§1. The Morse lemma

aim: any smooth function $f: \underset{0 \in}{\mathcal{U}} \overset{\subset \mathbb{R}^n}{\longrightarrow} \mathbb{R}$

with an non-degenerate critical point at $0 \in \mathcal{U}$

can be written locally as a quadratic form

after coordinate change.

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recall from analysis:

Def. 1.1: Let $\mathcal{U} \subset \mathbb{R}^n$ be open

i) a continuous map $\phi = (\phi_1, \dots, \phi_m): \mathcal{U} \to \mathbb{R}^m$

is smooth $\Leftrightarrow$ all $\phi_i: \mathcal{U} \to \mathbb{R}$ are smooth

ii) for $\phi: \mathcal{U} \to \mathbb{R}^m$ smooth (1-differentiable

suffices) $D\phi = \left(\partial_{x_k} \phi_j \right)_{jk} : \mathcal{U} \to M(m \times n, \mathbb{R})$

the Jacobian matrix

iii) if $m = n$ and if ϕ is a bijection $\phi: \mathcal{U} \to V \subset \mathbb{R}^n$

with V open. Then we call ϕ a

diffeomorphism $\Leftrightarrow$ ϕ^{-1} is also smooth.

ex.: - $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto -x$ is a diffeom.

- $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is smooth, bijective, not diffeomorphism: inverse map $x \mapsto \sqrt[3]{x}$ is not diff. at $x=0$

Theorem 1.1. (local inverse function theorem)

Let $\phi: \underset{\subset \mathbb{R}^n}{U} \to \mathbb{R}^n$ smooth, $p \in U$, then $\exists$ open neighborhoods U' of p and V' of $\phi(p)$ s.t. $\phi_{|U'}: U' \to V'$ is a diffeomorphism if and only if: $(D\phi)(p) \in Gl(n, \mathbb{R})$, i.e. $\det (D\phi)(p) \neq 0$. In that case, we call ϕ a local diffeomorphism at p.

Proof: in analysis lectures.

Corollary 1.2.: If $f: U \to \mathbb{R}$ and if

$p \in \mathrm{Crit}(f) := \{x \in U \mid (Df)(x) = 0\}$

is non-degenerate, then p is isolated.

Pf.: Let $\phi := Df : U \to \mathrm{Mat}(1 \times n, \mathbb{R}) \cong \mathbb{R}^n$

consider $D\phi = D^2 f : U \to \mathrm{Mat}(n \times n, \mathbb{R})$. p is

non-deg. critical point of $f \Leftrightarrow \det(D^2 f)(p)$

$= \det(D\phi)(p) \neq 0 \Rightarrow \phi$ is a local

diffeomorphism at p, i.e., $\exists\, p \in V \subset U$

s.t. $\phi_{|V}$ is injective. Hence $\phi^{-1}(\underset{\mathrm{Mat}(1 \times n, \mathbb{R})}{0}) = \{p\}$

$\Rightarrow \phi(q) = Df(q) \neq 0 \quad \forall q \in V \setminus \{p\}$ $\square$

overall aim of lecture: Classify functions „up to local diffeomorphisms". More precisely

Let $0 \in U \subset \mathbb{R}^n$, $f: U \to \mathbb{R}$ smooth and let $\varphi: U \to U$ be a local diffeomorphism s.t. $\varphi(0) = 0$. Put $g(\underline{y}) := (\varphi^* f)(\underline{y})$ $:= f(\varphi(\underline{y}))$, then $g: U \to \mathbb{R}$ is smooth and chain rule shows:

$$\underbrace{(Dg)(0)}_{\substack{\cap \\ M(1\times n, \mathbb{R})}} = \underbrace{(Df)(0)}_{\substack{\cap \\ M(1\times n, \mathbb{R})}} \cdot \underbrace{(D\varphi)(0)}_{\substack{\cap \\ M(n\times n, \mathbb{R})}}$$

φ local diffeom. at $0 \overset{1.1}{\Longrightarrow} (D\varphi)(0) \in Gl(n, \mathbb{R})$

hence $(Dg)(0) = 0 \iff (Df)(0) = 0$

g has crit. pt. at $0 \iff f$ has crit. at zero

Moreover: $(Df)(0)=0$, then:

$$(D^2 g)(0) = \left((D\varphi)(0)\right)^{tr} (D^2 f)(0) \cdot (D\varphi)(0)$$

(exercise)

again: $(D\varphi)(0) \in Gl(n, \mathbb{C})$ implies:

$\mathrm{rk}\,(D^2 g)(0) = \mathrm{rk}\,(D^2 f)(0)$, i.e. g has non-degenerate crit. pt. at $0 \iff f$ has

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+ Sylvester invariants of $(D^2 f)(0)$ and $(D^2 g)(0)$ are equal.

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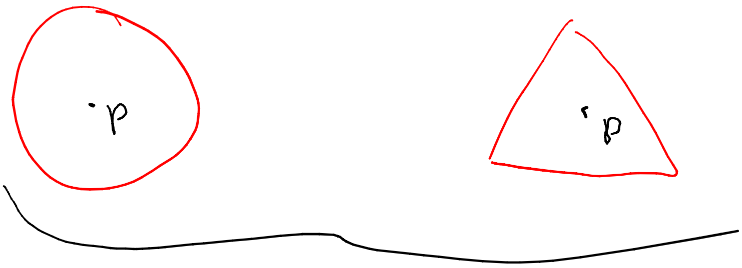
first aim: classify (up to local diffeom.)

a) non-critical pts. of smooth $f: U \to \mathbb{R}$

b) pts. of functions $f: \mathbb{R} \to \mathbb{R}$

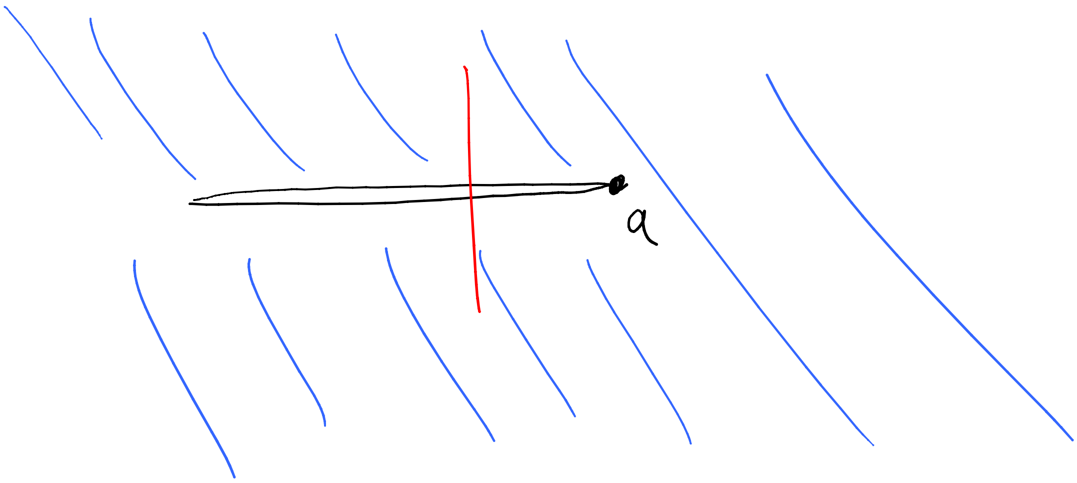
Definition 1.3: An open set $U \subset \mathbb{R}^n$ is called star-shaped with center $p \in U$ if $\forall q \in U$: $\{y \in \mathbb{R}^n : y = p + t(q-p), t \in [0,1]\} \subset U$.

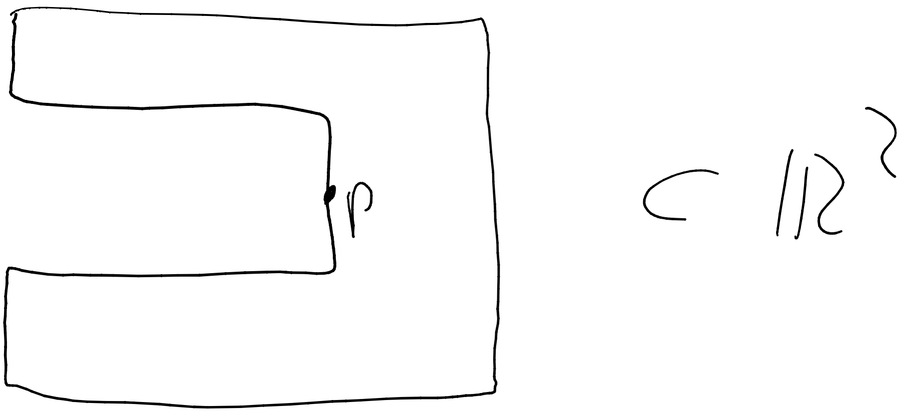
examples: obvious: convex $\Rightarrow$ star-shaped



convex ($\Rightarrow$ star shaped)

$\mathbb{R}^2 \setminus \{(-\infty, a) \mid a < 0\}$ star-shaped, not convex





not convex, not star-shaped

Lemma 1.4: Let $U \subset \mathbb{R}^n$ star-shaped with center $p \in U$ and let $f: U \to \mathbb{R}$ smooth. Let $k \in \mathbb{N}_{>0}$ s.t. $\forall \ell \in \{0, \dots, k-1\}$: $D^\ell f(p) = 0$

(i.e. $(\partial_{x_{i_1}} \dots \partial_{x_{i_\ell}} f)(p) = 0 \quad \forall i_1 \dots i_\ell \in \{1 \dots n\}$)

Then $\exists$ smooth functions $g_{i_1 \dots i_k}: U \to \mathbb{R}$ $\forall (i_1, \dots, i_k) \in \{1 \dots n\}^k$ s.t.

1.) $g_{i_1 \dots i_k} = g_{\sigma(i_1) \dots \sigma(i_k)} \quad \forall \sigma \in S_k$

2.) $f(\underline{x}) = \sum_{i_1, \dots, i_k = 1}^{n} g_{i_1 \dots i_k}(\underline{x}) \cdot \prod_{j=1}^{k} (x_{i_j} - p_{i_j})$

in part. $(k=1)$: $f(p)=0 \Rightarrow f(x)=\sum_{i=1}^{n} g_i(x)(x_i-p_i)$

$(k=2)$: $f(p)=0, (Df)(p)=0$

$$\Rightarrow f(x)=\sum_{i,j=1}^{n} g_{ij}(x)\cdot(x_i-p_i)(x_j-p_j)$$

$$g_{ij}: U \to \mathrm{Sym}(n,\mathbb{R})$$

3.) $g_{i_1\dots i_k}(p)=\frac{1}{k!}(\partial_{x_{i_1}}\dots\partial_{x_{i_k}} f)(p)$.

<u>Proof</u>: it suffices to prove case $p=0$

(otherwise $f(x)\mapsto f(x-p)$). Fix any

$x\in U$ and define $F:[0,1]\to\mathbb{R}$

by $F(t):=f(t\cdot x)$ (need U star shaped)

$\Rightarrow F(0)=f(0)=0$ and $F(1)=f(x)$

chain rule: $F'(t) = \sum_{i=1}^{n} x_i \cdot (\partial_{x_i} f)(t\underline{x})$

$$F''(t) = \sum_{i,j=1}^{n} x_i \cdot x_j \cdot (\partial_{x_i} \partial_{x_j} f)(t\underline{x}), \; \dots$$

$$\underline{F^{(\ell)}(x)} = \sum_{i_1 = \dots = i_\ell = 1}^{n} x_{i_1} \cdot \ldots \cdot x_{i_\ell} \cdot \left(\partial_{x_{i_1}} \cdots \partial_{x_{i_\ell}} f\right)(t\underline{x})$$

we have: $\forall\, \ell \leq k : F^{(\ell-1)}(s) = \underbrace{\int_0^s F^{(\ell)}(t)\,dt}_{= F^{(\ell-1)}(s) - \underbrace{F^{(\ell-1)}(0)}_{=0}}$

$\forall\, s \in [0,1]$

$\ell - 1 < k$

$\forall\; 0 \leq t_0 \leq t_1 \leq \ldots \leq t_k \leq 1$ we have

$$F^{(k-1)}(t_1) = \int_0^{t_1} F^{(k)}(t_0)\,dt_0, \quad F^{(k-2)}(t_2) = \int_0^{t_2} F^{(k-1)}(t_1)\,dt_1$$

$$\Rightarrow F(t_k) = \int_0^{t_k} F'(t_{k-1})\, dt_{k-1}$$

if $t_k = 1$: $F(1) = f(x) \overset{(*)}{=}$

$$\int_0^1 \dots \underbrace{\left(\int_0^{t_1} F^{(k)}(t_0)\, dt_0 \right)}_{F^{(k-1)}(t_1)} dt_1 \dots dt_{k-1}$$

Put: $\forall i_1 \dots i_k \in \{1, \dots, n\}$:

$$g_{i_1 \dots i_k}(x) := \int_0^1 \int_0^{t_{k-1}} \dots \int_0^{t_1} (\partial_{x_{i_1}} \dots \partial_{x_{i_k}} f)(t_0 \cdot x)\, dt_0 \dots dt_{k-1}$$

(recall $F^{(k)}(t) = \sum_{i_1 \dots i_k} (\partial_{x_{i_1}} \dots \partial_{x_{i_k}} f)(t \cdot x)\, x_{i_1} \dots x_{i_k}$)

then from $(*)$: $\boxed{f(x) = \sum_{i_1 \dots i_k \in \{1..n\}} g_{i_1 \dots i_k}(x) \cdot x_{i_1} \dots x_{i_k}}$

remains: $g_{i_1 \dots i_k}(x)$ are smooth: obvious since

∂_{x_j} and $\int_0^{t_k}$ interchange.

also: $g_{i_1 \dots i_k} = g_{\sigma(i_1) \dots \sigma(i_k)}$ since $\partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}$

$g_{i_1 \dots i_k}(0)$. use $\boxed{f(x) = \sum_{i_1 \dots i_k = 1}^{n} g_{i_1 \dots i_k}(x) \cdot x_{i_1} \dots x_{i_k}}$

$$\left(\partial_{j_1} \dots \partial_{j_k}\right)\left(g_{i_1 \dots i_k}(x) \cdot \underline{x_{i_1} \dots x_{i_k}}\right)(0) =$$

$$\begin{cases} \underline{g_{i_1 \dots i_k}(0)} & \text{if } (j_1 \dots j_k) = \sigma(i_1 \dots i_k) \\ & \text{for some } \sigma \in S_k \\ 0 & \text{if } (j_1 \dots j_k) \neq \sigma(i_1 \dots i_k) \end{cases}$$

since $|S_k| = k! \Rightarrow g_{i_1 \dots i_k}(0) = \frac{1}{k!}\left(\partial_{i_1 \dots i_k} f\right)(0)$

$\square$

Corollary 1.5. (Taylor development)

Let $f: U \to \mathbb{R}$ smooth, write

$$T^{k}_{f,p}(x) = \sum_{0 \leq |\alpha| \leq k} \frac{D^{\alpha} f}{\alpha!}(p)(x-p)^{\alpha}$$

for the k-th Taylor polynomial, with

$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$.

Then: $f(x) = T^{k-1}_{f,p}(x) + R(x)$

with $R \cancel{(x)} = \sum_{i_1, \dots, i_k = 1}^{n} g_{i_1 \dots i_k}(x)(x_{i_1} - p_{i_1}) \cdot \ldots \cdot (x_{i_k} - p_{i_k})$

and $g_{i_1 \dots i_k}(x) = \int_0^1 \int_0^{t_{k-1}} \cdots \int_0^{t_1} \underline{(\partial_{x_{i_1}} \cdots \partial_{x_{i_k}} f)(p + (x-p)t)}$

$dt_0 \dots dt_{k-1}$

and $|R(x)| \le \left(\sum_{i=1}^{n} |x_i - p_i|\right)^k \cdot \frac{1}{k!}$.

$$\sup_{\substack{t,\alpha,\ |\alpha|=k \\ t\in[0,1)}} \left| D^\alpha f(p-(x-p)t) \right|$$

<u>Proof</u>: apply lemma 1.4 to $\big(\underbrace{f - T^{k-1}_{f,p}}_{R}\big)(x)$

This yields formula (**) and

$$|R(x)| \le \underbrace{\sup_{(i_1,\dots,i_k)} |g_{i_1\dots i_k}(x)|}\cdot \overbrace{\left|\sum (x_{i_1}-p_{i_1})\cdot\ldots\cdot(x_{i_k}-p_{i_k})\right|}^{\le \left(\sum_{i=1}^{n}|x_i-p_i|\right)^k}$$

$$\le \sup_{t,\alpha} \left| D^\alpha f(p-(x-p)t) \right| \cdot \underbrace{\left| \int_0^1 \ldots \underbrace{\int_0^{t_1} dt_0}_{t_1} dt_1 \ldots dt_{k-1} \right|}_{= \frac{1}{k!}\int_0^1 t_{k-1}^{k-1}\, dt_{k-1} = 1/k!\ \square}$$

$\frac{1}{2}t_2^2$

$\vdots$

another application of 1.4.:

<u>Lemma 1.6.:</u> Let $0 \in U \subset \mathbb{R} = \mathbb{R}^1$ and let $f : U \to \mathbb{R}$ smooth. $\Rightarrow \exists\ 0 \in V \subset U$ and local diffeomorphism $\psi : V \to V$, $\psi(0) = 0$ s.t.: $f(\psi(y)) = \varepsilon^{k-1} \cdot y^k$, $k \in \mathbb{N}$, $\varepsilon \in \{1, -1\}$ if and only if $k = \min \{ l \in \mathbb{N} \mid f^{(l)}(0) \neq 0 \}$. If k even then $\varepsilon = f^{(k)}(0) / |f^{(k)}(0)|$.

<u>idea</u>: if some derivative of f at 0 does not vanish $\Rightarrow$ f is äquivalent to $\pm y^k$.