Singularity theory

§0. Introduction

singularity $\hat{=}$ exceptional state of a math. model $\hat{=}$ point in e.g. space/time in which model behaves differently than at ordinary points

examples of sing. in mathematics:

1.) linear algebra: a matrix $A \in M(n \times n, k)$ is called singular $\Leftrightarrow$ $\det(A) = 0$

2.) analysis 1: $f: \mathbb{R} \to \mathbb{R}$ differentiable map is called singular at $x \in \mathbb{R}$ $\Leftrightarrow$ $f'(x) = 0$

3.) analysis 2: $f = (f_1, \dots f_n): U \subset \mathbb{R}^n \to \mathbb{R}^n$ is call singular at $x \in U$ $\Leftrightarrow$ $\det(\underbrace{Df}_{\in M(n \times n,\ \text{functions on } U)})(x) = 0$

4.) differential equations: let $I \subset \mathbb{R}$ interval s.t. $0 \in I$ and $a: I \setminus \{0\} \to \mathbb{R}$ diff. function, then 0 is called a singular point of the differential equation: $y'(t) \overset{(*)}{=} a(t) \cdot y(t)$

ex.: $a(t) = \frac{k}{t}$, $k \in \mathbb{Z}$ => solutions of (*):

$y(t) = t^k$, i.e: if $k \geq 0$ then $y(t)$ is defined at $t=0$ but not if $k<0$

remark: for $a(t) = \frac{k}{t}$ the DE $y' = a \cdot y$ is said to have a regular singularity at $t=0$

if, e.g. $a(t) = -\frac{k}{t^2}$ => soln: $y(t) = e^{k/t}$

in this case, $y(t)$ is always (independent of k) not defined at $t=0$. Such DE's are called irregular

BUT: This lecture is NOT about DE's.

we will be concerned with singularities of functions and maps (like in ex. 2.) + 3.)).

Recall: Definition 0.1: Let $U \subset \mathbb{R}^n$ be open, $f: U \to \mathbb{R}$ continuous. Then

a) f is called smooth or of class C^∞, if all partial derivatives of arbitrary high order of f exist

b) for a smooth function $f: U \to \mathbb{R}$ we write

$$\left(\partial_{x_{i_1}} \cdots \partial_{x_{i_k}}\right)(f) = \frac{\partial^k}{\partial_{x_{i_1}} \cdots \partial_{x_{i_k}}} f =$$

$$\frac{\partial}{\partial x_{i_1}}\left(\frac{\partial}{\partial_{x_{i_2}}}\left(\cdots \left(\frac{\partial}{\partial_{x_{i_k}}} f\right) \cdots\right)\right)$$

for any tuple $i_1, \dots, i_k \in \{1, \dots n\}$, if $n=1$, we write $f^{(k)} = \partial_x^{(k)} f$.

The first derivative is written as a vector (gradient) $Df: U \longrightarrow M(1 \times n, \mathbb{R})$, i.e.

$\forall p \in \mathcal{U}$, $Df(p)$ is a row vector of length n.

The second derivative is called the Hessian $D^2 f = \mathrm{Hess}(f) = \left(\frac{\partial^2}{\partial_{x_i} \partial_{x_j}} \right)_{i,j \in \{1,\dots,n\}}$;

$\mathcal{U} \longrightarrow \mathrm{Mat}(n \times n, \mathbb{R})$.

<u>Recall</u>: order in which partial derivatives are taken does not matter, i.e. $D^2 f$ is a symmetric matrix.

<u>Def. 0.2</u>: a) A point $p \in \mathcal{U}$ is called a critical pt. or a singular point of a smooth function $f: \mathcal{U} \to \mathbb{R}$, if $(Df)(p) = 0$.

$\left(\Longleftrightarrow \forall i \in \{1,\dots,n\} : (\partial_{x_i} f)(p) = 0 \right)$.

b) A singular point p of f is called

degenerate, iff $(D^2 f)(p)$ is singular, i.e.
$\det (D^2 f)(p) = 0$

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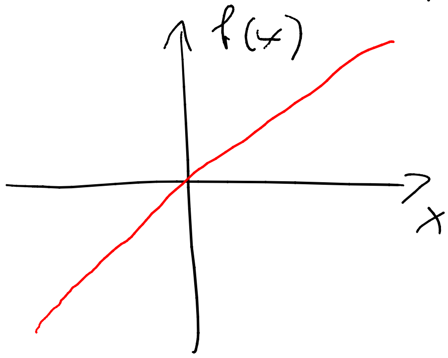
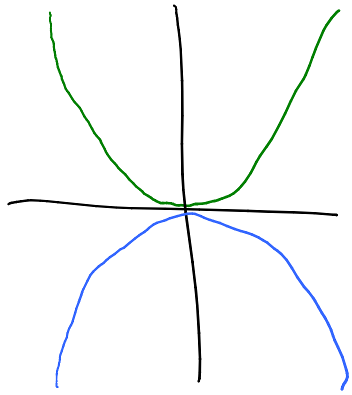
examples: first $U = \mathbb{R}$, in particular $n = 1$

1.) $f(x) = x$ $\Rightarrow$ graph



no singular point

2.) $f(x) = \pm x^2$ $\leadsto$ graph



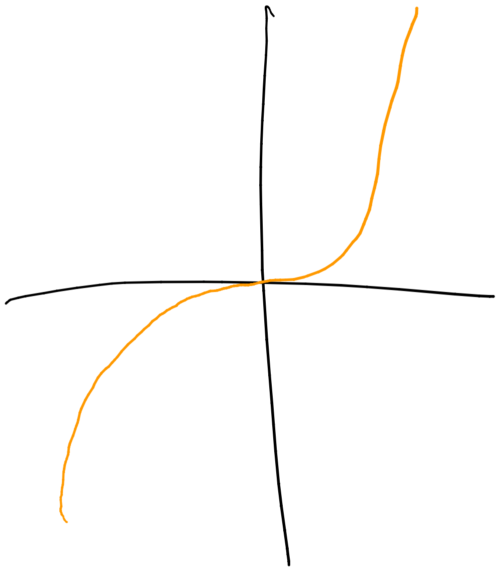
$f'(x) = \pm 2x$, $f'(0) = 0$

$f''(0) = \pm 2$

0 is critical pt., it is a local $\begin{cases} \text{min. } f = x^2 \\ \text{max } f = -x^2 \end{cases}$

non-degenerate

3.) $f(x)=x^3$ $\rightsquigarrow$ graph



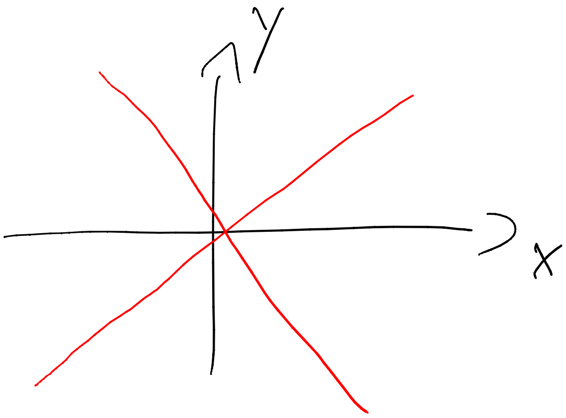
$f'(0)=0$

$f''(0)=0$

0 critical pt., it is saddle point, it degenerate

$U=\mathbb{R}^2$, $n=2$

4.) $f(x,y)=x^2-y^2$. Draw rather $f^{-1}(0)\subset\mathbb{R}^2$



$Df=\begin{pmatrix}2x & -2y\end{pmatrix}$, $D^2f=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$

$(Df)(0)=(0,0)$ $\rightsquigarrow$ $\underset{\mathbb{R}^2}{0}$ is the only crit. pt.

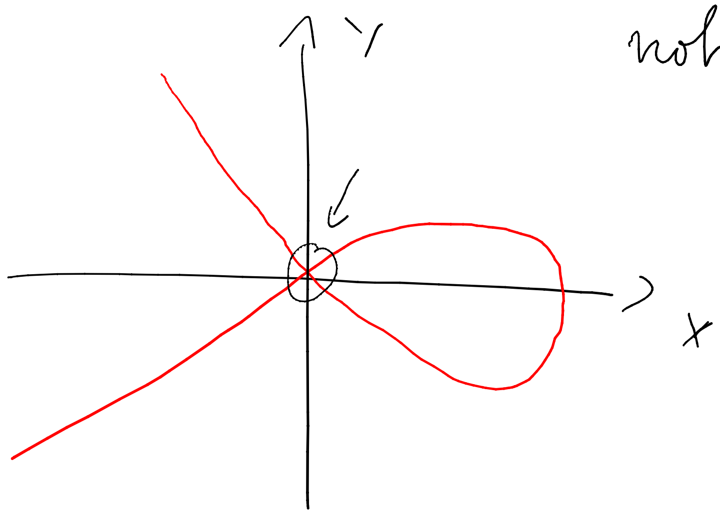
it is non-degenerate.

5.) $f(x,y) = y^2 - x^2(1-x)$, $Df = (3x^2-2x \quad 2y)$

$$D^2 f = \begin{pmatrix} 6x-2 & 0 \\ 0 & 2 \end{pmatrix}$$

critical points: $(0,0)$, $(\frac{2}{3}, 0)$

$f^{-1}(0)$:



notice $f(\frac{2}{3},0) \neq 0$

rk: if we restrict ex. 4.) + 5.) to small open sets $U \subset \mathbb{R}^2$ with $(0,0) \in U \Rightarrow f^{-1}(0)$ look quite similar: 

moral: Investigate functions „near" critical pts.!

<u>Def. 0.3.:</u> $f: U \to \mathbb{R}$ smooth, $p \in U$ a critical point of f. Then p is called <u>isolated</u>, if $\exists$ neighborhood V of p in U (i.e. $V \subset U$ open, $p \in V$) s.t. p is the only critical point of f in V

<u>ex.:</u> 1.) all examples above are isolated

2.) let $U = \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}} & \forall x > 0 \\ 0 & \forall x \leq 0 \end{cases}$

<u>Claim:</u> f is smooth

<u>idea:</u> $\lim\limits_{x \searrow 0} e^{-\frac{1}{x}} = 0$ (but $\lim\limits_{x \nearrow 0} e^{-\frac{1}{x}} = \infty$)

f continuous

$f'(x) = \begin{cases} \frac{1}{x^2} \cdot e^{-\frac{1}{x}} & x > 0 \\ 0 & \forall x \leq 0 \end{cases}$ $\quad \lim\limits_{x \searrow 0} \frac{1}{x^2} e^{-\frac{1}{x}} = \lim\limits_{y \to \infty} \underbrace{y^2}_{\to \infty} \underbrace{e^{-y}}_{\to 0} = 0$

$$f''(x) = \begin{cases} \left(\frac{2}{x^3} + \frac{1}{x^4}\right) \cdot e^{-\frac{1}{x}} & x > 0 \quad \lim_{x \downarrow 0} (\dots) = 0 \\ 0 & x \leq 0 \end{cases}$$

in general: $\boxed{f^{(k)}(x) = g(x) \cdot e^{-\frac{1}{x}} \quad g \in \mathbb{R}[\tfrac{1}{x}]}$

$$\Rightarrow \lim_{x \downarrow 0} g(x) \cdot e^{-\frac{1}{x}} = 0 \Rightarrow f \text{ is smooth}$$

obviously: $f'(x) = 0 \quad \forall x \leq 0$: i.e. all $x \leq 0$ are critical points of f

$\Rightarrow$ 0 is a non-isolated critical pt.

3.) (non-isolated critical pt. of a polynomial)

Let $U = \mathbb{R}^3$, $f(x, y, z) = x \cdot y \cdot z$

$$Df = (\partial_x f \;\; \partial_y f \;\; \partial_z f) = (yz \;\; xz \;\; xy)$$

$\{(0,0,z) \mid z \in \mathbb{R}\} \cup \{(0,y,0) \mid y \in \mathbb{R}\} \cup$

$\{(x,0,0) \mid x \in \mathbb{R}\}$ are the critical

points of f $\Longrightarrow$ $(0,0,0)$ is non-isolated critical point of f

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<u>Definition 0.9.:</u> (elementary catastrophes of René Thom)

The following smooth functions $f: \mathbb{R}^n \to \mathbb{R}$, $n \in \{1,2\}$ with isolated critical points are called elementary catastrophes:

$A_2: f(x) = x^3$ $\qquad$ $D_{+4}: f(x,y) = x^3 + y^3$

$A_3: f(x) = x^4$ $\qquad$ $D_{-4}: f(x,y) = x^3 - xy^2$

$A_4: f(x) = x^5$

$A_5: f(x) = x^6$ $\qquad$ $D_5: f(x,y) = x^2y + y^4$

Purpose of lecture: All singularities (of smooth functions $f: U \to \mathbb{R}$, $0 \in U \subset \mathbb{R}^n$ small) are up to coordinate change equivalent to one of the above list if Milnor number of $f \leq 5$.

Rk: similar statement for holomorphic functions $f: U \to \mathbb{C}$, $0 \in U \subset \mathbb{C}^n$, list even simpler: $D_{+4} \sim D_{-4} =: D_4$
