the notation $\mu(u, v)(dx)$.

The so defined Radon-measure-valued non-negative definite symmetric bilinear form μ on $D[a]$ will be called the *local energy measure* of a, or simply the *energy measure* of a.

The measures μ has indeed a *local* character in X, that is, the restriction of the measure $\mu(u, v)$ to any open subset A of X depends only on the restrictions to A of the functions in its argument. More precisely, if u_1, $u_2 \in D[a]$ are such that $\tilde{u}_1 = \tilde{u}_2$ m-a.e. on A (hence q.e. on A), then

$$\mathbf{1}_A(x)\, \mu(u_1, u_1)(dx) = \mathbf{1}_A(x)\, \mu(u_2, u_2)(dx) \qquad \text{on } X. \tag{3.7}$$

Moreover,

$$\mathbf{1}_A(x)\, \mu(u, v)(dx) = 0 \qquad \text{on } X, \tag{3.8}$$

if $u \in D[a]$ is constant on A and $v \in D[a]$ is arbitrary; see [LJ, Prop. 1.5.2(d); F, Lemma 5.4.6].

Let us point out some useful identities that immediately follow from the preceding definitions. From (3.5), we find that for every u, v, $\phi \in F_b$,

$$\int_X \tilde{\phi}(x)\, \tilde{\mu}(u, v)(dx) = \frac{1}{2}\{a^{\text{res}}(u\phi, v) + a^{\text{res}}(v\phi, u) - a^{\text{res}}(uv, \phi)\}, \quad (3.9)$$

where

$$a^{\text{res}}(u, v) := a(u, v) - \int_X \tilde{u}\tilde{v}k(dx), \quad (3.10)$$

$D[a^{\text{res}}] := D[a]$, is the *resurrected form* of a.

Clearly, in identity (3.9), $a^{\text{res}} = a$ whenever the killing measure of a vanishes. Moreover, if a is a local form, (3.9) holds for the energy measure μ of a, since then $\mu = \tilde{\mu}$ by (3.6).

In particular, if a is a diffusion, as defined in the preceding Section (i), then its energy measure μ satisfies the identity

$$\int_X \tilde{\phi}(x)\, \mu(u, v)(dx) = \frac{1}{2}\{a(u\phi, v) + a(v\phi, u) - a(uv, \phi)\}, \quad (3.11)$$

for every u, v, $\phi \in F_b$.

The energy measure μ of a regular Dirichlet form of diffusion type enjoys some additional important functional properties that will be described below.

(k) *The Leibniz Rule*

One of these properties is the *Leibniz rule*

$$\mu(uv, w)(dx) = u(x)\, \mu(v, w)(dx) + v(x)\, \mu(u, w)(dx) \qquad \text{on } X, \quad (3.12)$$

which holds for every for u, $v \in F_b$, and every $w \in F$ [LJ, Prop. 1.5.2(e); F, Lemma 5.4.2.]

(l) *The Chain Rule*

A second important property is the following *chain rule*, due to Le Jean [LJ, Prop. 2.1(a)]; see also [F, Th. 5.4.2]:

For every v, $u_1, u_2, ..., u_m \in F_b$ and for every $\eta \in C^1(\mathbb{R}^m)$ with $\eta(0) = 0$, we have $\eta(u_1, u_2, ..., u_m) \in F_b$ and

$$\mu(\eta(u_1, u_2, ..., u_m), v)(dx) = \sum_{i=1}^{m} \eta_{x_i}(u_1, u_2, ..., u_m)\, \mu(u_i, v)(dx), \quad (3.13)$$

and the formula extends to arbitrary $u_1, u_2, ..., u_m \in D[a]$, and then $\eta(u_1, u_2, ..., u_m) \in F$, provided the derivatives η_{x_i} are in addition uniformly bounded on $\mathbb{R}^m$. Note that, since the measure μ does not charge sets of capacity zero, the pointwise versions of the functions in the argument of η_{x_i} can be equivalently taken in open sets in the m-a.e. or the q.e. sense.

(m) *The Domination Principle*

The energy measure also obeys a domination principle, due to Le Jean [LJ, Prop. 1.5.5(b)]. Due to its importance in our present context we provide an independent proof, based on a classic Fourier transform argument, as given in [S] in connection with a variational compactness result for uniformly elliptic operators in euclidean spaces.

PROPOSITION. *Let a_1, a_2 be closable Markovian forms in H, with a common domain C, C being a dense subalgebra of $C_0(X)$. If $a_1(u, u) \leqslant a_2(u, u)$ for every $u \in C$, then*

$$\mu_1(u, u) \leqslant \mu_2(u, u) \qquad \textit{for every} \quad u \in C,$$

μ_1, μ_2 being the energy measures of the closure $\bar{a}_1$, $\bar{a}_2$ of a_1, a_2 in H, respectively.

Proof. Let ϕ be an arbitrary function of C. By the Leibniz rule (k) and the chain rule (l), for any positive $\lambda > 0$ we easily compute

$$\begin{aligned}&\mu(\phi \cos(\lambda u), \phi \cos(\lambda u))(dx) + \mu(\phi \sin(\lambda u), \phi \sin(\lambda u))(dx)\\ &\quad = \lambda^2 \phi^2 \mu(u, u)(dx) + \mu(\phi, \phi)(dx),\end{aligned}$$

both for $\mu = \mu_1$ and $\mu = \mu_2$. Therefore, for both forms $a = a_1$, $a = a_2$ we find

$$\begin{aligned}&a(\phi \cos(\lambda u), \phi \cos(\lambda u)) + a(\phi \sin(\lambda u), \phi \sin(\lambda u))\\ &\quad = \lambda^2 \int_X \phi^2 \mu(u, u)(dx) + \int_X \mu(\phi, \phi)(dx)) + \int_X \phi^2(x)\, k(dx)\\ &\quad + \iint_{X \times X - d} [\phi^2(x) + \phi^2(y) - 2\phi(x)\, \phi(y)(\cos \lambda u(x) \cos \lambda u(y)\\ &\quad + \sin \lambda u(x) \sin \lambda u(y))]\, j(dx, dy),\end{aligned}$$

with $\mu = \mu_1$, $k = k_1$, $j = j_1$ and $\mu = \mu_2$, $k = k_2$, $j = j_2$, respectively. Thus, by dividing the inequality

$$\begin{aligned}&a_1(\phi \cos(\lambda u), \phi \cos(\lambda u)) + a_1(\phi \sin(\lambda u), \phi(\sin \lambda u))\\ &\quad \leqslant a_2(\phi \cos(\lambda u), \phi \cos(\lambda u)) + a_2(\phi \sin(\lambda u), \phi \sin(\lambda u))\end{aligned}$$

by λ^2 and letting $\lambda \to \infty$, we find

$$\int_X \phi^2 \mu_1(u, u)(dx) \leqslant \int_X \phi^2 \mu_2(u, u)(dx). \quad \blacksquare$$

(n) *The Schwarz Rule*

Let u, $v \in F_b$. If $f \in L^2(X, \mu(u, u))$ and $g \in L^2(X, \mu(v, v))$, then fg is integrable with respect to the total variation of $\mu(u, v)$ and

$$\int_X |fg| \; |\mu(u, v)| \, (dx) \leqslant \left(\int_X |f|^2 \mu(u, u)(dx) \right)^{1/2} \left(\int_X |g|^2 \mu(v, v)(dx) \right)^{1/2},$$

moreover,

$$2 \, |fg| \; |\mu(u, v)| \leqslant |f|^2 \mu(u, u) + |g|^2 \mu(v, v)$$

as measures in X; see [F, Lemma 5.4.3].

(o) *The Truncation Lemma*

We need an explicit expression of the energy measure of *truncated functions*, given in the following lemma. We observe that for arbitrary functions u in the domain of the form, the set $\{\tilde{u} > 0\}$ is only a *quasi-open* subset of X and (3.14) below does not follow directely from (3.7).

LEMMA. *Let μ be the energy measure of a regular Dirichlet form of diffusion type a in H. Then for every u, $v \in F_b$ we have*

$$\mu(u^+, v)(dx) = \mathbf{1}_{\{\tilde{u} > 0\}}(x) \, \mu(u, v)(dx) \qquad \text{on } X. \tag{3.14}$$

Proof. Let $\phi \in D[a] \cap C_0(X)$ be arbitrary.

Let $\{\eta_\varepsilon\}_{\varepsilon > 0}$ be such that: $\eta_\varepsilon \in C^1(\mathbb{R})$, $\eta_\varepsilon(0) = 0$, η_ε converges uniformly, as $\varepsilon \downarrow 0$, to the function σ: $\sigma(r) = 0$ for $r \leqslant 0$, $= r$ for $r > 0$; moreover, $0 \leqslant \eta'_\varepsilon \leqslant 1$, $\eta'_\varepsilon(r) \to \xi(r)$ for every r, as $\varepsilon \downarrow 0$, where $\xi(r) = 0$ for $r \leqslant 0$, $= 1$ for $r > 0$.

Since $\eta_\varepsilon(u)$ is a normal contraction of u, $\eta_\varepsilon(u) \in F_b$ and $a_1(\eta_\varepsilon(u), \eta_\varepsilon(u)) \leqslant a_1(u, u)$ for every ε, where $a_1(u, u) = a(u, u) + (u, u)$.

By (3.11), for every ε we have

$$\int_X \phi(x) \, \mu(\eta_\varepsilon(u), v)(dx) = \frac{1}{2} \{ a(\eta_\varepsilon(u)\phi, v) + a(v\phi, \eta_\varepsilon(u)) - a(\eta_\varepsilon(u)v, \phi) \},$$

hence, by the chain rule,

$$\int_X \phi(x) \, \eta'_\varepsilon(\tilde{u}(x)) \, \mu(u, v)(dx) = \frac{1}{2} a(\eta_\varepsilon(u)\phi, v) + a(v\phi, \eta_\varepsilon(u)) - a(\eta_\varepsilon(u)v, \phi \}.$$

As $\varepsilon \downarrow 0$ in this identity, $\eta_\varepsilon(u)$ converges strongly to u^+ in the intrinsic norm (see the proof of (iii) of Theorem 1.4.2 in [F]) and in $L^\infty(X, m)$, hence $\eta_\varepsilon(u)\phi$, $\eta_\varepsilon(u)v$ converge to $u^+\phi$, u^+v, respectively. Therefore,

$$\int_X \phi \mathbf{1}_{\{\tilde{u} > 0\}} \mu(u, v)(dx) = \frac{1}{2} \{ a(u^+\phi, v) + a(v\phi, u^+) - a(u^+v, \phi) \},$$

which is to say, by (3.11),

$$\int_X \phi \mathbf{1}_{\{\tilde{u}>0\}} \mu(u, v)(dx) = \int_X \phi\mu(u^+, v)(dx).$$

Since $D[a] \cap C_0(X)$ is uniform dense in $C_0(X)$, this proves (3.14). ∎

It follows from the truncation lemma and the property of the killing measure not to charge sets of capacity zero, that for any local regular Dirichlet form, we have

$$a(u^+, v) = \int_{X \cap \{\tilde{u}>0\}} \mu(u, v)(dx) + \int_{X \cap \{\tilde{u}>0\}} \tilde{u}\tilde{v}k(dx), \tag{3.15}$$

for every $u, v \in F_b$.

(p) *Functions Locally in the Domain*

Let a be a local closable Markovian form with domain a dense separating subalgebra C of $C_0(X)$.

A (m-measurable) function u defined on X is said to be *locally* in $D[a]$, and we write $u \in D[a]_{\text{loc}}$, if for any relatively compact open subset U there exists a function $w \in D[a]$ such that $u = w$ m-a.e. in U. The space $D[a]_{b,\text{loc}}$ is defined analogously. Given $u \in D[a]_{\text{loc}}$, by (3.7) the measure $\mu(u, u)$ is well defined on X by putting $\mathbf{1}_U \mu(u, u) := \mathbf{1}_U \mu(w, w)$ for arbitary U and w as above.

The Leibniz and Schwarz rules clearly extend to arbitrary $u, v \in D[a]_{\text{loc}}$ and $w \in D[a]_{b,\text{loc}}$. Moreover, the chain rule also extends to every v, u_1, $u_2, ..., u_m \in D[a]_{b,\text{loc}}$ for every $\eta \in C^1(\mathbb{R}^m)$, the function $\eta(u_1, u_2, ..., u_m)$ being again in $D[a]_{b,\text{loc}}$, as well as to every v, $u_1, u_2, ..., u_m \in D[a]_{\text{loc}}$ provided that η_{x_i} are in addition uniformly bounded on $\mathbb{R}^m$ on compact subsets of X, and then again $\eta(u_1, u_2, ..., u_m) \in D[a]_{\text{loc}}$ [F, Thm. 5.4.3].

A function u defined on an open subset A of X is said to be *locally* in $D[a]$ *on* A, if there exists a function $w \in D[a]_{\text{loc}}$ such that $\phi u = \phi w$ for any $\phi \in C$ with support in A, where the function ϕu is extended to 0 on $X - A$. The functions *locally* in $D[a]_b$ *on* A are defined analogously. Given such a function u on A, the measure $\mu(u, u)$ is well defined on A by putting $\mu(u, u) := \mu(w, w)$ on A.

(q) *Energy Measures and Diffusion Forms in a Differentiable Setting*

Let us now suppose that X has, in addition, the structure of an orientable differentiable manifold and that there exist coordinate functions $x_1, x_2, ..., x_m$ locally in $D[a]$ on their domain A of definition. The measures $\mu(x_i, x_j)$ are then defined on A. By the chain rule in (l), any other coordinate functions $y_1, y_2, ..., y_m$ on a same open subset U of X are again locally in $D[a]$ on U, where the measure $\mu(y_h, y_k)$ are also defined, and the following change of variables formula holds on U:

$$\mu(y_h, y_k)(dy) = \sum_{ij=1}^{m} (\partial y_h/\partial x_i)(x_1, x_2, ..., x_m)(\partial y_k/\partial x_j)(x_1, x_2, ..., x_m)\, \mu(x_i, x_j)(dx).$$

Therefore,

$$v^{ij} := \mu(x_i, x_j), \qquad i, j = 1, ..., m, \tag{3.16}$$

is an invariantly defined Radon-measure-valued tensor $v = (v^{ij})$ on X, that is,

$$v^{hk}(dy) = \sum_{ij=1}^{m} (\partial y_h/\partial x_i)(x_1, x_2, ..., x_m)(\partial y_k/\partial x_j)(x_1, x_2, ..., x_m)\, v^{ij}(dx).$$

Moreover, again from the chain rule for functions locally in $D[a]$, by using a locally finite partition of unit in C, we find that every $u \in C^1(X)$ is locally in $D[a]_b$ on X, i.e., $u \in D[a]_{b,\mathrm{loc}}$. Therefore, the measures $\mu(u, u)$, hence also the measure $\mu(u, v)$, are well defined on X for every u, $v \in C^1(X)$ and they admit the coordinate invariant expression:

$$\mu(u, v)(dx) = \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, ..., x_m)\, v^{ij}(dx). \tag{3.17}$$

If now in the transformation rule, for fixed $\xi \in \mathbb{R}^m$, we choose $\eta(z) = \xi \cdot z$, $z \in \mathbb{R}^m$, and $u_1 = x_1$, $u_2 = x_2$, ..., $u_m = x_m$, we find that $\xi \cdot x$ is also locally in $D[a]_b$ and

$$\mu(\xi \cdot x, \xi \cdot x)(dx) = \sum_{ij=1}^{m} \xi_i \xi_j \mu(x_i, x_j)(dx), \tag{3.18}$$

and this implies that the condition

$$\sum_{ij=1}^{m} \xi_i \xi_j v^{ij}(dx) \geqslant 0 \qquad \text{for every} \quad \xi \in \mathbb{R}^m, \tag{3.19}$$

is satisfied on X, i.e., the (symmetric) tensor (3.16) is non-negative definite.

It follows that the diffusion part $a^{(c)}(u, v)$ admits the following invariant integral expression for every u, $v \in C_0^1(X)$,

$$a^{(c)}(u, v) = \int_X \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, v^{ij}(dx), \tag{3.20}$$

where the integral at the right hand side, invariantly defined, has to be intended reduced to the coordinate domains by means of a partition of unit in C.

Remark. In the present differentiable setting, the domination principle of Section 3(m) has the further consequence

$$\sum_{ij=1}^{m} \xi_i \xi_j \nu_1^{ij}(dx) \leqslant \sum_{ij=1}^{m} \xi_i \xi_j \nu_2^{ij}(dx) \qquad \text{in } X$$

for every $\xi \in \mathbb{R}^m$, where ν_1, ν_2 are the tensors (3.16) associated with the energy measures μ_1, μ_2, respectively. This follows from the Proposition of Section 3(m), by taking (3.17) and (3.18) into account.

4. Asymptotically Regular Dirichlet Forms

We now come back to the asymptotic theory of Section 2, which we shall develop in this section in the framework of regular Dirichlet forms on a locally compact space described in Section 3. In particular, X, m, and H are as in Section 3(a).

4.1. *Asymptotic Regularity*

We first state a compactness result for sequences of arbitrary Markovian forms a_h on H that satisfy an asymptotic regularity condition, in the sense of the following

Definition 4.1.1. We say that a sequence $\{a_h\}$ of Markovian forms in H is *asymptotically regular* in H, if the following property holds: There exists a dense subset $C \subset C_0(X)$, such that for every $u \in C$ we have

$$\liminf a_h(u_h, u_h) < \infty \text{ as } h \to \infty, \qquad \text{for some} \quad u_h \to u \text{ in } H \text{ as } h \to \infty.$$

We observe that no condition whatsoever is imposed on the forms a_h outside the set C.

Theorem 4.1.2. *Let $\{a_h\}$ be a sequence of Markovian forms in H, asymptotically regular on a dense subset C of $C_0(X)$. Then*:

(i) *There exist a densely defined Dirichlet form a in H and a subsequence $\{a_{h'}\}$ of $\{a_h\}$, such that $a_{h'}$ Γ-converges to a in H as $h' \to \infty$, moreover, $C \subset D[a]$.*

(ii) *There exist, and are unique, an energy measure μ in X, a killing Radon-measure k and a jumping Radon-measure j, such that a is given by*

$$a(u, v) = \int_X \mu(u, v)(dx) + \int_X \tilde{u}\tilde{v}k(dx)$$

$$+ \iint_{X \times X - d} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\, j(dx, dy),$$

for every $u, v \in D[\bar{a}_C]$, *where* $\bar{a}_C$ *is the closure of the restriction of* a *to* C *in* H. *Moreover,* $\bar{a}_C$ *is a regular Dirichlet form with core* C *in* H *and* a *is an extension of* $\bar{a}_C$.

(iii) *If* X *has in addition a differentiable structure and* $C = C_0^1(X)$, *then* a *takes on* $C^1(X) \cap D[\bar{a}_C]$ *the invariant expression*

$$a(u, v) = a^{(c)}(u, v) + \int_X uvk(dx)$$

$$+ \iint_{X \times X - d} (u(x) - u(y))(v(x) - v(y))\, j(dx, dy),$$

with $a^{(c)}(u, v)$ *given by* (3.20), *where* $v = (v^{ij})$ *is the tensor measure* (3.16) *in* X, *satisfying condition* (3.19).

Proof. The existence of a subsequence converging to a Dirichlet form in H follows from Theorem 2.8.1. By property (a) of Γ-convergence, $C \subset D[a]$, in particular, $D[a]$ is dense in H. Since the restriction a_C of a to C is closable in H, a_C admits a smallest closed extension, $\bar{a}_C$, and this is also a restriction of a, because a is closed. The domain of $\bar{a}_C$ is the completion of C with the intrinsic norm and is injected in H, see Section 1(d), thus $\bar{a}_C$ is a regular Dirichlet form in H. The rest of the theorem follows from Beurling–Deny's representation theory of regular Dirichlet forms, see Section 3(e) and 3(q). ∎

Remark. The measures occurring in the representation formula of a given regular Dirichlet form depend intrinsically only on the form itself and not on the underlying measure m of the space X. However, the form a obtained in the preceding theorem as a Γ-limit in H does indeed depend on the measure m, hence so do the measures uniquely associated with a by (ii) of Theorem 4.1.1. A simple example is described in Section 6.1, where the relaxation of a given form is seen to depend drastically on the choice of m. More generally, this shows that the choice of m should be expected to have an important role in the asymptotic definition of the "effective characteristics" of a composite medium, as described in Section 5 and exemplified in Section 6.

Remark. The limit form a may not be regular on its full domain, hence $a \neq \bar{a}_C$, as we shall see more explicitely in Section 5.5 and Example 6.5.1.

For such non-regular closed extensions no general integral representation theorem of Beurling–Deny's type is available. The structure of the form a on its full domain, as well as the structure of the energy measure of its regular restriction $\bar{a}_C$, must be investigated further in the specific case at hand. This in general will require a deeper analysis of the properties of *traces* and *generalized derivatives* of the functions belonging to the domain of the form. The intrinsic capacity, in particular, should be expected to have a basic role in this regard. Some examples will be given in Section 6.

4.2. *Spectral Compactness of Dirichlet Forms*

If the sequence of forms of Theorem 4.1.2, in addition to be asymptotically regular, is also asymptotically compact in H according to Definition 2.3.1, then Theorem 2.9.1 applies and we obtain the following stronger compactness result in the topology of Definition 2.1.1.

THEOREM 4.2.1. *Let the sequence $\{a_h\}$ of Theorem* 4.1.2 *be, in addition, asymptotically compact in H. Then the conclusions* (i), (ii), *and* (iii) *of Theorem* 4.1.2 *hold. Furthermore, $a_{h'}$ converges to a in H as $h' \to \infty$, according to Definition* 2.1.1, *and the resolvent operators $G_{h',\beta}$ of the relaxed form $\underline{a}_{h'}$ in H converge for each $\beta > 0$ to the resolvent operator G_β of the form a, in the strong operator topology of H, as $h' \to \infty$. If, in addition, the forms a_h are densely defined, then the related semigroups, spectral operators and spectral subspaces also converge, as described in Sections* 2.6, 2.7, *and Theorem* 2.9.1.

Remark. We point out that this remarkable spectral compactness property of Dirichlet forms refers to possibly *non-local* forms: Limit forms of local forms may indeed occur which are non-local, as seen for instance in Section 5.4 and Example 6.3.1. Compact families of local and strongly local forms will be described in Section 5.5 and Section 5.6.

4.3. *Closed Families of Local Forms*

As remarked at the end of the previous section, sequences of local, or even strongly local, closed forms may have a limit that is not local. It may also occur that just relaxing a strongly local form we get a form which is non-local; see Example 6.1.1 and Example 6.2.1. In this and in the following section, we give some conditions under which the local property, or the strong local property, are preserved by Γ-convergence, hence *a fortiori* by the stronger convergence of Definition 2.1.1.

We say that a sequence $\{\sigma_h\}$ of Radon measures on X is *bounded* if the sequence $\{\int_X \phi \, d\sigma_h\}$ is bounded for every $\phi \in C_0(X)$. We say that $\{\sigma_h\}$ is *absolutely continuous* in X (with respect to m), if we have $\sigma_h(B_h) \to 0$ as $h \to \infty$, for arbitrary Borel sets B_h in X, such that $m(B_h) \to 0$ as $h \to \infty$.

DEFINITION 4.3.1. We say that a sequence of energy measures $\{\mu_h\}$, defined on a common set $C \subset C_0(X)$, is *bounded* if, for every $u \in C$, the sequence of Radon measures $\{\mu_h(u, u)\}$ is bounded. We say that $\{\mu_h\}$ is *absolutely continuous* in X (with respect to m), if for every $u \in C$ the sequence $\{\mu_h(u, u)\}$ is absolutely continuous in X.

THEOREM 4.3.2. *Let $\{a_h\}$ be a sequence of local closable Markovian forms defined on a common domain $D[a_h] = C$, C being a dense separating subalgebra of $C_0(X)$. Let $\{a_h\}$ Γ-converge to a Dirichlet form a in H. If the sequence of energy measures $\{\mu_h\}$ is bounded and absolutely continuous on C in X, then the form a restricted to $D[a] \cap C$ is local. If, in addition, the sequence of killing measures $\{k_h\}$ is bounded, then $C \subset D[a]$ and the closure $\bar{a}_C$ of the restriction a_C of a to C is local. The measures μ_h, k_h are those associated with the closure of a_h in H.*

Proof. The smallest closed extensions $\bar{a}_h$ of the forms a_h are regular Dirichlet forms, admitting C as a common core, and $\bar{a}_h = \underline{a}_h$. As observed in Section 2.1, they Γ-converge to a in H as $h \to +\infty$. Moreover, they inherit the local property from a_h, as mentioned in Section 3(i). Therefore, it is not restrictive to prove the theorem by assuming that the a_h are local regular Dirichlet forms, admitting C as a common core, and that they have the expression

$$a_h(u, v) = \int_X \mu_h(u, v)(dx) + \int_X \tilde{u}\tilde{v} k_h(dx)$$

for every $u, v \in D[a_h]$.

We first prove that if the sequence of energy measures $\{\mu_h\}$ is bounded and absolutely continuous on C in X, then the form a restricted to $D[a] \cap C$ is local. Let $u, v \in D[a] \cap C$ be fixed, with disjoint compact supports. By Γ-convergence, there exist $u_h \to u$ in H, such that $a_h(u_h, u_h) \to a(u, u)$ as $h \to \infty$ and similarly, there exist $v_h \to v$ in H, such that $a_h(v_h, v_h) \to a(v, v)$ as $h \to \infty$. We can obviously assume that $u_h, v_h \in D[a_h]$ for every h. To simplify notation, we denote the quasi-continuous modification of u_h, v_h (with respect to the a_h capacity) again by u_h, v_h.

For every fixed h, we define $\underline{u}_h$ as

$$\underline{u}_h = \sup\{\inf\{u_h, u + 1/h\}, u - 1/h\} = u_h - (u_h - u - 1/h)^+ + (u - 1/h - u_h)^+,$$

and similarly,

$$\underline{v}_h = v_h - (v_h - v - 1/h)^+ + (v - 1/h - v_h)^+.$$

By our assumption on C, there exist ϕ, $\psi \in C$ with compact support such that: $\phi(x) \equiv 1$, $\psi(x) \equiv 0$ on a neighborhood of supp u; $\phi(x) \equiv 0$, $\psi(x) \equiv 1$ on a neighborhood of supp v; $\phi(x) + \psi(x) \equiv 1$ on X.

Since $\underline{u}_h\phi$, $\underline{v}_h\psi \in D[a_h]$, with disjoint supports, then

$$a_h(\underline{u}_h\phi + \underline{v}_h\psi, \underline{u}_h\phi + \underline{v}_h\psi) = a_h(\underline{u}_h\phi, \underline{u}_h\phi) + a_h(\underline{v}_h\psi, \underline{v}_h\psi), \tag{4.2}$$

for every h.

We write the first term at the right hand side, according to (4.1), as

$$a_h(\underline{u}_h\phi, \underline{u}_h\phi) = \int_X \mu_h(\underline{u}_h\phi, \underline{u}_h\phi)(dx) + \int_X (\underline{u}_h\phi)^2 k_h(dx), \tag{4.3}$$

and we estimate the diffusion term by the Leibniz rule,

$$\int_X \mu_h(\underline{u}_h\phi, \underline{u}_h\phi)(dx) = \int_X \underline{u}_h^2 \mu_h(\phi, \phi)(dx) + \int_X \phi^2 \mu_h(\underline{u}_h, \underline{u}_h)(dx)$$

$$+ 2\int_X \underline{u}_h \phi \mu_h(\underline{u}_h, \phi)(dx). \tag{4.4}$$

By the Schwarz rule of Section 3(n), the last term above can be estimated in terms of the first two terms at the right hand side,

$$\left|\int_X \underline{u}_h \phi \mu_h(\underline{u}_h, \phi)(dx)\right|^2 \leqslant \frac{1}{2}\left\{\int_X \underline{u}_h^2 \mu_h(\phi, \phi)(dx) + \int_X \phi^2 \mu_h(\underline{u}_h, \underline{u}_h)(dx)\right\}. \tag{4.5}$$

We estimate the first term at the right hand side of (4.4),

$$\begin{aligned}
&\int_X \underline{u}_h^2 \mu_h(\phi, \phi)(dx) \\
&\quad = \int_{\{u=0\}} \underline{u}_h^2 \mu_h(\phi, \phi)(dx) + \int_{\{u>0\}} \underline{u}_h^2 \mu_h(\phi, \phi)(dx) \\
&\qquad + \int_{\{u<0\}} \underline{u}_h^2 \mu_h(\phi, \phi)(dx) \\
&\quad \leqslant \frac{1}{h^2}\int_X \mu_h(\phi, \phi)(dx) + c\left(\int_{\{u>0\}} \mu_h(\phi, \phi)(dx) + \int_{\{u<0\}} \mu_h(\phi, \phi)(dx)\right) \\
&\quad \leqslant \frac{1}{h^2}\int_X \mu_h(\phi, \phi)(dx),
\end{aligned}$$

for some constant $c > 0$, independent of h.

Here we have first taken into account that the functions u_h are uniformly bounded in h and then that the second and third term vanish, by (3.8), because $\phi \equiv 1$ on the (open) integration set.

We now choose $\alpha \in C$, independent of h, with compact support and such that $0 \leqslant \alpha \leqslant 1$, $\alpha \equiv 1$ on supp ϕ. Then, by (3.7), (3.8),

$$\int_X \mu_h(\phi, \phi)(dx) = \int_{\operatorname{supp} \phi} \mu_h(\phi, \phi)(dx) \leqslant \int_X \alpha(x)\, \mu_h(\phi, \phi)(dx).$$

Since the sequence $\{\mu_h(\phi, \phi)\}$ is weakly bounded, this implies that $\int_X \mu_h(\phi, \phi)(dx)$ remains bounded as $h \to \infty$. Therefore,

$$\int_X \underline{u}_h^2 \mu_h(\phi, \phi)(dx) \to 0 \qquad \text{as} \quad h \to \infty. \tag{4.6}$$

We now estimate the second term at the right hand side of (4.4):

$$\begin{aligned}
\int_X \phi^2(x)\, \mu_h(\underline{u}_h, \underline{u}_h)(dx) &\leqslant \int_X \mu_h(\underline{u}_h, \underline{u}_h)(dx) \\
&= \int_{\{|u_h - u| < 1/h\}} \mu_h(\underline{u}_h, \underline{u}_h)(dx) \\
&\quad + \int_{\{|u_h - u| \geqslant 1/h\}} \mu_h(\underline{u}_h, \underline{u}_h)(dx) \\
&\leqslant \int_X \mu_h(u_h, u_h)(dx) + \int_{\{u_h - u \geqslant 1/h\}} \mu_h(\underline{u}_h, \underline{u}_h)(dx) \\
&\quad + \int_{\{u_h - u \leqslant -1/h\}} \mu_h(\underline{u}_h, \underline{u}_h)(dx).
\end{aligned}$$

We shall now prove that the last two terms above tend to zero as $h \to \infty$. We recall that $\underline{u}_h = u_h - (u_h - u - 1/h)^+ + (u - 1/h - u_h)^+$, and decompose $\mu_h(\underline{u}_h, \underline{u}_h)$ accordingly. By the truncation lemma of Section 3(o) and by (3.8),

$$\begin{aligned}
\mu_h(u_h, (u_h - u - 1/h)^+) &= \mathbf{1}_{\{u_h - u - 1/h > 0\}} \mu_h(u_h, u_h - u - 1/h) \\
&= \mathbf{1}_{\{u_h - u - 1/h > 0\}} \mu_h(u_h, u_h - u)
\end{aligned}$$

and similarly for the other terms arising from the decomposition of $\mu_h(\underline{u}_h, \underline{u}_h)$. By putting all these terms together again, since $\mathbf{1}_{\{u_h - u > 1/h\}} \mathbf{1}_{\{u - u_h > 1/h\}} = 0$, we find

$$\begin{aligned}
\mathbf{1}_{\{u_h - u - 1/h > 0\}} u_h(\underline{u}_h, \underline{u}_h) &= \mathbf{1}_{\{u_h - u - 1/h > 0\}} \{\mu_h(u_h, u_h) - 2\mu_h(u_h, u_h - u) \\
&\quad + \mu_h(u_h - u, u_h - u)\} = \mathbf{1}_{\{u_h - u - 1/h > 0\}} \mu_h(u, u).
\end{aligned}$$

Therefore,

$$\int_{\{u_h - u \geq 1/h\}} \mu_h(\underline{u}_h, \underline{u}_h)(dx) = \int_{\{u_h - u \geq 1/h\}} \mu_h(u, u)(dx).$$

As $h \to \infty$, $\underline{u}_h$ converges to u in $L^2(X, m)$, therefore $m(\{u_h - u > 1/h\} \to 0$, hence the last integral vanishes, by our assumption on the measures μ_h. The other term is shown to vanish in a similar way, therefore the second term at the right hand side of (4.4) is estimated in the limit as $h \to \infty$ by

$$\int_X \phi^2(x)\, \mu_h(\underline{u}_h, \underline{u}_h)(dx) \leq \int_X \mu_h(u_h, u_h)(dx) + o(1/h). \tag{4.7}$$

Going back to (4.4) and taking (4.5), (4.6), and (4.7) into account, we find

$$\int_X \mu_h(\underline{u}_h \phi, \underline{u}_h \phi)(dx) \leq \int_X \mu_h(u_h, u_h)(dx) + o(1/h). \tag{4.8}$$

As to the second integral at the right hand side of (4.3), since $|\underline{u}_h| \leq |u_h|$, we have

$$\int_X (\underline{u}_h \phi)^2\, k_h(dx) \leq \int_X u_h^2 k_h(dx). \tag{4.9}$$

By taking (4.8) and (4.9) into account, we finally get from (4.3) the following estimate of the first term at the right hand side of (4.2):

$$a_h(\underline{u}_h \phi, \underline{u}_h \phi) \leq \int_X \mu_h(u_h, u_h)(dx) + \int_X u_h^2 k_h(dx) + o(1/h),$$

that is,

$$a_h(\underline{u}_h \phi, \underline{u}_h \phi) \leq a(u_h, u_h) + o(1/h).$$

By similar arguments,

$$a_h(\underline{v}_h \psi, \underline{v}_h \psi) \leq a(v_h, v_h) + o(1/h).$$

Therefore, we get from (4.2)

$$a_h(\underline{u}_h \phi + \underline{v}_h \psi, \underline{u}_h \phi + \underline{v}_h \psi) \leq a(u_h, u_h) + a(v_h, v_h) + o(1/h),$$

hence, by our choice of u_h and v_h

$$\limsup a_h(\underline{u}_h \phi + \underline{v}_h \psi, \underline{u}_h \phi + \underline{v}_h \psi) \leq a(u, u) + a(v, v) \qquad \text{as} \quad h \to \infty.$$

On the other hand, since $\underline{u}_h\phi + \underline{v}_h\psi$ converges to $u+v$ in $L^2(X, m)$, by the first condition of Γ-convergence we get

$$a(u+v, u+v) \leqslant \liminf a_h(\underline{u}_h\phi + \underline{v}_h\psi, \underline{u}_h\phi + \underline{v}_h\psi) \quad \text{as} \quad h \to \infty.$$

Therefore,

$$a(u+v, u+v) \leqslant a(u, u) + a(v, v),$$

which, by the polarization identity, gives

$$a(u, v) \leqslant 0. \tag{4.10}$$

By exchanging v with $-v$ from the beginning of the previous argument, we obtain the opposite inequality in (4.10), hence $a(u, v) = 0$.

This concludes the proof that the form a possesses the local property on $D[a] \cap C$. In order to complete the proof of the theorem, we first observe that for every $u \in C$ we have:

$$a(u, u) \leqslant \liminf a_h(u, u) \leqslant \limsup \left\{ \int_X \mu_h(u, u)(dx) + \int_X u^2 k_h(dx) \right\}$$

$$\leqslant \limsup \int_X \alpha(x)\, \mu_h(u, u)(dx) + \limsup \int_X u^2 k_h(dx),$$

as $h \to \infty$, where $\alpha \in C$, $0 \leqslant \alpha \leqslant 1$, $\alpha \equiv 1$ on a compact neighborhood of supp u. Since α, $u \in C$, both these two terms are bounded, by our assumption on the energy and killing measures. Therefore, $D[a]$ contains C. Moreover, as noted in Section 3(i), the closure $\bar{a}_C$ is also local. ∎

4.4. *Closed Families of Diffusions*

We now prove that under the same assumptions of Theorem 4.3.2 the strong local property also is preserved by Γ-convergence.

THEOREM 4.4.1. *Let $\{a_h\}$ be a sequence of strongly local, closable Markovian forms defined on a common domain $C = D[a_h]$, C being a dense separating subalgebra of $C_0(X)$. Let a_h Γ-converge to a Dirichlet form a in H, as $h \to +\infty$. If the sequence of energy measures $\{\mu_h\}$ is bounded and absolutely continuous with respect to m in X, then $C \subset D[a]$ and the closure $\bar{a}_C$ of the restriction a_C of a to C is also strongly local.*

Proof. Theorem 4.3.2 applies, with $k_h = 0$ for every h. Therefore, $C \subset D[a]$ and $\bar{a}_C$ is local. In order to prove that $\bar{a}_C$ is strongly local it suffices to prove, as remarked at the end of Section 3(i), that a_C is strongly local, that is, $a(u, v) = 0$ if u, $v \in C$ and v is constant, say $v \equiv 1$, on a neighborhood of the support of u.

We first construct sequences $\{u_h\}$, $\{\underline{u}_h\}$ and $\{v_h\}$, $\{\underline{v}_h\}$ as in the proof of Theorem 4.3.2.

For every h, we now choose $\alpha_h \in C^1(R)$, such that $\alpha_h(r) \equiv 0$ if $|r| \leqslant 1/h$, $\alpha_h(r) \equiv r$ if $|r| > 1/\sqrt{h}$, $0 \leqslant \alpha'_h \leqslant 1/(1-1/\sqrt{h})$. We also define $\beta_h(r) = \alpha_h(r-1)+1$ for every r. We have $\alpha_h \circ \underline{u}_h \in D[a_h] \cap C_0(X)$, $\beta_h \circ \underline{v}_h \in D[a_h] \cap C_0(X)$ for every h. Moreover, it is easy to check that $\beta_h \circ \underline{v}_h \equiv 1$ on a neighborhood of the support of $\alpha_h \circ \underline{u}_h$, hence $a_h(\alpha_h \circ \underline{u}_h, \beta_h \circ \underline{v}_h) = 0$, therefore by the polarization identity,

$$a_h(\alpha_h \circ \underline{u}_h + \beta_h \circ \underline{v}_h, \alpha_h \circ \underline{u}_h + \beta_h \circ \underline{v}_h) = a_h(\alpha_h \circ \underline{u}_h, \alpha_h \circ \underline{u}_h) + a_h(\beta_h \circ \underline{v}_h, \beta_h \circ \underline{v}_h).$$

Since $\alpha_h(r)$ converges to r uniformly and $\underline{u}_h$ converges to u in $L^2(X, m)$, then $\alpha_h \circ \underline{u}_h$, also converges to u in $L^2(X, m)$, and similarly $\beta_h \circ \underline{v}_h$ converges to v in $L^2(X, m)$. Therefore, from the first condition of Γ-convergence, we find that

$$\begin{aligned} a(u+v, u+v) \leqslant & \limsup a_h(\alpha_h \circ \underline{u}_h, \alpha_h \circ \underline{u}_h) \\ & + \limsup a_h(\beta_h \circ \underline{v}_h, \beta_h \circ \underline{v}_h) \qquad \text{as} \quad h \to \infty. \end{aligned}$$

By the chain rule, we have

$$\begin{aligned} a_h(\alpha_h \circ \underline{u}_h, \alpha_h \circ \underline{u}_h) &= \int_X \mu_h(\alpha_h \circ \underline{u}_h, \alpha_h \circ \underline{u}_h)(dx) \\ &= \int_X \alpha_h'^2(\underline{u}_h(x))\, \mu_h(\underline{u}_h, \underline{u}_h)(dx) \\ &\leqslant (1/(1-1/\sqrt{h}))^2 \int_X \mu_h(\underline{u}_h, \underline{u}_h)(dx). \end{aligned}$$

We now observe that the last integral is the same integral that, in the proof of Theorem 4.3.2, was already estimated according to

$$\begin{aligned} \int_X \mu_h(\underline{u}_h, \underline{u}_h)(dx) &\leqslant a_h(u_h, u_h) + \int_{\{|u_h - u| \leqslant 1/h\}} \mu_h(u, u)(dx) \\ &= a_h(u_h, u_h) + o(1/h). \end{aligned}$$

Therefore,

$$\limsup a_h(\alpha_h \circ \underline{u}_h, \alpha_h \circ \underline{u}_h) \leqslant a(u, u) \qquad \text{as} \quad h \to \infty,$$

and, similarly,

$$\limsup a_h(\beta_h \circ \underline{v}_h, \beta_h \circ \underline{v}_h) \leqslant a(v, v).$$

Thus,

$$a(u+v, u+v) \leqslant a(u, u) + a(v, v),$$

hence

$$a(u, v) \leqslant 0.$$

In order to prove the opposite inequality, we apply our previous argument to the function $-v$, after traslating α_h and β_h by replacing r with $r+2$. ∎

4.5. *Compact Families of Local Forms and Diffusions*

By taking the closure theorems of Sections 4.3 and 4.4 into account, we get from Theorem 4.1.2 and Theorem 4.2.1 the following compactness result for families of local forms:

THEOREM 4.5.1. *Let $\{a_h\}$ be a sequence of local closable Markovian forms defined on a common domain $C = D[a_h]$, C being a dense separating subalgebra of $C_0(X)$. In addition, let the sequence of energy measures $\{\mu_h\}$ of the closures $\bar{a}_h$ of a_h in H be bounded and absolutely continuous with respect to m in X, and let the sequence of killing measures $\{k_h\}$ of $\bar{a}_h$ be also bounded. Then*:

(i) *The conclusion* (i), (ii), (iii) *of Theorem* 4.1.2 *hold with $j=0$ and $\bar{a}_C$ is local. Furthermore, if $k_h = 0$ for every h then both j and k vanish and $\bar{a}_C$ is strongly local.*

(ii) *If, in addition, the sequence $\{a_h\}$ is asymptotically compact in H, then $\{a_{h'}\}$ converges to a in the stronger sense of Definition* 2.1.1. *Furthermore, related resolvents, semi-groups, spectral operators and subspaces also converge, as described in Sections* 2.4, 2.6, *and* 2.7.

Proof. We recall from Section 3(i) that a regular form is local if and only if its jumping measure vanishes. Therefore, for every h and every $u \in C$, we have

$$a_h(u, u) = \int_X \mu_h(u, u)(dx) + \int_X u^2 k_h(dx).$$

Since the sequences $\{\mu_h\}$, $\{k_h\}$ are bounded (Section 4.3), lim sup $a_h(u, u) < +\infty$ as $h \to \infty$. Therefore, the sequence $\{a_h\}$ is asymptotically regular on C in H and Theorem 4.1.2 applies. Again by our assumption on $\{\mu_h\}$, $\{k_h\}$, Theorem 4.3.2 also applies, hence $\bar{a}_C$ is local and the jumping measure j in (ii) of Theorem 4.1.2 vanishes. If, in particular, the forms a_h are strongly local then, by Theorem 4.4.1, the form $\bar{a}_C$ is also strongly local and both measures j and k vanish in (ii) of Theorem 4.1.2. This proves (i) above. Finally, if the sequence $\{a_h\}$ is asymptotically compact in H, then Theorem 4.2.1 also applies and this gives (ii). ∎

5. Energy Forms on Composite Media

In this section we shall apply the results of Section 4 to the variational theory of composite media.

In this theory, as already mentioned in the Introduction, the *effective characteristics* of a composite body are obtained from an asymptotic variational principle, that involves suitable converging sequences of approximate energies.

The diffusion part of these approximate energy forms can be constructed from given families of *admissible energy measures*, as described in Section 5.1. Moreover, various types of boundary conditions and potentials can be incorporated in the energy, once we allow general *Borel measures* to be in the role of the killing and jumping measures of the approximate Dirichlet forms, as it is explained in Section 5.2.

The convergence of the approximate energy forms, up to subsequences, can then be deduced from the general compactness results of Section 4, and this provides a *measure-valued* definition of the effective characteristics of the body, as explained in Section 5.3.

In the subsequent Section 5.4 to 5.7, we describe more explicitely, in a differentiable setting, some general examples of approximate diffusions.

In Sections 5.4 and 5.6, the approximate conductivities are allowed to develop *singularities* with respect to the underlying measure m, and possibly generate *non-local* potentials. In Sections 5.5 and 5.7, the local character of the energy is preserved in the limit.

On the other hand, while the diffusion forms considered in Sections 5.4 and 5.5 may be highly degenerate and only converge in the sense of Γ-convergence, those of Sections 5.6 and 5.7 are submitted to additional coerciveness assumptions that lead to the stronger convergence of Definition 2.1.1, hence also to convergence of the resolvent operators, semigroups and spectral families associated with the forms. In particular, in Section 5.7 a spectral compactness result for coercive families of diffusions is established, namely, Theorem 5.7.1.

5.1. *Admissible Energy Measures*

We say that μ is an *admissible energy measure* in X, if μ is a non-negative definite, symmetric bilinear Radon-measure-valued form in X, defined on a dense separating subalgebra C of $C_0(X)$, such that the form

$$\tilde{\mathcal{a}}(u, v) = \int_X \mu(u, v)(dx), \qquad D[\tilde{\mathcal{a}}] = C, \tag{5.1}$$

is a *strongly local, closable Markovian* form in H. Then, the closure of $\tilde{\mathcal{a}}$ in H is a regular Dirichlet form of diffusion type in H, that we shall denoted by $\mathcal{A}$. The energy measure of $\mathcal{A}$, still denoted by μ, is now defined on the whole of $D[\mathcal{A}]$ and coincides with the initial μ on C. We thus have

$$\mathcal{A}(u, v) = \int_X \mu(u, v)(dx), \qquad \text{for every} \quad u, v \in D[\mathcal{A}]. \tag{5.2}$$

The form $\mathcal{A}$ is uniquely associated with the given μ and admits C as a core. It defines a *capacity* for subsets of X and every function $u \in D[\mathcal{A}]$ admits a quasi-continuous version, denoted by $\tilde{u}$ below, as mentioned in Section 3(d).

In a differentiable setting for X, as in Section 3(q), we will also say that an invariantly defined, non-negative definite symmetric Radon-measure-valued tensor $\nu = (\nu^{ij})$ in X is an *admissible energy measure*, if the form

$$\tilde{\mathcal{A}}(u, v) := \int_X \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, \nu^{ij}(dx), \tag{5.3}$$

with domain $D[\tilde{\mathcal{A}}] = C_0^1(X)$, is *closable* in H.

A given tensor ν is an admissible energy measure in X if and only if the measure

$$\mu(u, v)(dx) := \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, \nu^{ij}(dx), \tag{5.4}$$

invariantly defined on $C = C_0^1(X)$, is an admissible energy measure in X according to our preceding definition. The closure $\mathcal{A}$ in H of the form (5.3), uniquely determined by the given ν, is then a regular diffusion and can be written as the form (5.2). Its energy measure μ, defined on $D[\mathcal{A}]$, coincides on $C_0^1(X)$ with the initial μ, given by (5.4).

5.2. *Admissible Killing Measures and Jumping Measures*

Given an arbitrary admissible energy measure μ in X, as defined in the previous section, we denote by $K[\mu]$ the family of all positive *Borel* measures on X, with extended real values, that do not charge subsets of X of capacity zero. By $J[\mu]$ we denote the family of all symmetric positive *Borel* measures on $X \times X$-diag, that do not charge subsets of $X \times X$ whose projection on X has capacity zero. The capacity involved is the capacity of the form (5.2) associated with μ.

Given an admissible μ, and given measures $k \in K[\mu]$, $j \in J[\mu]$, we define the form

$$a(u, v) = \int_X \mu(u, v)(dx) + \int_X \tilde{u}\tilde{v}k(dx)$$
$$+ \iint_{X \times X - d} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\, j(dx, dy)), \tag{5.5}$$

with domain

$$D[a] = \{u \in D[\mathcal{A}] : \tilde{u} \in L^2(X, k(dx)),$$
$$(\tilde{u}(x) - \tilde{u}(y)) \in L^2(X \times X, j(dx, dy))\}.$$

Here $\mathcal{A}$ is the regular Dirichlet form of diffusion type (5.2), associated with the given μ.

LEMMA 5.2.1. *The form* (5.5) *is a Dirichlet form in* H.

Proof. We prove that a is closed in H. In fact, if $\{u_n\}$ is a Cauchy sequence for the intrinsic metric of a, then $\{u_n\}$ is a Cauchy sequence for the intrinsic metric of $\mathcal{A}$, therefore there exists $u \in D[\mathcal{A}]$ such that u_n converges to u in the metric of $\mathcal{A}$, as $n \to \infty$. Then, by Theorem 3.1.4 of [F], a subsequence of $\{u_n\}$ exists, that we still denote by $\{u_n\}$, such that $\tilde{u}_n$ converges q.e. to $\tilde{u}$ in X. On the other hand, the initial sequence $\{u_n\}$ is also a Cauchy sequence in the space $L^2(X, k(dx))$, therefore it converges in this space to some function v. Since the measure k does not charge subsets of $\mathcal{A}$-capacity zero, $v = \tilde{u}$ k-a.e. in X, hence $\tilde{u}_n$ converges to $\tilde{u}$ in $L^2(X, k(dx))$. A similar argument shows that $(\tilde{u}_n(x) - \tilde{u}_n(y)) \in L^2(X \times X, j(dx, dy))$ converges to $(\tilde{u}(x) - \tilde{u}(y)) \in L^2(X \times X, j(dx, dy))$ in $L^2(X \times X, j(dx, dy))$. Thus, $u \in D[a]$. The Markovianity property, in the form of the last condition of section 1(f), is also easily checked. ∎

Special forms of type (5.5) are those involving killing measures $k := \infty_E$, where E is an arbitrary Borel subset of X of positive capacity and $\infty_E(B) := +\infty$ if $B \cap E$ has positive capacity, $\infty_E(B) := 0$ if $B \cap E$ is a null set. These measures are related to *homogeneous Dirichlet conditions* on possibly irregular subsets of X. They were first introduced in [DMM1, DMM2], in connection with so-called *relaxed Dirichlet problems*, and their probabilistic interpretation was given in [BDMM].

In this regard, we point out that if in (5.5) we choose $k := \infty_E$, with E a given closed subset of X, then $u \in D[a]$ clearly implies $\tilde{u}(x) = 0$ q.e. on E, therefore, by Theorem 4.4.2(i) of [F], $D[a]$ is contained in the closure of $C_0(X - E) \cap D[a]$ for the intrinsic norm of a. Now, the restriction of a to such a closure is the regular Dirichlet form associated with the variational *Dirichlet problem* for the given form a in the open subset $\Omega = X - E$ of X, with homogeneous Dirichlet condition on the boundary of Ω. From the point of view of composite media, infinite Borel killing measures of this kind can thus be seen as describing the presence in the body of *perfectly conductive inclusions*, on which the potential is kept equal to zero.

Another important example is that of infinite Borel jumping measures of the type $j = \infty_{E \times F}$, where E and F are Borel subsets of X with disjoint closures and we define $\infty_{E\times F}(B_1 \times B_2)) := +\infty$ if $B_1 \cap E$ and $B_2 \cap F$ have positive capacity, $\infty_{E\times F}(B_1 \times B_2)) := 0$ otherwise. These measures, related to "*matching conditions*" between separate regions of the body, were first introduced in [DMGM], in connection with the spectral analysis of certain Riemannian manifolds of complicated topological type.

If in (5.5) the measure $j(dx, dy)$ is of this type, then $u \in D[a]$ implies $\tilde{u}(x) = \tilde{u}(y)$ for q.e. $x \in E$ and q.e. $y \in F$, what clearly amounts to a change in the topological type of the domain X, as in the elementary case of *periodic boundary conditions.* Again from the point of view of composites, infinite Borel jumping measures of this kind can be seen as *coupled inclusions* in the body, connected by *perfect conductors* that keep the potential at the same value on the coupled regions (*short-circuits*).

Always from the point of view of composite media, we should also note that *perfect insulating inclusions* E in X, on the boundary of which a generalized condition of *Neumann type* for the potential is expected to hold, can also be taken into account in (5.5), by choosing admissible energy measures that vanish on E.

Remark. The form (5.5), in general, will not be regular, nor densely defined in H. In fact, if the measures k and j are too singular, for example they are infinite on some large subset of X, then $D[a]$ may be small and the space $C_0(X) \cap D[a]$ may not be dense in $C_0(X)$.

5.3. *Effective Characteristics of Composite Media*

We are now in a position to define the *effective characteristics* of a composite material in a measure-valued sense, following the lines of the asymptotic variational approach outlined in the Introduction.

We suppose that, for every h, we give an admissible energy measure μ_h in X, defined on a dense separating subalgebra of $C_0(X)$. Uniquely associated with μ_h there is a regular Dirichlet form of diffusion type in H,

$$\mathcal{A}_h(u, v) = \int_X \mu_h(u, v)(dx), \qquad u, v \in D[\mathcal{A}_h], \tag{5.6}$$

as described in Section 5.1. Furthermore, we give arbitrary measures $k_h \in K[\mu_h]$ and $j_h \in J[\mu_h]$.

We then consider the sequence of forms

$$a_h(u, v) = \int_X \mu_h(u, v)(dx) + \int_X \tilde{u}(x)\, \tilde{v}(x)\, k_h(dx)$$

$$+ \iint_{X\times Y-d} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y))\, j_h(dx, dy)$$

with domains

$$D[a_h] = \{u \in \mathcal{C}_h : \tilde{u} \in L^2(X, k_h(dx)), (\tilde{u}(x) - \tilde{u}(y)) \in L^2(X \times X, j_h(dx, dy))\}.$$

As seen in Section 5.2, these forms are Dirichlet forms in H, therefore Theorem 4.1.2 and Theorem 4.2.1 apply and we can state the following

THEOREM 5.3.1. *For every h, let μ_h, $k_h \in K[\mu_h]$, $j_h \in J[\mu_h]$ be arbitrary admissible measures in X and let a_h be the form* (5.7). *Let the sequence $\{a_h\}$ be asymptotically regular on a dense set C of $C_0(X)$ in H. Then, a densely defined Dirichlet form a and measures μ, k, j exist, for which the conclusions* (i), (ii), *and* (iii) *of Theorem* 4.1.2 *hold. If, in addition, the sequence $\{a_h\}$ is asymptotically compact in H, then the further conclusions of Theorem* 4.2.1 *also hold.*

The measures μ, k and j are *effective characteristics* for the body X. In the differentiable setting of Section 3(q), in particular, the energy measure μ is uniquely determined by the *effective conductivity* tensor ν, given by (3.16).

We also point out that these effective measures result from the combined asymptotic effects of all the terms occurring in the approximate forms (5.7). The separate contributions arising from the diffusion parts of (5.7) will be further analysed in the following sections. Special compactness results for families of killing and jumping measures associated with uniformly elliptic operators in euclidean spaces have been given in [DMM1, DMGM]. More general cases will be dealt with in [M6].

5.4. *Singular Conductivities*

In this and in the subsequent sections we shall apply Theorem 5.3.1 and Theorem 4.5.1 to the asymptotic study of special forms of diffusion type, in the differentiable setting of Section 3(q).

In these examples, the asymptotic regularity is a consequence of suitable *equi-continuity* properties of the forms, as assured by estimates of the forms *from above*. By the comparison criteria of Section 2.10, this gives control of the domains *from below*.

We consider a sequence $\{\nu_h\}$ of admissible energy measures ν_h in X, in the sense of Section 5.1, and we suppose, in addition, that for every h the following *degenerate ellipticity condition* is satisfied,

$$0 \leqslant \sum_{ij=1}^{m} \xi_i \xi_j \nu_h^{ij}(dx) \leqslant \Lambda \, |\xi|^2 \, \beta_h(dx) \qquad \text{in } X \text{ for every } \xi \in \mathbb{R}^m, \tag{5.8}$$

for some $\Lambda > 0$ independent of h, with β_h a positive Radon measure in X, such that

$$\beta_h \text{ converges weakly to a Radon measure } \beta \text{ in } X \quad \text{as} \quad h \to \infty. \tag{5.9}$$

The forms

$$a_h(u, v) = \int_X \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, \nu_h^{ij}(dx), \tag{5.10}$$

with domains $D[a_h] = C_0^1(X)$, as a consequence of (5.9), are asymptotically regular on $C = C_0^1(X)$ in H, hence so are their closures $\bar{a}_h$ in H, which are regular diffusions in H.

Therefore, from Theorem 5.3.1 and the comparison criteria of Section 2.10, we obtain:

THEOREM 5.4.1. *Under the assumptions* (5.8), (5.9), *let* $\{a_h\}$ *be the sequence of forms* (5.10). *Then*:

(i) *There exist a densely defined Dirichlet form a in H and a subsequence* $\{a_{h'}\}$ *of* $\{a_h\}$, *such that* $a_{h'}$ *Γ-converges to a in H as* $h' \to \infty$; *moreover* $C_0^1(X) \subset D[a]$.

(ii) *There exist, and are unique, an energy measure* $\nu = (\nu^{ij})$ *in X, satisfying the ellipticity condition*

$$0 \leqslant \sum_{ij=1}^{m} \xi_i \xi_j \nu^{ij}(dx) \leqslant \Lambda\, |\xi|^2\, \beta(dx) \qquad \text{on } X \text{ for every } \xi \in \mathbb{R}^m, \tag{5.11}$$

a Radon measure k and a Radon measure j, such that the form a is represented for every u, $v \in C_0^1(X)$ *as in* (iii) *of Theorem* 4.2.1. *Moreover, for arbitrary u,* $v \in D[\bar{a}_C]$, *the form a is represented according to* (ii) *of Theorem* 4.1.2, *where* μ *is the energy measure of the form* $\bar{a}_C$.

Proof. By taking Theorem 5.3.1 into account, we only must prove the ellipticity condition (5.11), that does not follow from the domination principle of Section 3(m), via the comparison criterion of Corollary 2.10.3, because the form associated with $\beta(dx)$ on $C_0^1(X)$ may not be closable. However, as we show below, a similar proof as the one given in Section 3(m) can be carried on also in the present asymptotic case. We know in fact that local coordinates $x = (x_1, x_2, ..., x_m)$ belong locally to $D[\bar{a}_C]_b$, therefore $u = \xi \cdot x$ is also locally in $D[\bar{a}_C]_b$ and by the chain rule,

$$\mu(\xi \cdot x, \xi \cdot x)(dx) = \sum_{ij=1}^{m} \xi_i \xi_j \nu^{ij}(dx).$$

For every $\phi \in C = C_0^1(X)$, by the Leibniz rule and the chain rule:

$$\begin{aligned}
0 &\leqslant \lambda^2 \int_X \phi^2 \mu(u, u)(dx) + \int_X \mu(\phi, \phi)(dx) + \int_X \phi^2(x)\, k(dx) \\
&\quad + \iint_{X \times X - d} [\phi^2(x) + \phi^2(y) - 2\phi(x)\,\phi(y)(\cos \lambda u(x) \cos \lambda u(y) \\
&\quad + \sin \lambda u(x) \sin \lambda u(y))]\; j(dx, dy) \\
&= a(\phi \cos(\lambda u), \phi \cos(\lambda u)) + a(\phi \sin(\lambda u), \phi \sin(\lambda u)) \\
&\leqslant \liminf \{ a_{h'}(\phi \cos(\lambda u), \phi \cos(\lambda u)) + a_{h'}(\phi \sin(\lambda u), \phi \sin(\lambda u)) \} \\
&\leqslant \Lambda \liminf \left\{ \lambda^2 \int_X \phi^2\, |Du|^2\, \beta_{h'}(dx) + \int_X |D\phi|^2\, \beta_{h'}(dx) \right\} \\
&= \Lambda \left\{ \lambda^2 \int_X \phi^2\, |Du|^2\, \beta(dx) + \int_X |D\phi|^2\, \beta(dx) \right\}.
\end{aligned}$$

By dividing the previous inequality by λ^2 and letting $\lambda \to \infty$, we find

$$\int_X \phi^2 \mu(u, u)(dx) \leqslant \Lambda \int_X \phi^2\, |Du|^2\, \beta(dx),$$

therefore,

$$0 \leqslant \int_X \phi^2 \sum_{ij=1}^{m} \xi_i \xi_j v^{ij}(dx) \leqslant \int_X \phi^2\, |\xi|^2\, \beta(dx).$$

This implies

$$0 \leqslant \sum_{ij=1}^{m} \xi_i \xi_j v^{ij}(dx) \leqslant \Lambda\, |\xi|^2\, \beta(dx) \qquad \text{in } X \text{ for every } \xi \in \mathbb{R}^m. \quad \blacksquare$$

Remark. Assumption (5.9) above is weak enough to allow the energy measures v_h to develop strong singularities in X as $h \to \infty$. An asymptotic singular energy measure is the one occurring in the *conductive thin-layer* model of Example 6.5.1. More surprisingly, as already mentioned in Section 4.2, the energy measures v_h may even generate in the limit non-trivial jumping measures and bring to a loss of locality in the asymptotic form. This interesting phenomenon is illustrated by Examples 6.1.1 to 6.3.1 of the following Section 6.

5.5. *Compactness for Degenerate Diffusions*

The diffusion character of the forms (5.10) will be preserved in the limit provided assumption (5.9) of Theorem 5.4.1 is verified in the strenghtened form of condition (5.12) below.

We assume that condition (5.8) is now satisfied with $\beta_h(dx) = \beta_h(x)\, m(dx)$, $\beta_h \in L^1_{\text{loc}}(X, m)$, $\beta_h(x) \geqslant 0$ m-a.e. in X, such that

$$\beta_h \text{ converges weakly in } L^1_{\text{loc}}(X, m) \text{ to } \beta \in L^1_{\text{loc}}(X, m), \qquad \text{as} \quad h \to \infty. \tag{5.12}$$

Clearly, the assumptions of Theorem 4.5.1 are now satisfied and we get the following:

THEOREM 5.5.1. *Under the assumptions* (5.8) *and* (5.12), *let* $\{a_h\}$ *be the sequence of forms* (5.10). *Then the conclusions* (i), (ii) *of Theorem* 5.4.1 *hold, with* $j = 0$ *and* $k = 0$. *Furthermore*:

(iii) *There exist, and is unique, an invariantly defined tensor* $a = (a^{ij})$ *on* X, *with* $a^{ij} = a^{ji} \in L^1_{\text{loc}}(X, m)$, $i, j = 1, ..., m$, *satisfying the degenerate ellipticity condition*

$$0 \leqslant \sum_{ij=1}^{m} \xi_i \xi_j a^{ij}(x) \leqslant \Lambda\, |\xi|^2\, \beta(x) \qquad m\text{-a.e. on } X \text{ for every } \xi \in \mathbb{R}^m, \tag{5.13}$$

such that the form a is represented by

$$a(u, v) = \int_X \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, a^{ij}(x)\, m(dx) \tag{5.14}$$

for every u, $v \in C^1_0(X)$ *and by*

$$a(u, v) = \int_X \mu(u, v)(dx) \tag{5.15}$$

for arbitrary u, $v \in D[\bar{a}_C]$, *the energy measure* μ *being uniquely determined by the tensor* $a = (a^{ij})$.

We recall that $\bar{a}_C$ denotes the closure in H of the restriction of a to $C = C^1_0(X)$, which is a regular Dirichlet form of diffusion type in H, and μ is its energy measure in X. Moreover, a is an extension of $\bar{a}_C$.

Proof of Theorem 5.5.1. As a consequence of the assumptions (5.8) and (5.12), the sequence $\{\mu_h\}$ of the (admissible) energy measures

$$\mu_h(u, v) = \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, \nu^{ij}_h(dx),$$

defined for u, $v \in C = C^1_0(X)$, is bounded and absolutely continuous in X, according to Definition 4.3.1. Therefore, Theorem 4.5 applies, with $C = C^1_0(X)$ and $k_h = 0$ for every h. In particular, (i), (ii), and (iii) of Theorem 4.1.2 hold, with $j = 0$ and $k = 0$. Moreover, by Theorem 5.4.1(ii), the energy tensor ν of a satisfies the ellipticity condition (5.11). By (5.12), $\beta(dx) = \beta(x)\, m(dx)$ with $\beta \in L^1_{\text{loc}}(X, m)$, therefore, by Radon–Nikodym theorem, (5.11) implies that (5.13) holds, with $a^{ij} = d\nu^{ij}/dm$. Moreover, (5.14) and (5.15) follow from (iii) and (ii) of Theorem 4.1.2. ∎

Remark. Also in Theorem 5.5.1 we may have $a \neq \bar{a}_C$ as already remarked in Section 4.1, see Example 6.5.1.

Remark. The equi-integrability condition (5.12) and simplified versions of Theorem 5.5.1 for elliptic operators in euclidean spaces were first given in [BDM, CS, MS].

5.6. *Coercive Forms*

In both Theorems 5.4.1 and 5.5.1, no condition is imposed to the forms *from below*, except of course that of being non-negative. Therefore, they may become highly degenerate in X and develop quite big domains in H, loosing their regularity and their diffusion character on "large" subsets of X. A nontrivial example is the already mentioned *insulating thin-layer* model, described in the following Section 6.5.

We now describe some examples of asymptotically regular forms that are also *asymptotically compact* in H, according to Definition 2.3.1. These are forms that, in addition to be estimated from above as in the previous section, can also be estimated *from below*, due to suitable *equi-coerciveness* properties, what provides additional control of the domains *from above*. The domain $D[a]$ of the limit form is injected in some special subspace of H and the functions $u \in D[a]$ inherit additional properties, for example, "weak" differentiability properties, as in Example 6.4.1.

We suppose that, for every h, $\nu_h = (\nu_h^{ij})$ is an admissible energy measure in X, that satisfies the condition

$$\lambda \sum_{i=1}^{m} \xi_i^2 \alpha^{ii}(dx) \leqslant \sum_{ij=1}^{m} \xi_i \xi_j \nu_h^{ij}(dx) \leqslant \Lambda\, |\xi|^2\, \beta_h(dx) \qquad \text{in } X \text{ for every } \xi \in \mathbb{R}^m, \tag{5.16}$$

where $0 < \lambda \leqslant \Lambda$ are given constants,

$$\alpha = (\alpha^{ii}(dx)\delta^{ij}) \qquad \text{is an admissible energy measure in } X, \tag{5.17}$$

independent of h, and $\{\beta_h\}$ is a sequence of positive Radon measures in X satisfying condition (5.9).

The form

$$\alpha(u, v) := \int_X \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, \alpha^{ij}(dx), \tag{5.18}$$

$D[\alpha] = C_0^1(X)$, is then closable in H, and its closure $\bar{\alpha}$ is a regular Dirichlet form of diffusion type in H.

We then have the following:

THEOREM 5.6.1. *Under the assumptions* (5.9), (5.16), *and* (5.17), *let* a_h *be the forms* (5.10). *Then, the conclusions* (i) *and* (ii) *of Theorem* 5.4.1 *hold; moreover,*

$$C_0^1(X) \subset D[a] \subset D[\bar{\alpha}],$$

and the energy measure $\nu = (\nu^{ij})$ *satisfies the condition*

$$\lambda \sum_{i=1}^{m} \xi_i^2 \alpha^{ii}(dx) \leqslant \sum_{ij=1}^{m} \xi_i \xi_j \nu^{ij}(dx) \leqslant \Lambda\, |\xi|^2\, \beta(dx) \qquad \text{on } X \text{ for every } \xi \in \mathbb{R}^m, \tag{5.19}$$

in X. Furthermore, (iii) *below holds:*

(iii) *If, in addition,* $D[\bar{\alpha}]$ *is compactly injected in H, then* $a_{h'}$ *converges to a in H as* $h' \to \infty$ *according to Definition* 2.1.1, *and the resolvent operator* $G_{h',\beta}$ *of the form* $\bar{a}_{h'}$ *converges for each* $\beta > 0$ *to the resolvent operator* G_β *of the form a in the strong operator topology of H, as* $h' \to \infty$. *Moreover, semigroups and spectral resolutions also converge, as described in Sections* 2.6, 2.7.

Proof. Clearly, Theorem 5.4.1 applies. By (i) of Theorem 5.4.1 and by Corollary 2.10.3, we have, in particular, $C_0^1(X) \subset D[a] \subset D[\bar{\alpha}]$ and

$$\lambda \bar{\alpha}(u, u) \leqslant a(u, u) \qquad \text{for every} \quad u \in H.$$

By the domination principle of Section 3(m) and the Remark at the end of Section 3(q), this gives the first inequality of (5.19). Moreover, (ii) of Theorem 5.4.1 holds, in particular, the second inequality of (5.19) also holds. Finally, if $D[\bar{\alpha}]$ is compactly injected in H, then by (5.16) the sequence $\{a_h\}$ is asymptotically compact in H, therefore Theorem 4.2.1 applies and this proves (iii) above (recall, in particular, that the forms a_h are densely defined in H). ∎

Remark. For a stochastic description of the energy measures of the functions that belong to $D[\bar{\alpha}]$ we refer to [F, (5.4.36)].

5.7. *Spectral Compactness for Coercive Diffusions*

We now consider a special case of Theorem 5.5.1 and 5.6.1, in which, according to Theorem 4.5.1, the asymptotic form inherits the diffusion property on $C_0^1(X)$.

We assume that $a_h^{ij} = a_h^{ji} \in L^1_{\text{loc}}(X, m)$, $i, j = 1, ..., m$, satisfy the condition:

$$\lambda\,|\xi|^2\,\beta(x) \leqslant \sum_{ij=1}^{m} \xi_i \xi_j a_h^{ij}(x) \leqslant \Lambda\,|\xi|^2\,\beta(x) \qquad m\text{-a.e. in } X \text{ for every } \xi \in \mathbb{R}^m, \tag{5.20}$$

for some constants $0 < \lambda \leqslant \Lambda$ and every h, with $\beta \in L^1_{\text{loc}}(X, m)$, $\beta(x) \geqslant 0$, such that $\beta(dx) := (\beta(x)\, m(dx)\delta^{ij})$ is an admissible energy measure in X, that is, such that the form

$$\tilde{\beta}(u, v) = \int_X Du\, Dv\, \beta(x)\, m(dx), \tag{5.21}$$

with domain $D[\tilde{\beta}] = C_0^1(X)$, is closable in H. The closure of $\tilde{\beta}$ in H will be denoted by β.

THEOREM 5.7.1. *Under the assumptions* (5.20), (5.21), *let* $\{a_h\}$ *be the sequence of the diffusion forms* (5.10) *in* H, *where* $v_h^{ij}(dx) = a_h^{ij}(x)\, m(dx)$. *Then*:

(i) *There exist a regular Dirichlet form of diffusion type* a *in* H, *with* $D[a] = D[\beta]$ *and core* $C_0^1(X)$, *and a subsequence* $\{a_{h'}\}$ *of* $\{a_h\}$, *such that* $a_{h'}$ Γ-*converges to* a *in* H *as* $h' \to \infty$.

(ii) *There exist and are unique* $a^{ij} = a^{ji} \in L^1_{\text{loc}}(X, m)$ *in* X, *verifying the condition*

$$\lambda\,|\xi|^2\,\beta(x) \leqslant \sum_{ij=1}^{m} \xi_i \xi_j a^{ij}(x) \leqslant \Lambda\,|\xi|^2\,\beta(x) \qquad m\text{-a.e. on } X \text{ for every } \xi \in \mathbb{R}^m, \tag{5.22}$$

such that

$$a(u, v) = \int_X \sum_{ij=1}^{m} (\partial u/\partial x_i)(x_1, x_2, ..., x_m)(\partial v/\partial x_j)(x_1, x_2, ..., x_m)\, a^{ij}(x)\, m(dx)$$

for every $u, v \in C_0^1(X)$. *Moreover,*

$$a(u, v) = \int_X \mu(u, v)(dx)$$

for every $u, v \in D[a]$, *where* μ *is the energy measure of* a.

(iii) *In addition, if* $D[\beta]$ *is compactly injected in* $L^2(X, m)$, *then* $a_{h'}$ *converges to* a *as* $h' \to \infty$ *according to Definition* 2.1.1 *and, furthermore, resolvents, semigroups, spectral operators and spectral subspaces also converge, as described in Sections* 2.4, 2.6, 2.7 *and Theorem* 4.2.1.

Proof. Both Theorems 5.5.1 and 5.6.1 clearly apply. In particular, (i), (ii) of Theorem 5.4.1 hold, with $j=0$ and $k=0$, and, moreover, (5.19) of Theorem 5.6.1 holds with

$$\alpha^{ii}(dx)=\beta(dx)=\beta(x)\,m(dx),\ \beta\in L^1_{\mathrm{loc}}(X,m),\qquad \text{for every } i=1,\ldots,m.$$

Therefore, the form a, provided by (i) of Theorem 5.4.1, is a regular Dirichlet form of diffusion type in H, with domain $D[a]=D[\beta]$ and core $C_0^1(X)$.

Moreover, by Radon–Nikodym theorem, (5.22) follows from (5.19). This proves (i) and (ii) above. Finally, (iii) above follows from (iii) of Theorem 5.6.1. ▮

Remark. For conditions ensuring the admissibility of β, that is, the closability of the form (5.21), we refer to [AR].

Remark. Special results of the type of Theorem 5.7.1 for uniformly elliptic operators in euclidean spaces were first obtained by S. Spagnolo [S] and E.Y. Hruslov [H].

6. Examples

6.1. *Nonlocal Relaxations*

Example 6.1.1. This example describes the possible effect of point singularities of the measure m on a relaxed form. We take $X=\mathbb{R}^m$ and fix $n+1$ points $\{z_o,\ldots,z_n\}\in X$, connected by curves z_{hk} of lenght $|z_{hk}|$ meeting only at their end points, and a non-negative symmetric $n\times n$ matrix (c_{hk}). We then consider the form

$$a(u,u)=\frac{1}{2}\sum_{hk=0}^{n} c_{hk}\int_{z_{hk}}\left|\frac{du}{ds}\right|^2 ds,$$

with domain $D[a]=\{u\in C_0^1(X): u(z_o)=0\}$. We define $m(dx)=dx+\sum_{h=o}^{n}\delta_{\{z_h\}}(dx)$, where dx is the Lebesgue measure in $\mathbb{R}^N$ and $\delta_{\{z\}}(dx)$ the Dirac mass at z, and take the relaxation $\underline{a}$ of a in the space $L^2(X,m)$. Then,

$$\begin{aligned}\underline{a}(u,u)&=\frac{1}{2}\sum_{hk=o}^{n}\frac{|u(z_h)-u(z_k)|^2}{R_{hk}}\\&=\frac{1}{2}\sum_{h\neq k=1}^{n}\frac{|u(z_h)-u(z_k)|^2}{R_{hk}}+\sum_{h=1}^{n}\frac{|u(z_h)|^2}{R_{ho}},\end{aligned}$$

with

$$D[\underline{a}] = \{u \in L^2(X, m) : u(z_o) = 0\},$$

where

$$R_{hk} = \frac{|z_{hk}|}{c_{hk}} \qquad \text{for every} \quad h, k.$$

We note that the initial form a is not closable in $L^2\ (X, m)$, in agreement with Theorem 2.1.2 of [F], already mentioned in Section 3(i). Moreover, we observe that the relaxation of a in $L^2(X, dx)$ is identically zero.

The (non-local) forms $\underline{a}$ were introduced by Beurling and Deny [BD1], as the basic example of their theory. We refer to that fundamental paper also for physical interpretations of the energy form $\underline{a}$.

6.2. *Singular Weights and Nonlocal Potentials*

EXAMPLE 6.2.1. This example shows that the intrinsic capacity has an important role in connection with traces and nonlocalities. We take $X = \Omega$, a bounded open subset of $\mathbb{R}^N$. We fix $\{z_o, ..., z_n\}$, z_{hk} and (c_{hk}) as in Example 6.1.1, with $z_o \in \partial\Omega$. We take a *weight* w in the Muckenhoupt A_2-class, that is, a function w, $w^{-1} \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying the condition

$$⨍_B w\, dx ⨍_B w^{-1} dx \leqslant c\, |B^2| \qquad \text{uniformly on the euclidean balls } B \text{ of } \mathbb{R}^N,$$

where $|B|$ is the N-dimensional volume of B. We further assume that w is such that

$$\begin{aligned} w\text{-cap}\{z_h\} > 0 \qquad &\text{for every } h,\\ w\text{-cap}\{z_{hk}^{(\varepsilon)}\} = 0 \qquad &\text{for every } h, k \text{ and every } \varepsilon > 0, \end{aligned} \tag{6.1}$$

where $\{z_{hk}^{(\varepsilon)}\}$ denotes the open portion of the curve z_{hk} off an ε-neighborhood of its end points and w-cap denotes the capacity of the form (5.21), where $\beta(x) = w(x)$. We consider the form

$$a(u, u) = \int_X |Du|^2\, w(x)\, dx + \frac{1}{2} \sum_{hk=0}^{n} c_{hk} \int_{z_{hk}} \left|\frac{du}{ds}\right|^2 ds, \tag{6.2}$$

with domain $D[a] = C_0^1(X)$, and we take the relaxation $\underline{a}$ of a in $L^2(X, dx)$. We find

$$\underline{a}(u, u) = \int_X |Du|^2\, w(x)\, dx + \frac{1}{2} \sum_{h \neq k = 1}^{n} \frac{|\tilde{u}(z_h) - \tilde{u}(z_k)|^2}{R_{hk}} + \sum_{h=1}^{n} \frac{|\tilde{u}(z_h)|^2}{R_{ho}}, \tag{6.3}$$

with domain $D[\underline{a}] = \{u \in L^2(X, dx) : Du \in L^2(X, w\,dx),\ \tilde{u}(z_o) = 0\}$ and R_{hk} as in Example 6.1.1.

An example of a weight w with the required property is given by

$$w(x) = \sum_{h=o}^{n} |x - z_h|^{\beta_h}, \qquad \text{with} \quad -N < \beta_h < -(N-2), \tag{6.4}$$

with $N \geqslant 3$, as it follows from the following formula due to [FJK],

$$w\text{-cap}(B_r(z), B_R(Z)) = \left[\int_r^R \frac{s^2}{w(B_s(z))} \frac{ds}{s} \right]^{-1},$$

where $w(B_s) = \int_{B_s} w\,dx$.

6.3. *Singular Conductivities and Nonlocal Potentials*

EXAMPLE 6.3.1. We take X, $\{z_o, ..., z_n\}$, z_{hk}, (c_{hk}) and w as in Example 6.2.1. For every $\varepsilon > 0$ and every h, k, we take $T_\varepsilon(z_{hk})$ to be the open ε-tubular neighborhood of the curve z_{hk} at distance greater than ε from the end points of z_{hk}. We consider the sequence of forms

$$a_\varepsilon(u, u) = \int_X |Du|^2\, w(x)\, dx + \frac{c}{2\varepsilon^{N-1}} \sum_{hk=0}^{n} c_{hk} \int_{T_\varepsilon(z_{hk})} |Du|^2\, dx, \qquad \varepsilon > 0, \tag{6.5}$$

with domain $D[a_\varepsilon] = C_0^1(X)$. Then, a_ε is a closable form in $L^2(X, dx)$ and, for a suitable value of the constant c, a_ε converges to the form (6.3) according to Definition 2.1.1, as $\varepsilon \to 0$. Moreover, resolvents, semigroups and spectral families also converge, in the sense of Sections 2.5, 2.6, and 2.8.

For $N \geqslant 4$ and w of the form (6.4) this example was first given in [BDM], as a development of previous examples of the same type due to [CS, MS], intended to show that an equivalent property of locality, namely the *additive* character of the forms with respect to the integration domain, may be lost by Γ-convergence. To our knowledge, however, nonlocal examples of this type were not explained by a general theory.

We observe that the "conductivity coefficients"

$$a^{ij}(x) := \left[w(x) + \frac{c}{2\varepsilon^{N-1}} \sum_{hk=0}^{n} c_{hk} \mathbf{1}_{T_\varepsilon(z_{hk})}(x) \right] \delta^{ij}, \qquad x \in X,$$

remain bounded in $L^1(X, dx)$ as $\varepsilon \to 0$, however they converge weakly only in the sense of measures in X.

6.4. *Conductive Thin-Layers*

EXAMPLE 6.4.1. The *conductive thin-layer*. We take X to be a bounded open subset of $\mathbb{R}^N$ and write $x=(x', x_N)$, $x'=(x_1, ..., x_{N-1})$. For every $\varepsilon>0$ we consider the layer $\Sigma_\varepsilon=\{x\in X : |x_N|<\varepsilon\}$ of thickness ε around $\Sigma=\{x\in X : x_N=0\}$ and define the form

$$a_\varepsilon(u, u)=\int_X |Du|^2\,dx+\frac{c}{\varepsilon}\int_{\Sigma_\varepsilon} |Du|^2\,dx,$$

with $D[a_\varepsilon]=C_0^1(X)$. Then, a_ε is a closable form in $L^2(X, dx)$ and a_ε converges to the form

$$a(u, u)=\int_X |Du|^2\,dx+c\int_\Sigma |D_\Sigma u(x', 0)|^2\,dx', \tag{6.6}$$

as $\varepsilon\to 0$, in the sense of Definition 2.1.1, together with resolvents, semigroups and spectral families, as in Example 6.3.1. The domain of the form (6.6) is $D[a]=\{u\in H_0^1(X) : u|_\Sigma\in H_0^1(\Sigma)\}$, where $u|_\Sigma$ denotes the trace of u on Σ, and $D_\Sigma=(\partial/\partial x_1, ..., \partial/\partial x_{N-1})$.

We notice that (6.6) has *measure-valued* conductivities

$$v^{11}(dx)=\cdots=v^{(N-1)(N-1)}(dx)=dx+c\,dx'\delta_{\{0\}}(dx_N), \qquad v^{NN}(dx)=dx.$$

Because of the surface energy on Σ, the self-adjoint operator associated with the form (6.6) is the Laplace operator in $X-\Sigma$, with homogeneous Dirichlet condition on the boundary of X and with the *second order* transmission condition

$$\left[\frac{\partial u}{\partial n_\Sigma}\right]=c\Delta_\Sigma \qquad \text{on } \Sigma,$$

on the layer Σ, where

$$\Delta_\Sigma=\sum_{h=1}^{N-1}\frac{\partial^2 u(x', 0)}{\partial x_h^2}$$

is the tangential Laplacian on Σ and $[\partial u/\partial n_\Sigma]$ denotes the jump of the normal derivative of u accross Σ.

We observe that, as in Example 6.3.1, the conductivity coefficients remain bounded in $L^1(X, dx)$ as $\varepsilon\to 0$ and, for a suitable value of c, converge weakly in the sense of measures to $\beta=dx+|\Sigma\cap X|^{-1}\,\mathbf{1}_{\Sigma\cap X}(x', 0)\,dx'$.

This example is due to H. Pham Huy [PHSP] and E. Sanchez-Palencia [SP]. The diffusion process associated with the form (6.6) has been independently studied by N. Ikeda-S. Watanabe [IW] and, more recently, by M. Tomisaki [T].

6.5. *Insulating Thin-Layers*

EXAMPLE 6.5.1. The *insulating thin-layer*. We take X, Σ_ε and Σ as in Example 6.4.1, and we define now the form a_ε by

$$a_\varepsilon(u, u) = \int_{X - \Sigma_\varepsilon} |Du|^2 \, dx + c\varepsilon \int_{\Sigma_\varepsilon} |Du|^2 \, dx,$$

with $D[a_\varepsilon] = C_0^1(X)$. Then, a_ε is a closable form in $L^2(X, dx)$ and a_ε Γ-converges as $\varepsilon \to 0$ to the form

$$a(u, u) = \int_X |Du|^2 \, dx + c \int_\Sigma [u]_\Sigma^2 \, dx', \tag{6.7}$$

with domain $D[a] = \{u \in L^2(X, dx) : u \in H^1(\{x_N > 0\}) \cap H^1(\{x_N < 0\})\}$, where $[u]_\Sigma$ denotes the jump of u accross Σ.

The form (6.7) is a non-regular, densely defined Dirichlet form in $L^2(X, dx)$. The closure of its restriction to $C = C_0^1(X)$ is the form

$$a(u, u) = \int_X |Du|^2 \, dx, \qquad D[a] = H_0^1(X),$$

with energy measure $\mu(u, u)(dx) = |Du|^2 \, dx$ for $u \in C_0^1(X)$. We observe that the form (6.7) can still be written as

$$\int_X \mu(u, u)(dx)$$

on its full domain, provided we introduce the *singular* energy measure:

$$\mu(u, u)(dx) = |Du|^2 \, dx + c[u]_\Sigma^2 \, \mathbf{1}_\Sigma(x', 0) \, dx'. \tag{6.8}$$

This provides an explicit example of an energy measure μ defined on a space of discontinuous functions and shows, in particular, that the domain $D[a]$ of the form (6.7) is contained in the space BV and the gradient of arbitrary u in $D[a]$ can only be defined as a measure.

Moreover, this example also shows that in Theorem 5.5.1 we may have indeed $a \neq \bar{a}_C$.

Finally, by comparing this example with Example 6.4.1, of which it represents somehow the dual case, we find that now the *reciprocals* of the conductivity coefficients remain bounded in $L^1(X, dx)$ as $\varepsilon \to 0$ and they converge weakly to the same measure β of Example 6.4.1.

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