Def. / Lemma 3.2:

(right equivalence)

a) Let $\mathcal{R}_{n,n} := \mathcal{R}_n^n = \underbrace{\mathcal{R}_n \oplus \ldots \oplus \mathcal{R}_n}_{n\text{-times}}$ $\mathcal{R}_n$-mod.

$m^{\oplus n} = m \oplus \ldots \oplus m \subset \mathcal{R}_{n,n}$ submodule of $\mathcal{R}_{n,n}$

$$\mathcal{G}_n := \Big\{ \underbrace{[\varphi]}_{\mathcal{E}^n(U) \quad \mathcal{O}^n(U)} = ([\varphi_1], \ldots, \overset{\in m \subset \mathcal{R}_n}{[\varphi_n]}) \in m^{\oplus n} \;\Big|\; \det (D\varphi)(0) \neq 0 \Big\} \subset m^{\oplus n}$$

b) $(\mathcal{G}_n, \circ)$ is group where $\circ$ is composition of map germs $(\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$

c) $\mathcal{G}_n$ acts on $\mathcal{R}_n$ via

$$\mathcal{G}_n \times \mathcal{R}_n \to \mathcal{R}_n$$
$$(\varphi, f) \mapsto (\varphi^{-1})^* f : U \to \mathbb{K}, \; x \mapsto f(\varphi^{-1}(x))$$

d) $(\mathcal{G}_n, \circ)$ is called group of

local diffeomorphisms $(\mathcal{R}_n = \mathcal{E}_n)$

local analytic coordinate changes)

biholomorphic — " —

for $\mathcal{R}_n = \mathcal{O}_n$

e) Two germs $f, g \in \mathcal{R}_n$ are called <u>right equivalent</u> iff they are in the same $\mathcal{G}_n$-orbit, i.e. iff $\exists \varphi \in \mathcal{G}_n : f = \varphi^* g = g \circ \varphi \in \mathcal{R}_n$

write $f \underset{R}{\sim} g$

action from the right

Pf: (b): clear from chain rule:

$$D(\varphi\circ\psi)(0) = (D\varphi)(0)\cdot(D\psi)(0)$$

(c) exercise.

examples: a) let $f \in m_{R_1}^{k-1} \subset R_1$ but $f \notin m_{R_1}^{k}$ (i.e. the class of f in $m_{R_1}^{k-1}/m_{R_1}^{k}$ is non-zero), then (lemma 1.6.)

$$f \underset{R}{\sim} \pm x^k \quad \left(\text{if } R_1 = \theta_1 \quad f \underset{R}{\sim} x^k\right)$$

b) $f \in m_{R_n} \subset R_n$, $(Df)(0) \neq 0$, the $f \underset{R}{\sim} x_1$

(lemma 1.7.)

(c) $f \in m_{R_n}^2 \subset R_n$, i.e. f has critical pt. at 0

$rk(D^2 f(0)) = n$ (i.e. f is non-deg at 0)

$\Longrightarrow f \underset{R}{\sim} D^2 f$

Morse lemma, Th. 1.8

(only proved for $\mathcal{R}_n = \mathcal{E}_n$, but it holds for $\mathcal{R}_n = \mathcal{O}_n$)

<u>Def. 3.3.:</u> Let $f \in \mathcal{R}_n$

a) Let $k \in \mathbb{N}$. f is called k-determined if $\forall g \in \mathcal{R}_n$ s.t. $T_f^k = T_g^k \Longrightarrow f \underset{R}{\sim} g$.

(clear: f k-determined $\Longrightarrow$ $k+1$-det.)

b) f is called finitely determined if $\exists k \in \mathbb{N}$: f is k-determined

c) if f is finitely determined,

then the minimal k s.t. f is k-determined is called the determinacy of f.

<u>Notation:</u> $f \in \mathcal{R} = \mathcal{R}_n$, write $j_k(f) := \overline{f} \in \mathcal{R}/_{m_{\mathcal{R}}^{k+1}}$ and call it the k-jet of f. Using the isomorphism (of vect. spaces)

$$\mathcal{R}/_{m^{k+1}} \longrightarrow \mathbb{K}[x_1 \dots x_n]_{\leq k}$$

we have $j_k(f) = T_f^k$.

<u>Def. 3.4:</u> a) Let $f \in \mathcal{R}_n$, then write $J_f \subset \mathcal{R}_n$ for the ideal generated by the partial derivatives of f, $J_f = (\partial_{x_1} f, \dots, \partial_{x_n} f)$

J_f is called Jacobi ideal or Jacobian ideal of f.

b) R/J_f is called Jacobian or Milnor algebra (it is a $\mathbb{K}$-alg.)

c) If $\mu := \dim_{\mathbb{K}}(R/J_f) < \infty$ then μ is called Milnor number of f, sometimes written $\mu(f)$.

<u>Lemma 3.5:</u> Let k be a field and (R, m) be a local k-algebra, e.g.

$$R = \begin{cases} R \\ \mathbb{K}[[x]] \end{cases}$$

and let $I \subset R$ be an ideal

Suppose that $\dim_k (R/I) =: r < \infty$

Then $m^r \subset I$.

recall R/m_R^k are f.d. $\mathbb{k}$-v.spaces

$\simeq \mathbb{K}[x]_{\leq k-1}$

Pf: $\forall k \in \mathbb{N}$: $m^k + I \subset R$ is an ideal

and $m^{k+1} + I \subset m^k + I$, consider

quotient map $\varphi_k : R/m^{k+1}+I \twoheadrightarrow R/m^k+I$

$\bar{a} \longrightarrow [a]$

(surjective!). Then we have: $\ker \varphi_k \simeq$

$m^k+I \big/ m^{k+1}+I$

$\Big($ "$\supseteq$": $\bar{a} \in m^k + I \Leftrightarrow \varphi_k(\bar{a}) = [a] = 0$

"$\subseteq$": $\bar{a} \in \ker \varphi_k \Leftrightarrow \varphi_a(\bar{a}) = [a] = 0 \Leftrightarrow a \in m^k + I \Big)$

Put $v_k := \dim_k \ker \varphi_k$, then

$$\boxed{v_k = 0} \Longleftrightarrow \underbrace{\frac{m^k + I}{m^{k+1} + I}}_{\displaystyle \overset{\|}{\frac{m^k}{m^k \cap I + m^{k+1}}}} = \{0\}$$

$$= \{0\}$$

$$\Longleftrightarrow m^k \subset m^k \cap I + \underset{\substack{\| \\ m \cdot m^k}}{m^{k+1}} = m^k \cap I + m \cdot m^k$$

$$\underset{\substack{\text{Nakayama} \\ (b)}}{\Longleftrightarrow} m^k \subset m^k \cap I \Longleftrightarrow \boxed{m^k \subset I}$$

hence: $v_k = 0 \Longleftrightarrow m^k \subset I$

so if $v_k = 0 \Longrightarrow v_{k+1} = 0$ (since $m^{k+1} \subset m^k$)

put $N := \min_k (v_k = 0)$

Now: $r := \dim_k(R/I) \overset{!}{=} \sum_{k \geq 0} v_k$

$$= \sum_{k=0}^{N-1} v_k$$

since $N = \min_k (v_k = 0) \Rightarrow \forall k < N : v_k > 0$

$$\Longrightarrow r \geq N \Rightarrow m^r \subset m^N$$

$$v_N = 0 \Longleftrightarrow m^N \subset I$$

Hence: $m^r \subset I$ $\square$

Theorem 3.6. (Mather 1968): Let $f \in \mathcal{R}_n$ let $k \in \mathbb{N}$ s.t. $m_R^{k+1} \subset m_R^2 \cdot J_f \Rightarrow f$ is k-determined.

Corollary 3.7.: $f \in \mathcal{R}_n$, $\mu = \mu(f) < \infty$

$\Longrightarrow$ f is $\mu+1$-determined.

(beware that determinacy of f can be much smaller)

Proof: from 3.5 we know that $m_{\mathcal{R}}^{\mu} \subset J_f$. This implies that

$$m_{\mathcal{R}_n}^{(\mu+1)+1} \subset m_{\mathcal{R}_n}^2 \cdot J_f$$

$\xRightarrow{3.6}$ f is $\mu+1$-determined.

$\square$

examples:

a) let $f = x^2 - y^{k+1} \in \mathcal{R}_2$, then

$$J_f = (\partial_x f, \partial_y f) = (2x, -(k+1)\cdot y^k)$$

$$= (x, y^k), \text{ obviously } J_f = (x, \underbrace{xy^{k-1}, x^2y^{k-2}, \ldots, x^{k-1}y, x^k, y^k}_{= m_{R_2}^k})$$

hence, $m_{R_2}^k \subset J_f$, so $\underbrace{R_2/m_{R_2}^k \cong \mathbb{K}[[x,y]]/(x,y)^k \cong \mathbb{K}[x,y]/(x,y)^2}_{=:A} \twoheadrightarrow R_2/J_f$

A is finite-dim. $\mathbb{K}$-vector space

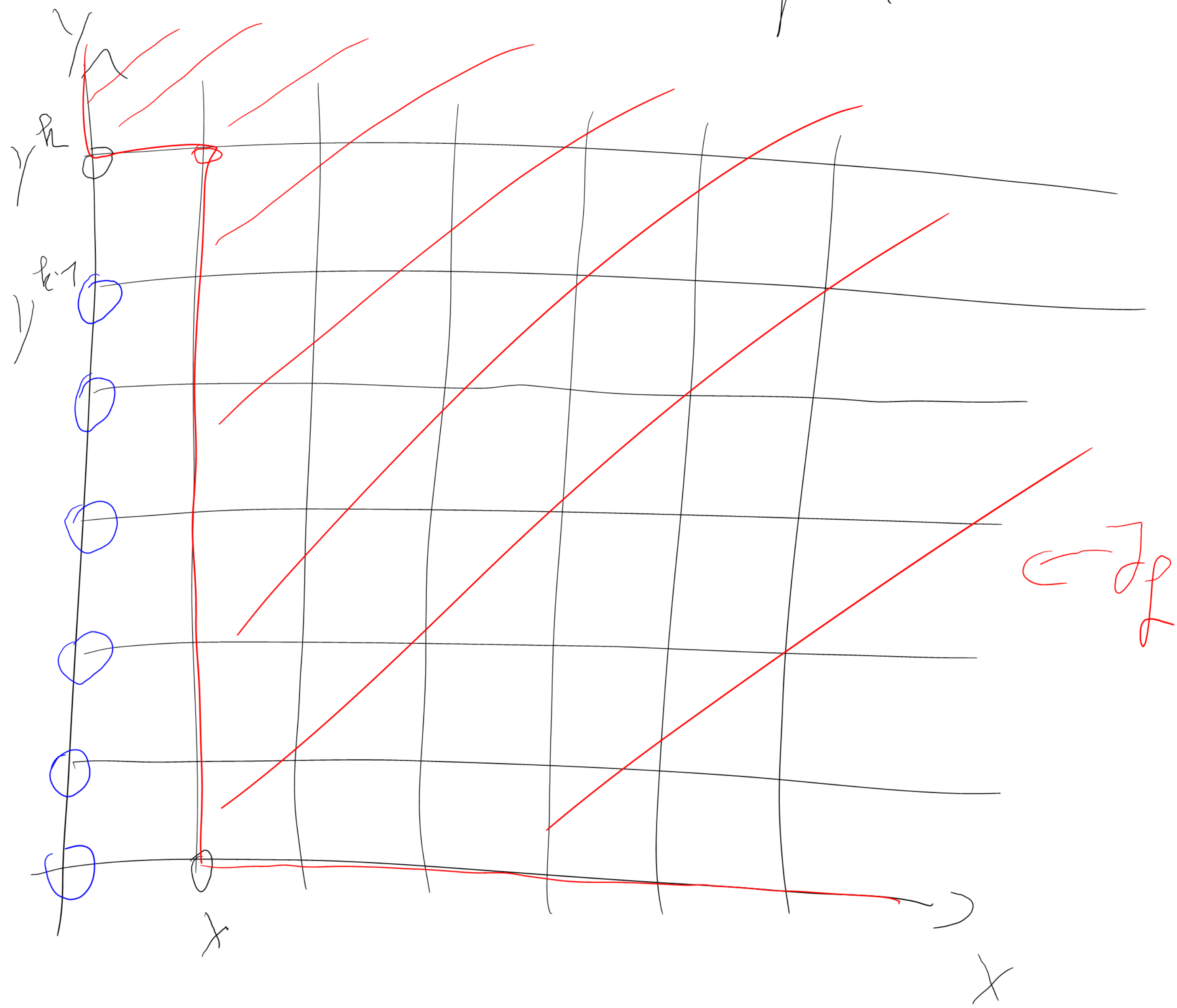
hence, also R_2/J_f is finite-dim. over $\mathbb{K}$ and so $\mu(f) < \infty$

Notice that we also have $A \twoheadrightarrow \mathbb{K}[x,y]/(x,y^k)$ (since $(x,y)^k \subset (x,y^k) \subset \mathbb{K}[x,y]$) but inclusion $\mathbb{K}[x,y] \hookrightarrow R_2$ induces inclusion $\mathbb{K}[x,y]/(x,y^k) \xhookrightarrow{\phi} R_2/J_f$, hence

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, two heads] & R_2/J_f \\ \mathbb{K}[x,y]/(x,y^k) \arrow[ur, hook, "\phi"'] & \end{tikzcd}$$

hence, ϕ is also surjective and so ϕ is isomorphism of $\mathbb{K}$-algebras

$\mathbb{K}[x,y]$; monomial diagram



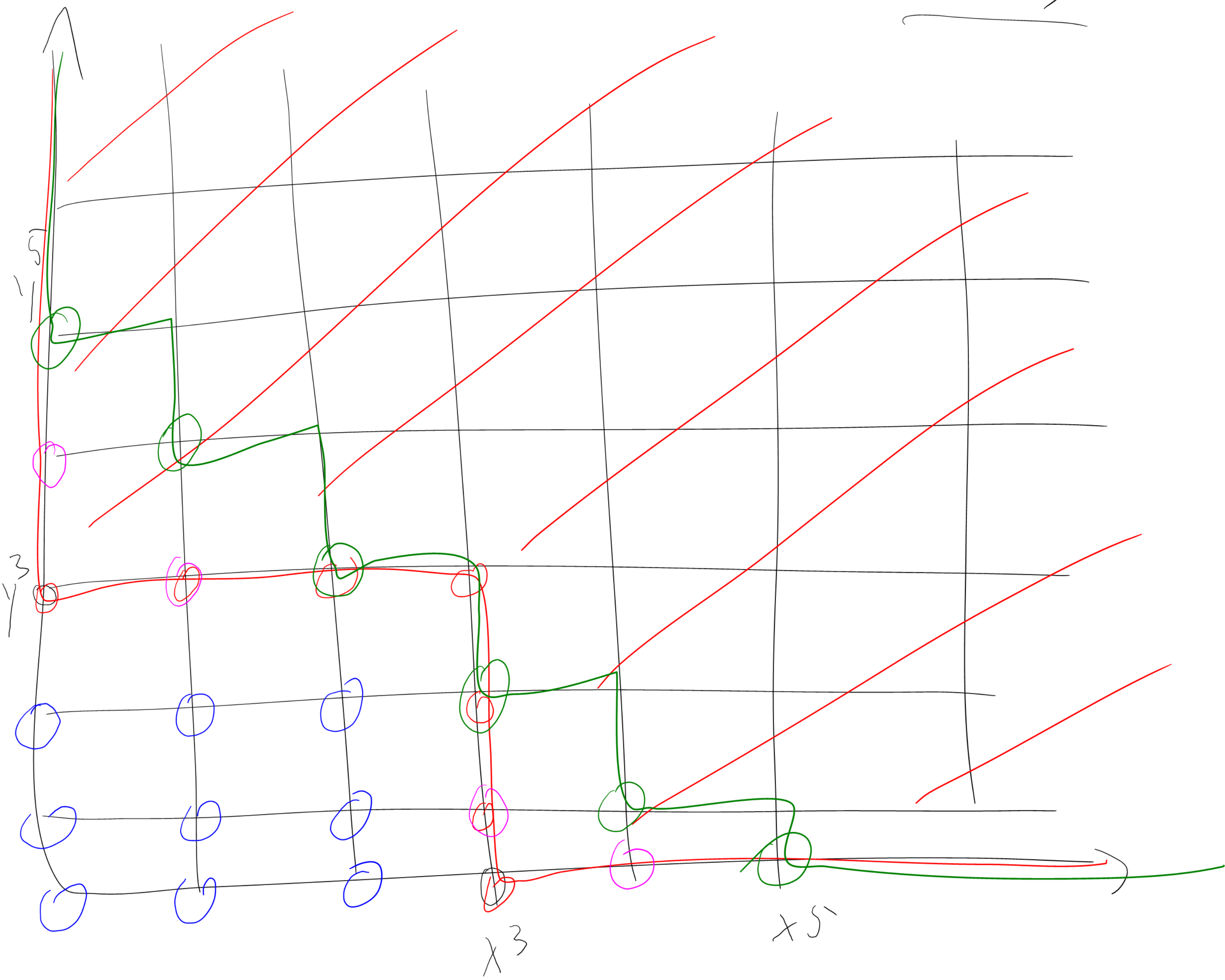
$$R_2/J_f \simeq \mathbb{K}[x,y]/(x,y^k) = \mathbb{K}[1] \oplus \mathbb{K}[y] \oplus \dots \oplus \mathbb{K}[y^{k-1}]$$

hence: $\mu(f) = \dim_{\mathbb{K}}(R_2/J_f) = k$

$\overset{3.7.}{\Longrightarrow}$ f is $k+1$-determ.

actually determinacy <u>is</u> $k+1$

b) $\underline{f = x^4 + y^4}$, $J_f = (4x^3, 4y^3) = \underline{(x^3, y^3)}$



$\mu(f) = 9 \stackrel{3.7}{\Longrightarrow} f$ is 10-determined.

$\underline{\text{But}}$: $m^5 \underset{\subset}{=} m^2 \cdot J_f \stackrel{3.6}{\Longrightarrow} f$ is 4-det.

det(f) (= Determinancy of f) = 4

since: if f were 3-determined

we would have $f \sim T^3_f = 0$ $\lightning$

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next time: Pf. of 3.6. in 7 steps.