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Exercises Singularity Theory

1. *(1 point)* Show that $x^p + y^p$ and $x^p + y^p + x^2y^2$ $(p \geq 5)$ are not right equivalent in $\mathcal{R}_2$.

2. *(2 points)* Let $f = x^3 + y^3 + z^3 + 3 \cdot c \cdot xyz \in \mathcal{R}_3$, $c \in \mathbb{R}$, $c \neq 0, -1$.

   (a) Compute the Milnor number and determinacy of f.

   (b) Explain why $c = -1$ is special.

3. *(4 points)* Let $f = x^n$ and $F(x,t) := f_t(x) := x^n + tx^{n+1}$.

   (a) Determine a smooth function $H(t,x) : I \times (-\epsilon, \epsilon) \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an intervall containing 0), such that $h_t(x) := H(-,t)$ for all $t \in (-\epsilon, \epsilon)$ is a coordinate change (i.e., a local diffeomorphism at $0 \in \mathbb{R}$), and such that we have $f \circ h_t = f_t$.

   (b) Find a smooth function $\xi(x,t)$ which satisfies the following equation
   $$\xi(x,t) \cdot \partial_x F(X,t) + \partial_t F_t(x,t) = 0$$

   (c) Write down explicitly the following differential equation.
   $$\partial_t H(x,t) = \xi(H(X,t),t).$$

   (d) Show that a solution $H(x,t)$ of the equation $\partial_t H(x,t) = \xi(H(X,t),t)$ satisfies the identity
   $$\frac{dF(H(x,t),t)}{dt} = 0.$$

4. *(3 points)* Let $f = x^3 + xy^p \in \mathcal{R}_2$, $p > 2$. Find the *smallest* integer k such that $\mathbf{m}^k \subset \mathbf{m}J_f$.

   (Remark: This calculation is the main step towards establishing the precise value of the determinacy of f. Namely, one has the following theorem: *If f is a germ and k is the smallest integer such that $\mathbf{m}^k \subset \mathbf{m}J_f$ then the determinacy of f is either k or $k-1$.*)