recall:

Theorem 2.16. (Lemma of Nakayama):

Let (R, m) be a local ring, then

(a) Let A be a finitely generated R-module s.t.

$A \subset m \cdot A \implies A = \{0\}$

(b) Let A be an R-module, $B, C \subset A$ s.t.

B is finitely generated and s.t.

$B \subset C + m \cdot B \implies B \subset C$

(c) Let A be a finitely generated R-module, then

$A = \sum_{i=1}^{k} R \cdot a_i$ (i.e., $a_1, \dots, a_k$ are generators of A as R-module) $\iff A/mA = \mathrm{Span}_{R/m}([a_1], \dots, [a_k])$

(i.e. the classes $[a_1], \dots, [a_k]$ are a generating set of the R/m-vector space A/mA)

Proof:

(a) Let $a_1, \dots, a_k$ be a generating set of A as R-module $\Longrightarrow \forall i \in \{1, \dots, k\}$:

$$\underbrace{A \ni a_i = \sum_{j=1}^{k} b_{ij} \cdot a_j}_{} \quad \text{with } b_{ij} \in \mathfrak{m} \quad \forall i \in \{1, \dots, k\}$$

(since $A \subset \mathfrak{m} \cdot A$)

using matrices, this reads:

$$\boxed{\left(\mathrm{Id}_k - (b_{ij}) \right) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = 0}$$

$M(k \times k, R)$ Leibniz formula

$$\Longrightarrow \det\left(\mathrm{Id}_k - (b_{ij}) \right) \overset{\downarrow}{=} 1 + c$$

where $c \in m$. Hence (ex.)

$$\det(\mathrm{Id}_k - (b_{ij})) \in R^*$$

$\Rightarrow \exists$ inverse matrix $Y \in M(k \times k, R)$

(rk: R ring, $X \in M(k \times k, R)$, $\det(X) \in R^*$

$\Leftrightarrow \exists\, Y \in M(k \times k, R) : XY = YX = \mathrm{Id}_k$)

$$\Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = \underbrace{Y \cdot (\mathrm{Id}_k - (b_{ij}))}_{\mathrm{Id}_k} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = 0$$

$\Rightarrow a_1 = \ldots = a_k = \{0\} \Rightarrow A = \{0\}$

(b) $B, C \subset A$, B f.g. $B \subset C + mB$

to show: $B \subset C$

Consider $B+C = \{b+c \mid b \in B, c \in C\} \subset A$ (R-submodule of A) and $C \subset B+C$ hence, we can consider quotient $B+C/C$ (still R-module), isomorphic to B, hence, finitely generated. Now:

$$B \subset C + m \cdot B \Rightarrow \underline{B+C} \subset C + mB + C = \underline{C + mB}$$

$$\Rightarrow \frac{B+C}{C} \subset \frac{C+mB}{C} \overset{!}{\simeq} m \cdot \frac{C+B}{C}$$

so: $\frac{B+C}{C} \subset m \cdot \frac{C+B}{C}$ (and $B+C/C$ is f. g.)

$$\xRightarrow{(a)} \frac{B+C}{C} = \{0\} \Rightarrow B+C = C$$

$$\Rightarrow B \subset C$$

(c) „$\Rightarrow$": follows from 2.15 c)

„$\Leftarrow$": using (b) it suffices to show:

$$A \subset \sum_{i=1}^{k} R a_i + m A$$

Let $a \in A \underset{\substack{\uparrow \\ \text{assumption} \\ \text{of (c)}}}{\Longrightarrow} \exists\, r_1, \dots, r_k \in R$ s.t.

$$\underbrace{[a]}_{\in A/mA} = \sum_{i=1}^{k} \underbrace{[r_i]}_{\substack{\in R/m \\ \text{field}}} \cdot \underbrace{[a_i]}_{A/mA}$$

hence $a - \sum_{i=1}^{k} r_i a_i \in m \cdot A$

hence $a \in \sum r_i \cdot a_i + mA$

$\square$

example of application of Nakayama's lemma: check inclusion of ideals in $R = \begin{cases} \mathcal{E} \\ \mathcal{O} \end{cases}$ or $\mathbb{K}[[\underline{x}]]$ using 2.16. b):

<u>ex</u>: let $n=2$, $I = (x,y) \subset \mathbb{K}[[x,y]]$

$J = (x + \sin(xy),\ y + x^2 + y^3 \cos(x))$.

<u>Then</u>: $I \subset J$!

remarkable, since a priori it is unclear how to express x and y as lin. comb. of generators of J.

From 2.16. b) we know that it is suff. to show $I \subset J + m \cdot I = J + m^2$ (since $I = m$). What is $J + m^2$?

We have $J+m^2 = \left(x+\sin(xy),\, y+\underline{x^2}+y^3\cdot\cos(xy)\right) + \left(x^2, xy, y^2\right)$

$$= \left(x+\sin(xy),\, y+\underline{y^3\cos(xy)},\, x^2,\, xy,\, \underline{y^2}\right)$$

$$= \left(x+\sin(xy),\, y,\, x^2,\, \underline{xy},\, \underline{y^2}\right)$$

$$= \left(x+\sin(xy),\, y,\, x^2\right) \underset{\uparrow}{=} \left(x, y, x^2\right)$$

$$\sin(xy) \subset (xy) \subset (y)$$

$= (x,y) = I$. Hence $J+m^2 = I \Rightarrow I \subset J$.

<u>Lemma 2.17.:</u> Let $f \in R$, let $T_f^k = \sum_{|\nu| \leq k} \frac{D^\nu f(0)}{\nu!} x^\nu$ be its kth Taylorpolynomial:

(a) $\forall k \in \mathbb{N}$: $I_k := \{f \in R \mid T_f^{k-1} = 0\}$ is ideal in R, generated by $\{x^\nu \mid |\nu| = k\}$

(b) $I_1 = \{ f \in R \mid T_f^0 = 0 \} = \{ f \in R \mid f(0) = 0 \} = m$

(c) $I_k = m^k := \underbrace{m \cdot \ldots \cdot m}_{k\text{-times}} = (a_1 \cdot \ldots \cdot a_k \mid a_i \in m)$

(d) $\boxed{T : R \to \mathbb{K}[[x_1 \ldots x_n]]}$ induces ($\forall k \in \mathbb{N}$)

isomorphism of $\mathbb{K}$-algebras:

$$\boxed{R / m_R^k} \xrightarrow[\text{iso of } \mathbb{K}\text{-alg.}]{} \mathbb{K}[[\underline{x}]] / m^k_{\mathbb{K}[[\underline{x}]]}$$

and $\mathbb{K}[[\underline{x}]] / m^k_{\mathbb{K}[[\underline{x}]]} \overset{\downarrow}{\simeq} \dfrac{\mathbb{K}[x_1, \ldots, x_n]}{(x_1, \ldots x_n)^k}$

$$\underset{\uparrow}{\simeq} \mathbb{K}[x_1, \ldots, x_n]_{<k} = \{ g \in \mathbb{K}[x_1 \ldots x_n] \mid \deg(g) < k \}$$

iso of

$\mathbb{K}$-vect. sp.

(e) $\dim_{\mathbb{K}} \mathbb{K}[[x_1 \dots x_n]] / (x_1 \dots x_n)^k = \dim \mathbb{K}[x_1 \dots x_n]_{<d}$

$= \binom{n+k}{k}$ and $\dim \dfrac{m_R^k}{m_R^{k+1}}$ ← comm. rings without 1; ← ideal in m_R^k

$= \dim \left(\mathbb{K}[\underline{x}]_{<k+1} / \mathbb{K}[\underline{x}]_{<k} \right) = \binom{n+k-1}{k}$

(f) $\operatorname{Ker}(T : R \to \mathbb{K}[[\underline{x}]]) = \bigcap_{k \in \mathbb{N}} m_R^k =: m_R^\infty$

(this is ideal in R) and $R/_{m_R^\infty} \simeq \operatorname{Im}(T)$ isom. of $\mathbb{K}$-algebras

(g) $R = \mathcal{E}$: T surjective $\Rightarrow \mathcal{E}/_{m_\mathcal{E}^\infty} \simeq \mathbb{R}[[x]]$

$R = \mathcal{O}$: T inj. $\Rightarrow m_\mathcal{E}^\infty = (0) \Rightarrow \mathcal{O} \simeq \mathbb{C}\{x\}$

$\mathcal{O} \simeq \operatorname{Im}(T : \mathcal{O} \to \mathbb{C}[[x]])$

<u>Proof</u>: (a) I_k is ideal: $c, f, g \in \mathcal{R}$

s.t. $T_f^{k-1} = T_g^{k-1} = 0 \Rightarrow T_{f+c\cdot g}^{k-1} = 0$

using Lemma 1.3 (holds also for $\mathcal{R} = \mathcal{O}$):

$T_f^{k-1} = 0 \Rightarrow \forall \underline{\nu} \in \mathbb{N}^n : |\nu| = k \Rightarrow \exists g_\nu \in \mathcal{R}$:

$$f = \sum_{|\nu| = k} g_\nu x^\nu \Rightarrow f \in (x^\nu)_{|\nu| = k}$$

(b) clear: special case of (a) for $k = 1$

(c) clear: by definition: $m^k = (x_1, \dots, x_n)^k$

$$= (x^\nu)_{|\nu| = k} \overset{(a)}{=} I_k$$

(d) recall that $T : \mathcal{R} \to \mathbb{K}[[x]]$ is local, i.e.

$T(m_{\mathcal{R}}) \subset m_{\mathbb{K}[[x_1 \dots x_n]]}$ & T is homom. of $\mathbb{K}$-alg (i.e. $T(f\cdot g) = T(f)\cdot T(g)$)), hence:

$T(m_R^k) \subset m^k_{\mathbb{K}[[x]]}$. Then the

induced map: $\boxed{T : R/_{m_R^k} \to \mathbb{K}[[x]]/_{m^k_{\mathbb{K}[[x]]}}}$

is a well-defined map of $\mathbb{K}$-algebras

(that it is an iso will be shown in f(x)g))

clear: $\mathbb{K}[x_1 \dots x_n] \subsetneq \mathbb{K}[[x_1, \dots, x_n]]$ hence

$$\mathbb{K}[x_1 \dots x_n]/_{(x_1, \dots, x_n)^k} \subset \mathbb{K}[[x_1 \dots x_n]]/_{m^k_{\mathbb{K}[[x]]}}$$

$\forall \sum_\nu a_\nu x^\nu \in \mathbb{K}[[x]]$ we have: $[\sum a_\nu x^\nu] =$

$$\left[\sum_{|\nu|<k} a_\nu x^\nu + \sum_{|\nu|\geq k} a_\nu x^\nu\right] = \left[\sum_{|\nu|<k} a_\nu x^\nu\right] + \left[\sum_{|\nu|\geq k} a_\nu x^\nu\right] \text{ in } m^k_{\mathbb{K}[[x]]}$$

$$= \Big[\underbrace{\sum_{|\nu|<k} a_\nu x^\nu}_{\substack{\cap \\ \mathbb{K}[x_1 \dots x_n]_{<d}}} \Big] \text{ in } \mathbb{K}[\![x]\!] / m^k_{\mathbb{K}[\![x]\!]}$$

e) ex.

(f) $f \in \ker(T) \iff (D^\nu f)(0) = 0 \quad \forall \nu \in \mathbb{N}^n$

$$\iff T^k_f = 0 \quad \forall k \in \mathbb{N} \iff f \in \underset{\substack{\| \\ m^k_{\mathcal{R}}}}{I_k} \quad \forall k \in \mathbb{N}$$

$$\iff f \in m^k_{\mathcal{R}} \quad \forall k \iff f \in m^\infty_{\mathcal{R}} .$$

$$\implies T : \mathcal{R}/_{m^\infty_{\mathcal{R}}} \simeq \operatorname{Im}(T)$$

<u>notice:</u> $m^k_{\mathcal{R}} + m^\infty_{\mathcal{R}} = m^k_{\mathcal{R}}$, hence induced map

$$T : \left(\mathcal{R}/_{m^\infty_{\mathcal{R}}}\right)/_{m^k_{\mathcal{R}}} \simeq \mathcal{R}/_{m^k_{\mathcal{R}}} \underset{(*)}{\overset{\simeq}{\implies}} \operatorname{Im} T/_{m^k_{\mathbb{K}[\![x]\!]}}$$

is isomorphism.

(g) $\underline{R = \mathcal{E}}$ $\Rightarrow \operatorname{Im}(T) = \mathbb{R}[[x]] \Rightarrow T: \mathcal{E}/_{m_{\mathcal{E}}^\infty} \simeq \mathbb{R}[[x]]$

$$\xRightarrow{(*)} \mathcal{E}/_{m_{\mathcal{E}}^k} \simeq \mathbb{R}[[x]]/_{m^k_{\mathbb{R}[[x]]}}$$

$\underline{R = \mathcal{O}}$: $m_{\mathcal{O}}^\infty = (0)$. Moreover:

$$\mathbb{C}[[x]]/_{(x_1 \ldots x_n)^k} \simeq \underbrace{\mathbb{C}\{x\}}_{\operatorname{Im} T}/_{(x_1 \ldots x_n)^k}$$

$\underline{\text{hence (by *)}}$: $T: \mathcal{O}/_{m_{\mathcal{O}}^k} \xrightarrow{\simeq} \mathbb{C}[[x]]/_{m^k_{\mathbb{C}[[x]]}}$

$\square$

§3. Finite determinacy

Aim: (a) define an equivalence relation, idea: $f, g \in \mathcal{R}$, $f \sim g \iff$ $\exists$ coordinate change ϕ : $f = g \circ \phi$

(b) Def: f is k-determined iff

$$\forall g : T^k_f = T^k_g \implies f \sim g$$

(c) f has an isolated sing. at 0

$\implies$ f is finitely determined

$\exists k$: f is k-determined

reminder : group actions

Def. 3.1.: Let G be a group (not necc. abelian), M set

a) a map $G \times M \to M$ is called group action iff: i) $(g \cdot h)(x) = g(h(x))$ $\forall x \in M, g, h \in G$

ii) $e(x) = x$ $\quad$ e neutral el. of G.

write $g \bullet x := g \cdot x := g(x)$

b) relation $x \sim y \Leftrightarrow \exists g \in G : x = g y$ is equivalence relation on M, set of all elements of M equivalent to x is called orbit of x, written $G \cdot x$

the set of orbits (orbit space) is denoted M/G

ex.: (a) $G = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$

$M = \mathbb{R}^2$, $G \times M \to M$, $(r, x) = r \cdot x$

$$\forall x \in \mathbb{R}^2 \Rightarrow Gx = \begin{cases} \{\text{line through } x\} \setminus \{0\} & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

$$M/G = \underbrace{\text{set of lines through } 0}_{\mathbb{P}^2_{\mathbb{R}}} \cup \{0\}$$

(b) $G = gl(n, \mathbb{C})$, $M = M(n \times n, \mathbb{C})$

$G \times M \to M$: $(T, A) = TAT^{-1}$

$G \cdot A = \{\text{matrices conjugate to } A\}$

$M/G = \{\text{Jordan normal forms}\}$

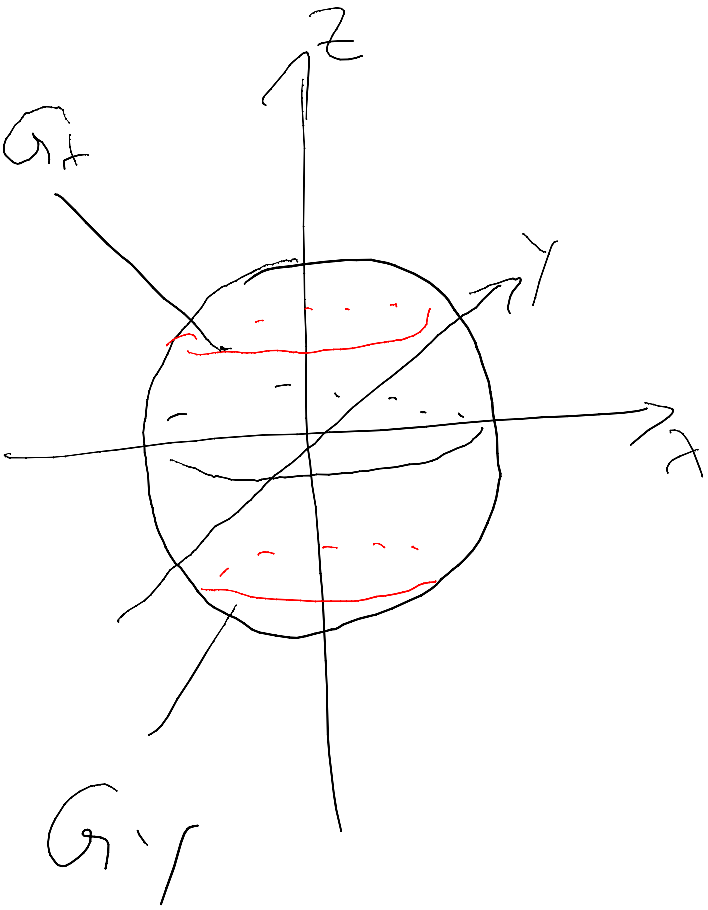
92

c) $G = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \,\middle|\, \varphi \in [0, 2\pi) \right\}$

$\subset O(2, \mathbb{R})$

$M = S^2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}$

$G \times M \to M : \left(A, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \mapsto \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array} \right) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$M/G \simeq [-1, 1]$